The Abdus Salam International Centre for Theoretical Physics

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Advanced Workshop on Anderson Localization, Nonlinearity and Turbulence: a Cross-Fertilization

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Random Conformal Curves in 2D in and out-of Equilibrium

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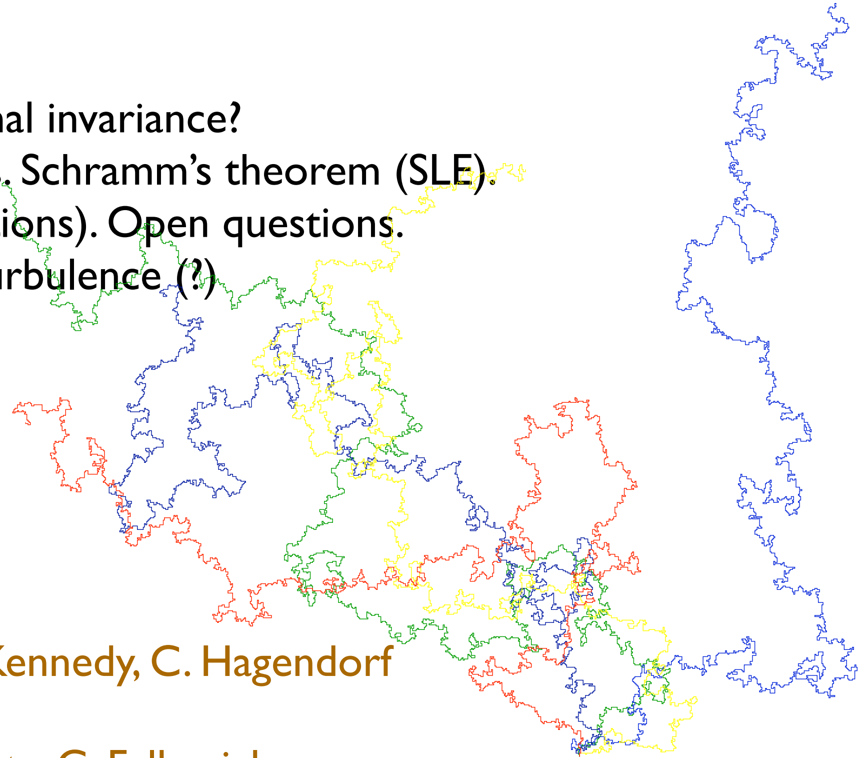
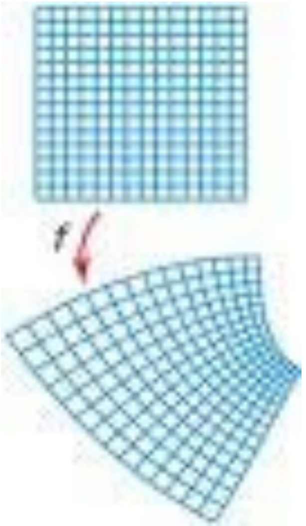
Random conformal curves in 2D in and out-of equilibrium

-- Random curves. Conformal invariance?
-- Dynamics of static curves. Schramm's theorem (SLE).
-- Equilibrium SLE: (Applications). Open questions.
-- Out-of-equilibrium: 2D turbulence (?)



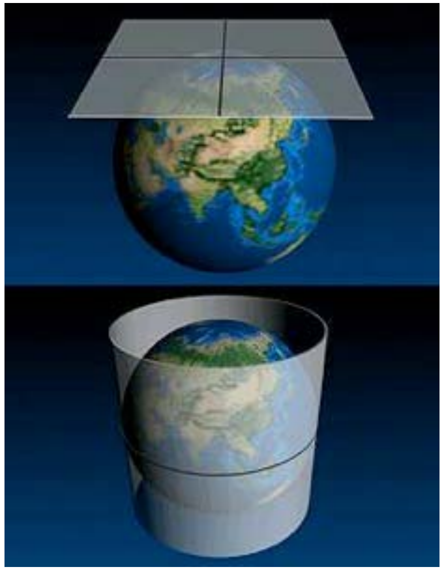
with M. Bauer
K. Kytola, L. Cantini, T. Kennedy, C. Hagendorf

with A. Celani, G. Boffetta, G. Falkovich,



Conformal transformations:

