The Abdus Salam International Centre for Theoretical Physics

2162-15

Advanced Workshop on Anderson Localization, Nonlinearity and Turbulence: a Cross-Fertilization

23 August - 3 September, 2010

Freezing Transition in Decaying Burgers Turbulence and Random Matrix Dualities

Yan V. FYODOROV

School of Mathematical Sciences
The University of Nottingham
U.K.

Freezing Transition in Decaying Burgers Turbulence

and

Random Matrix Dualities

Yan V Fyodorov

School of Mathematical Sciences
The University of Nottingham, UK

30th of August 2010, ICTP , Trieste, Italy

References:

Y V F, **P Le Doussal , and A Rosso** : Europhys. Letters **90** (2010) 60004.

Decaying Burgers Turbulence:

- The problem of analysis of solutions of the (unforced) Burgers equation

$$\partial_t \mathbf{v} + (\mathbf{v}\nabla)\mathbf{v} = \nu\nabla^2\mathbf{v}, \quad \mathbf{v}(\mathbf{x}, t=0) = -\nabla\Psi_0(\mathbf{x}),\ \nu > \mathbf{0}$$

with random initial condition, usually assumed to be Gaussian and specified in terms of the two-point correlation function $\overline{\mathbf{v}(\mathbf{x},0)\mathbf{v}(\mathbf{x}',0)}$, or alternatively $\overline{\Psi_0(\mathbf{x})\Psi_0(\mathbf{x}')}$. General reference: **Bec & Khanin** Physics Reports **447** (2007), 1

- The problem appears as an important reference model not only in fluid dynamics, but also in such diverse physical contexts as statistics of growing interfaces (**Kardar, Parisi, Zhang '95**), statistical mechanics of systems with quenched disorder **Balents, Bouchaud, Mezard '95; Le Doussal '08**), and formation of large scale structures in cosmology (**Gurbatov, Saichev, Shandarin '89; Vergassola, Dubrulle, Frish, Noullez '94)** . In particular, the cosmological applications stimulated interest in **dBt** for **vanishing viscosity** $\nu \to 0$ and **scale-free** power-law random initial conditions:

$$\overline{\mathbf{v}(\mathbf{x},0)\mathbf{v}(\mathbf{x}',0)} \sim |\mathbf{x}-\mathbf{x}'|^{-n-1} \quad \text{at large distances}$$

Cole-Hopf solution, mapping to Statistical Mechanics:

Exact solution to unforced Burgers Equation with any (sufficiently fast decaying) initial condition can be written in terms of the **"effective potential"**

$$\mathcal{H}_x(\mathbf{y}) = \Psi_0(\mathbf{y}) + \frac{1}{2t}(\mathbf{x}-\mathbf{y})^2$$

as

$$\mathbf{v}(\mathbf{x},t) = -2\nu\nabla_x \ln Z, \quad Z(\mathbf{x},t) = \int e^{-\frac{1}{2\nu}\mathcal{H}_x(\mathbf{y})}\frac{d\mathbf{y}}{(4\pi\nu t)^{d/2}}$$

Obviously Z can be interpreted as the **partition function** emerging in Statistical Mechanics of a particle equilibrated in the energy potential $\mathcal{H}_x(\mathbf{y})$ at **effective temperature** $T = 2\nu$. In the **inviscid** (= zero temperature) limit finding the velocity profile thus amounts to solving the **minimization** problem

$$\mathbf{v}(\mathbf{x},t) = \nabla_x\left[\min_{\mathbf{y}}\mathcal{H}_x(\mathbf{y})\right] = \frac{1}{t}\left[\mathbf{x}-\mathbf{y}_{min}(\mathbf{x},t)\right]$$

which generically leads to formation of **shock patterns** - surfaces of discontinuities in the velocity field. **Nothing interesting is generally expected to happen for finite viscosity** $\nu > 0$.

Scale-free energy spectrum in 1D:

scale-free power-law random initial conditions:

$$\overline{v(x,0)v(x',0)} \sim |x - x'|^{-\mathbf{n}-1} \quad \text{at large distances}$$

It is conventional characterize the velocity field $v(x,t)$ via the "energy spectrum"

$$E(k,t) = \frac{1}{2\pi}\int \overline{v(x,t)v(0,t)} \exp ikx \, dx.$$

The **scale-free** random initial conditions for $v(x,0) = \partial_x \Psi_0(x)$ imply for the initial energy spectrum $E(k,0) \sim |k|^{\mathbf{n}}$, $k \to 0$ and falls off quickly for $k \gg 1$. For $\mathbf{n} > 1$ the corresponding **potential** $\Psi_0(x)$ is a **random stationary function** with finite variance $\overline{[\Psi_0(x)]^2} = \int E(k,0)\frac{dk}{k^2} < \infty$ ("**short-ranged**" disorder). For $-1 < \mathbf{n} < 1$ ("**long-ranged**" disorder) $\Psi_0(x)$ has **stationary increments**: $\overline{[\Psi_0(x) - \Psi_0(x')]^2} < \infty$. For $\mathbf{n} < -1$ only the initial **velocity** field $v(x,0)$ may have stationary increments (e.g Brownian-motion type velocity).

The picture of decaying Burgers turbulence in $d = 1$:

Self-similar initial conditions: $\overline{\Psi_0(x)\Psi_0(x')} \sim |x - x'|^{-\mathbf{n}+1}$, **Inviscid** limit $\nu = 0$.

Exactly solvable cases:

(i) Brownian-motion initial potential ($\mathbf{n} = 0$) (Burgers '74) . (ii) Delta-correlated initial potential ($\mathbf{n} = 2$) (Kida '79). (iii) Brownian-motion type initial velocity ($\mathbf{n} = -2$) (Sinai '92).

For general $\mathbf{n}$ it is commonly accepted that:
(i) one-point p.d.f. for the velocity $v(x,t)$ is **always Gaussian**
(ii) the energy decay $E(t) = \overline{v(x,t)^2} = \int E(k,t), dk$ is governed by the **self-similar evolution** $E(k,t) = \frac{L^3(t)}{t^2}\tilde{E}\left[kL(t)\right]$ with the **characteristic lengthscale** $L(t)$.

General qualitative arguments (see **Gurbatov et al.** '91;'97)) yield that in the **long-ranged** case $\mathbf{n} < \mathbf{1}$ the lengthscale evolves as $L(t) \sim t^{2/(3+n)}$ so that $E(t) = \overline{v(x,t)^2} \sim \dot{L}^2 \sim t^{-\frac{2(n+1)}{n+3}}$. In the the **short-ranged** potential $\mathbf{n} > \mathbf{1}$ we have, in contrast, the **Kida**'s decay laws: $L(t) \sim t^{1/2}/(\ln t)^{1/4}$ and the energy decay $E(t) \sim 1/t\sqrt{\ln t}$. Some subtle violation of self-similarity occurs for $1 < \mathbf{n} < 2$.
The limiting case $\mathbf{n} = 1$ **has not been studied yet** and is our main object of interest.

Logarithmically-correlated random potential and freezing:

For $\mathbf{n} = 1$ the initial velocity decays as $< v(x,0)v(x',0) >\sim |x-x'|^{-2}$ which implies for the potential

$$\overline{\Psi_0(x)\Psi_0(x')} = -2\ln\left[|x-x'|/L\right], \quad \epsilon < |x-x'| < L$$

where $L \gg 1$ and $\epsilon \ll 1$ are the infrared and ultraviolet cutoff scales. We further assume $\overline{\Psi_0(x)\Psi_0(x')} = 2\ln L/\epsilon$ for $|x-x'| \leq \epsilon$ and $\overline{\Psi_0(x)\Psi_0(x')} = 0$ for $|x-x'| \geq L$.
Statistical mechanics of a single particle in random logarithmically correlated landscape was recently under intensive investigation (**Chamon, Mudry et al.** '96;'97 **Carpentier** $\&$ **Le Doussal** '01; **YVF** $\&$ **Bouchaud** '08) which provided evidences in favour of existence of a **freezing transition** at the temperature $T = T_c = 1$.
A pecular mechanism for such freezing was revealed by **YVF, Le Doussal** $\&$ **Rosso** '09 : it was conjectured that all thermodynamic quantities which in the high-temperature phase $T > T_c$ happen to be **invariant** with respect to the **duality transformation** $T \to 1/T$ **freeze** at the self-dual point $T_c = 1$, that is **retain down to zero temperature** the value they acquired at the critical point $T = T_c$.