planar transformations preserving angles,
i.e. local rotations and dilatations,

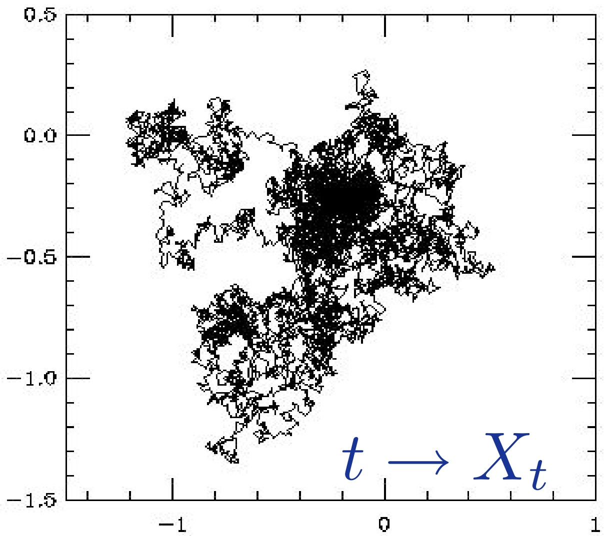


Mercator world map *Nova et Aucta Orbis Terrae Descriptio ad Usum Navigatium Emendate* (1569)

Conformally invariant random curves

- The simplest example: 2D Brownian motion

scaling limit of random walks



Why conformal invariance?
by cut and paste property (Levy)

Gaussian $< X_t^2 >= t$
Rotation and scaling $X_{\mu^2 t} \simeq \mu X_t$
I.I.D. increments

$$X_t - X_s \simeq X_{t-s}$$

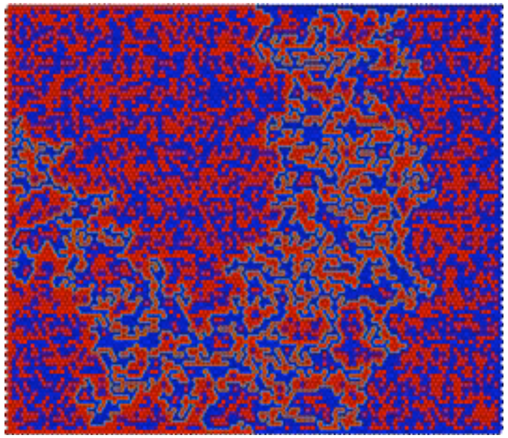
Concatenation of increments:
Conformal transformation
= Time reparametrization

Scale invariance + locality => conformal invariance

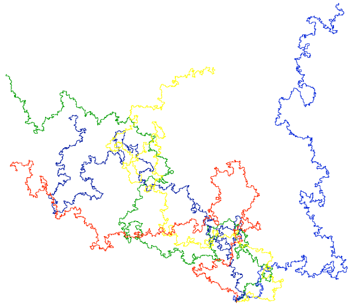
Critical interfaces at equilibrium

Interfaces in 2D critical systems, or geometrical curves,
in planar domain with boundary conditions

Percolation

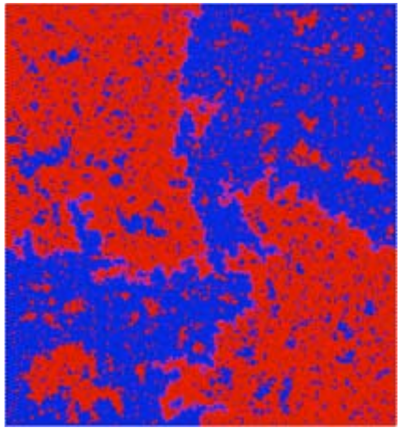


Self avoiding walks (SAW)
Polymers (dilute phase)



Non intersecting walks with weights:
$W_\gamma = y_c^{||\gamma||}$ with fugacity y_c

2D Ising



The measure? Boltzmann rules

What is the continuum limit?

But non-local object (local Boltzmann weights, difficult in field theory)

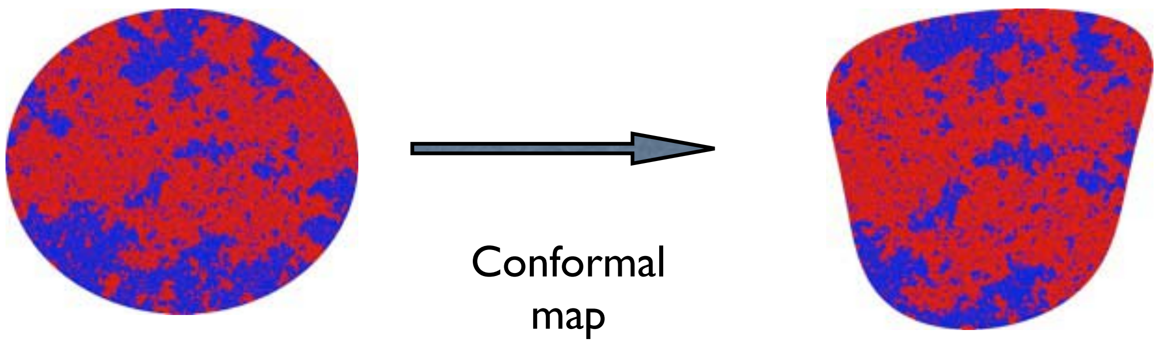
What is conformal invariance? (Schramm)

Conformal invariance for random curves, in planar domains, with marked points (originating from geometry or stat. mech. models)

☆ Conformal Transport

But (almost) a tautology......

Comparing two measures in two different domains.



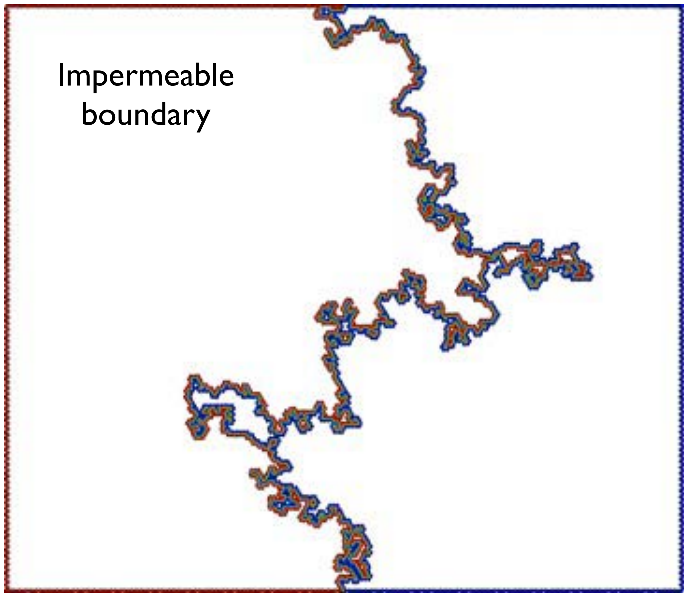
$$\mathbf{P}_D[\gamma \subset U] = \mathbf{P}_{f(D)}[\gamma \subset f(U)]$$

The image of any sample should be distributed as a sample in the image space

Interlude: remembering the past (boundary)

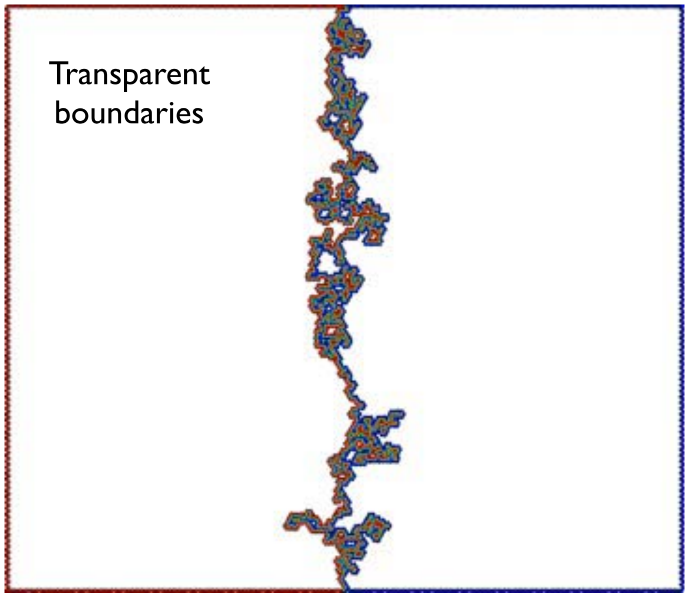
On a hexagonal lattice, with red/blue boundary,
a line is drawn on the frontier between blues and reds