Statistical mechanics in random potential and Burgers velocity:

In the language of statistical mechanics with $T = 2\nu$ the velocity p.d.f. is given by

$$\mathcal{P}(v) = \overline{\delta\left(v + \frac{1}{t} \prec y \succ_T\right)} \quad \text{where} \quad \prec \mathcal{O} \succ_T = Z^{-1} \int \frac{dy}{\sqrt{2\pi T t}} \mathcal{O}(y) e^{-\mathcal{H}_0(y)/T}$$

where $\mathcal{H}_0(y) = \Psi_0(y) + \frac{y^2}{2t}$ and we set $x = 0$ in view of the **translational invariance** of the disorder. To understand better **thermodynamics** of our system and the nature of the anticipated **freezing transition** it turns out to be instructive to consider also a different object:

$$\mathcal{P}_Y(Y) = \overline{\prec \delta(Y - y) \succ_T} = \overline{\frac{1}{Z} e^{-\mathcal{H}_0(Y)/T}}$$

interpreted as the averaged p.d.f. of the coordinate of a particle **equilibrated** at a given temperature T in the random energy landscape $\mathcal{H}_0(Y)$. At $T \to 0$ the thermal average is obviously dominated by the **deepest minimum** of the landscape whose position $\mathbf{Y}_{\min}$ fluctuates from one realization of disorder to the other. This mechanism immediately implies for velocity p.d.f. in **zero viscosity** (= $T \to 0$) limit the relation

$$\mathcal{P}(v)|_{T=0} = t\mathcal{P}_Y(vt)|_{T=0}$$

Statistical mechanics in random potential and λ- Hermite ensemble:

The disorder averaging procedure for $\mathcal{P}_Y(Y)$ can be performed via the standard **replica trick** after representing $Z^{-1} = Z^{n-1}|_{n\to 0}$ and using the Gaussian nature of the random potential $\Psi_0(y)$ by employing $\overline{\Psi_0(y)\Psi_0(y')} = -2\ln\left[|y-y'|/L\right]$. One finds

$$\mathcal{P}_Y(Y) = \lim_{n\to 0}\left\langle \frac{1}{n}\sum_{j=1}^{n}\delta\left(Y - z_j\sqrt{Tt}\right)\right\rangle_{n,-\gamma}$$

where $\gamma = 1/T^2 > 0$ and we have defined for $1 \le n < 1/\gamma$

$$\langle \ldots \rangle_{n,\lambda} = \frac{1}{S_n(\lambda)}\int_{-\infty}^{\infty}(\ldots)\prod_{i<j}^{n}|z_i - z_j|^{2\lambda}\prod_{j=1}^{n}\frac{dz_j}{\sqrt{2\pi}}e^{-\frac{z_j^2}{2}},$$

with $S_n(\lambda) = \prod_{j=1}^{j=n}\left[\Gamma(1+j\lambda)/\Gamma(1+\lambda)\right]$ being the famous **Selberg** integral. For finite integer $n \geq 1$ and $\lambda > 0$ the above expression is nothing else but the mean density of the so-called λ-**Hermite** ensemble of $n \times n$ random matrices introduced by **Dumitriu** $\&$ **Edelman** '02 .

Note that the corresponding random matrix-like integrals are still **convergent** for $\lambda = -\gamma$ as long as $0 < \gamma < 1$. The **replica limit** implies $n \to 0$.

Statistical mechanics in random potential and λ- Hermite ensemble:

Although a closed-form expression for the eigenvalue density for λ-**Hermite** ensemble does not seem to be available **Dumitriu** $\&$ **Edelman** '02,'06 developed analytic tools allowing one to calculate **a few lower moments** of that density. Performing the analytical continuation $n \to 0$ and $\lambda \to -\gamma$ we obtained the lower nonvanishing moments $M_{2q} = \int \mathcal{P}_Y(Y) Y^{2q} dY$ up to $2q = 16$. We present below the corresponding **cumulants** C_{2q}:

$$C_2 = t\left(T + T^{-1}\right), \quad C_4 = -t^2, \quad C_6 = 2t^3\left(T + T^{-1}\right)$$

$$C_8 = -t^4\left[26 + 6\left(T^2 + T^{-2}\right)\right], \quad C_{10} = t^5\left[300\left(T + T^{-1}\right) + 24\left(T^3 + T^{-3}\right)\right]$$

and similar but longer expressions for $C_{2q},\ q = 6, 7, 8$.

The main feature apparent from the above (and proved in full generality) is that **all the cumulants** (and hence the whole function $\mathcal{P}_Y(Y)$) are **invariant** with respect to the **duality transformation** $T \to 1/T$. Employing the **freezing conjecture** we thus predict that **the whole probability distribution** $\mathcal{P}_Y(Y)$ **freezes** at the critical point $T = T_c = 1$ providing a vivid picture of what freezing entails.

Freezing scenario vs. numerics for zero viscosity velocity moments:

If this scenario were correct, the values of the above cumulants evaluated at $T = 1$ should immediately provide, in view of the discussed zero-temperature correspondence, the **cumulants of the velocity p.d.f.** in **zero viscosity** limit:

$$\overline{v^2}|_{\nu=0} = \frac{2}{t}, \quad \overline{v^4}^c = \left[\overline{v^4} - 3\overline{v^2}^2\right]|_{\nu=0} = -\frac{1}{t^2}, \; etc.$$

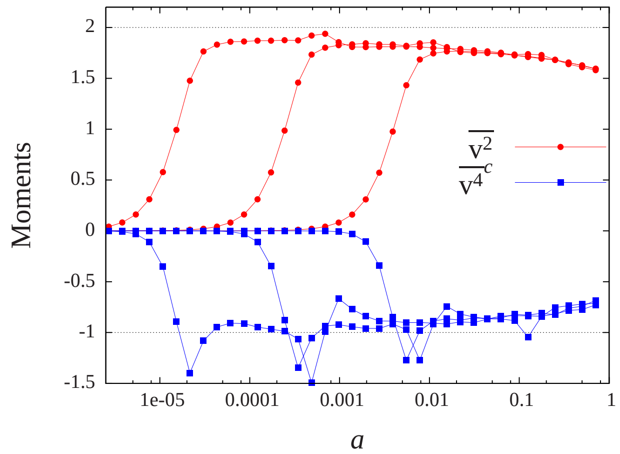

Figure 1: **Numerical evaluation of $\overline{v^2}$ and $\overline{v^4}^c$ in the inviscid limit $\nu = 0$ for discretized Burgers equation (number of points $M = 2^{10}, 2^{14}, 2^{18}$) with periodic version of the logarithmically correlated initial potential compared with the theoretical prediction at $t = 1$ (averaging is peformed over 10^6 samples). Small oscillations are observed in $\overline{v^4}^c$ when periodic boundary conditions cannot be neglected, and disappear with increase of M.**

Freezing scenario vs. numerics for second moment at all temperatures:

Moreover, by employing the **exact** relation $\overline{\prec y^2 \succ_T} = t^2 \overline{\prec y \succ_T^2} + Tt$ valid at **any** temperature due to **statistical translational invariance** of the random potential $\Psi_0(x)$ the above results predict $\overline{v^2} = \overline{\prec y \succ_T^2} = \frac{1}{t}(2 - T)$ in the whole low-temperature phase $T < T_c = 1$.

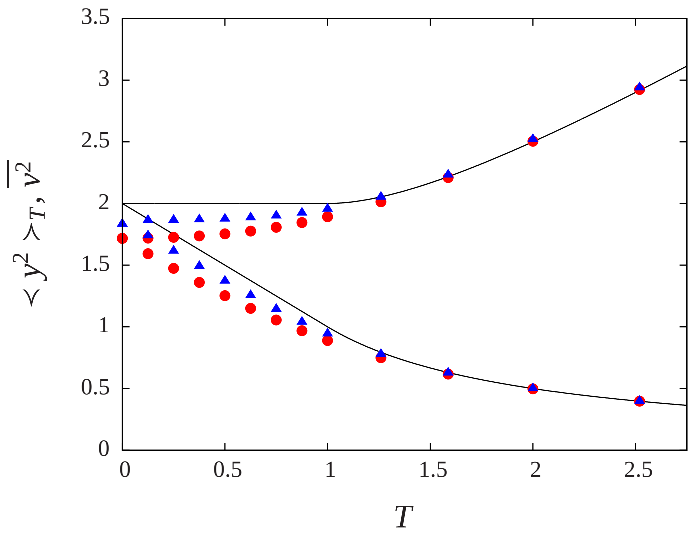

Figure 2: **The top solid line is the analytical prediction for $\overline{\prec y^2 \succ_T}$ (from C_2 in the text, with $t = 1$), and the bottom solid line $\overline{v^2} = \overline{\prec y^2 \succ_T} - T$. Circles are simulations with $M = 2^{14}$ points, triangles are simulations with $M = 2^{18}$ points, (averaged over 5×10^4 samples).**

Freezing manifestations in p.d.f. of local velocities:

We expect that the **freezing transition** induces changes in the shape of the p.d.f. for velocity $v(x,t) = -2\nu\partial_x \ln Z(x,t)$ at finite critical viscosity $\nu_c(\equiv T_c/2) > 0$. Consider the generating function calculated via a variant of the replica trick:

$$G(q) = \overline{\ln\left[1 - iq\partial_x \ln Z(x,t)\right]} = \lim_{n\to 0} \frac{1}{n}(W_n(q) - W_n(0))$$

where for any integer $n \geq 1$

$$W_n(q) = \sum_{k=0}^{n} \left(\begin{array}{c} n \\ k \end{array}\right) (iqT)^k \overline{(\partial_x Z)^k \, Z^{n-k}}$$

Following the same steps as before we arrive at the identity

$$\frac{W_n(q)}{L^{n^2\gamma}(\sqrt{T}t)^{-n(n-1)\gamma} S_n(-\gamma)} = \left\langle \prod_{j=1}^{n} (iq\sqrt{\frac{T}{t}}\, z_j + 1) \right\rangle_{n,-\gamma}, \quad \gamma = 1/T^2$$

To continue to $n = 0$ we exploit a duality relation for λ- Hermite ensemble

$$\left\langle \prod_{j=1}^{n} (z_j + \tau) \right\rangle_{n,\lambda} = \int_{-\infty}^{\infty} \frac{dw}{\sqrt{2\pi}}\, e^{-\frac{w^2}{2}} \left(\tau + i\sqrt{\lambda} w\right)^n$$

where τ is an arbitrary parameter. Though originally proved by Dumitriu $\&$ Edelman '02 for integer $n \geq 1$ and $\lambda > 0$, we analytically continue $\lambda = -\gamma$ with $0 < \gamma < 1$ and perform $n \to 0$. In this way we recover the velocity p.d.f. $\mathcal{P}(v)$ which turns out to be a simple Gaussian with zero mean and variance $\overline{v^2} = 1/Tt$. Such a calculation is valid only for $T > T_c = 1$.

Freezing is equivalent to the spontaneous breakdown of replica symmetry:

It turns out that **freezing** in the low-temperature phase $T < T_c = 1$ can be successfully accounted for by incorporating the mechanism of **spontaneouus replica symmetry breaking** (RSB). In doing this we adopt to the continuum model the scheme of incorporating **one-step RSB** mechanisms proposed in **Bouchaud & Mezard '97** for the simplest case of discrete Random Energy Model without spatial correlations.

The basic idea behind this scheme is that for $T < T_c$ and $0 < n < 1$ the configurations which give the leading-order contributions to the random-matrix integral over $z_1, \ldots, z_n$ are obtained by grouping n replica indices into $k = n/m$ groups of m replica each, and assuming that all coordinates z_i for the replica indices $i_1, \ldots, i_m$ inside the same group are "frozen" around the common value, i.e. approximately equal: $z_{i_1} \approx z_{i_2} \approx \ldots \approx z_{i_m}$. At the same time k coordinates of the centres of masses of different groups play the role of new effective degrees of freedom and can take any values. Integrating out the "frozen" coordinates and taking into account the number of ways we can built the groups we find that the expression becomes proportional to the $m-$ dependent large factor $\exp - \left[n\left(\frac{1}{m} + \frac{m}{T^2}\right)\ln\epsilon\right]$. The parameter m is then found from extremizing (in fact, minimizing) this factor, which selects $m = T$ as long as $T < T_c = 1$.

Up to factors tending to unity in the replica limit $n \to 0$ we arrive at the relation

$$W_n(q) \sim \left\langle \prod_{l=1}^{k} \left(1 + iq\sqrt{t} z_l\right)^m \right\rangle_{k=\frac{n}{T}, -\gamma m^2 = -1}, \quad m = T \qquad (1)$$

Finally, we notice that one can perform the replica limit $n \to 0$ by exploiting one more, rather non-trivial, duality relation for $\lambda-$Hermite RMT ensemble discovered by Desrosiers '09 for k, m positive integer, $\lambda > 0$ and any complex s:

$$\left\langle \prod_{l=1}^{k} (z_l + s)^{-m\lambda} \right\rangle_{k,\lambda} = \left\langle \prod_{l=1}^{m} (z_l + s)^{-k\lambda} \right\rangle_{m,\lambda}$$

We conjecture that the relation remains valid if continued to $\lambda \to -1$ and furthermore to $0 < k, m < 1$.

This allows to perform straightforwardly the **replica limit $n \to 0$** leading to the expression of the velocity p.d.f

$$\mathcal{P}(v) = \lim_{m \to T} \lim_{\lambda \to -1} \left\langle \delta(v + \frac{z_1}{\sqrt{t}}) \right\rangle_{m,\lambda}$$

as eigenvalue density in the **$\lambda-$Hermite** ensemble. As a consequence, at $T = 0$ it indeed simply relates to the corresponding limit of the distribution $\mathcal{P}_Y(Y)$ in full agreement with the **freezing scenario**.

At any $0 \leq T \leq 1$ we then can find a few low velocity cumulants explicitly. In particular

$$\overline{v^2}|_{T<1} = \frac{1}{t}(2-T), \ \left[\overline{v^4} - 3\overline{v^2}^2\right]|_{T<1} = -\frac{1}{t^2}(1-T)^2$$

which fully agrees with the zero-viscosity limit and also matches the high-temperature phase moments at the transition point $T_c = 1$. These results are in agreement with numerics, and show that the velocity p.d.f. $\mathcal{P}(v)$ is non-Gaussian everywhere in the low-viscosity phase. The shape is consistent with a negative kurtosis and the difference increases at low temperature.

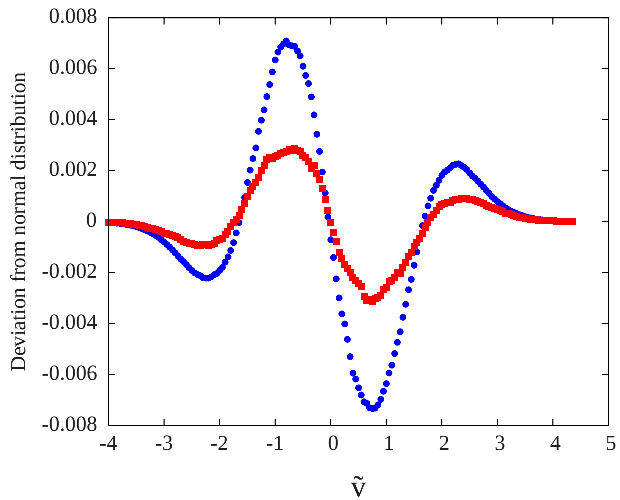

Figure 3: Non-Gaussian character of the rescaled velocity $\tilde{v} = v/\sqrt{\overline{v^2}}$ below the freezing temperature, as shown by the difference between their cumulative distributions. Simulations are performed over 10^6 samples of $M = 2^{18}$ points and $a = 0.002$. Circles are data at $T = 0$, Squares are data at $T = 0.5 < T_c$.

Conclusions & Discussion:

- **Combining the methods of statistical mechanics with insights from the random matrix theory we reveal a phase transition with decreasing viscosity ν at finite $\nu = \nu_c > 0$ in one-dimensional decaying Burgers turbulence with a power-law correlated random profile of Gaussian-distributed initial velocities $\overline{v(x,0)v(x',0)} \sim |x - x'|^{-2}$. The low-viscosity phase exhibits non-Gaussian one-point probability density of velocities, reflecting a spontaneous one step replica symmetry breaking (RSB) in the associated statistical mechanics problem. We obtain the low orders cumulants analytically which favourably agree with numerical simulations.**

- **RSB mechanisms in Burgers turbulence were exploited earlier by Bouchaud and Mezard, '96'97 in their mean-field treatment of the short-ranged correlated Kida model in zero-viscosity limit. Our model (and results) are essentially non mean-field in nature. In particular, the shock size distribution computed numerically is found to be different from the Kida model. We also got some analytical insights for the finite viscosity $\nu > 0$ behaviour of the velocities in the Kida model (velocity distribution, energy decay) and reveal some freezing-like crossover. For example for the energy decay the approach gives**

  $\overline{v^2} = \frac{1}{t}\left(\left(\frac{\sigma}{\ln(2\pi t/\rho)}\right)^{1/2} - 2\nu\right)$ **as long as** $t < t_c = \frac{1}{4\pi\nu}e^{\frac{\sigma}{4\nu^2}}$, **and** $v = 0$ **for** $t > t_c$

  Here $\rho = m/2\nu$ satisfies $\sigma\rho^2 = 1 + \ln(2\pi t/\rho)$, and m must be in the interval $[0, 1]$

Our method was based on a few assumptions, most importantly
(i) freezing scenario and its manifestation via **replica symmetry breaking**
(ii) ability to analytically continue **random matrix duality identities** beyond their conventional range.

Although we believe numerics convincingly confirms validity of our analytical results it remains **a challenge to justify** those steps by *bona fide* mathematical methods.

work in progress:
Statistics of shocks in **decaying Burgers turbulence** with correlated initial conditions $< v(x)v(x') > \sim |x - x'|^{-2}$.