Harmonic explorer (Shramm-Scheffield)



conformally invariant

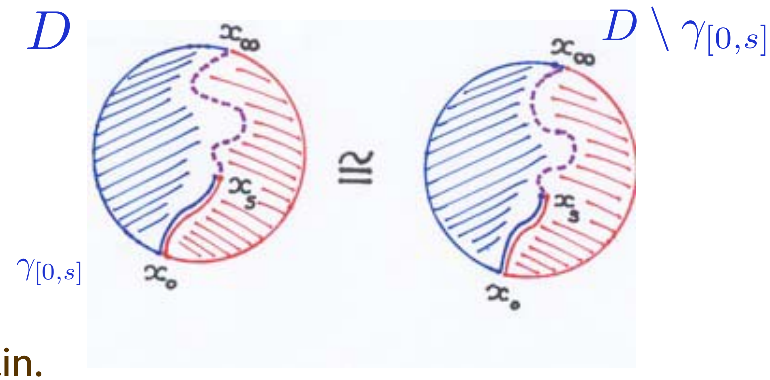
Boundary explorer



Not conformally invariant

After N step, a random walker RW is send from the hexagon on the tip of the curves, the color of the hexagon is assigned to be that of the boundary hit by RW.

What is conformal invariance? (Schramm)



★ Domain Markov property

Two possible setups with data:

$D, \; x_0, \; x_\infty, \; \gamma_{[0,s]}$

Comparing conditioning and cutting the domain.

$$\mathbf{P}_D[\cdots|\gamma_{[0,s]}] \equiv \mathbf{P}_{D\setminus\gamma_{[0,s]}}[\cdots]$$

★ Domain Markov property + Conformal Transport

→ determine the measure par iteration

Enough to specify the statistics of the germs of the curves

--- How to code the evolution of curves?

Curves coded into a conformal maps
which satisfy Loewner equation with source:

$$\frac{dg_t(z)}{dt} = \frac{2}{g_t(z) - \xi_t}$$

Schramm-Loewner Evolutions (SLE).

Conformally invariant measure on curve in planar domains

«Schramm's theorem»:

Conformal transport +
domain Markov properrty

$$t \to \xi_t = \sqrt{\kappa}\ B_t$$

= 1D Brownian motion

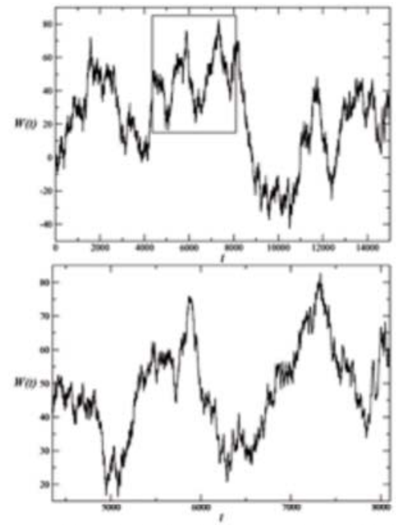


Figure 7. Self-similarity of a Brownian motion path. In (a) we plot a path of a Brownian motion with 15000 time steps. The curve in (b) is a blow-up of the region delimited by a rectangle in (a), where we have rescaled the x axis by a factor 4 and the y axis by a factor 2. Note that the graphs in (a) and (b) "look the same," statistically speaking. This process can be repeated indefinitely.

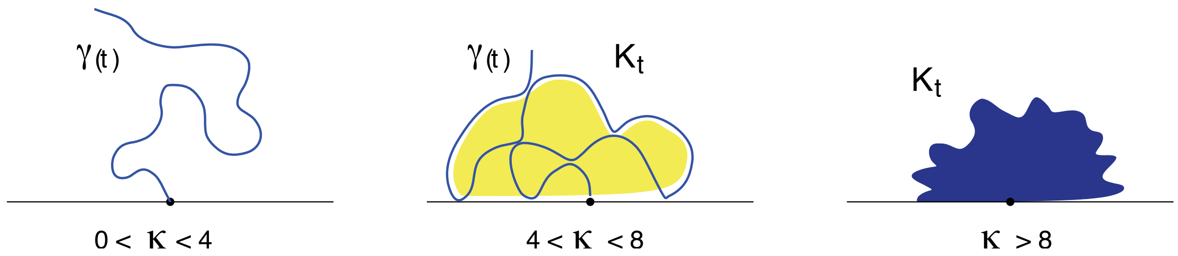
I.I.D. of the Brownian motion increments translates into
the domain Markov property (conformal invariance)

In practice: knowledge on 1D brownian motion
= knowledge on the curves

Classification of conformal invariant curves with properties

One parameter: κ

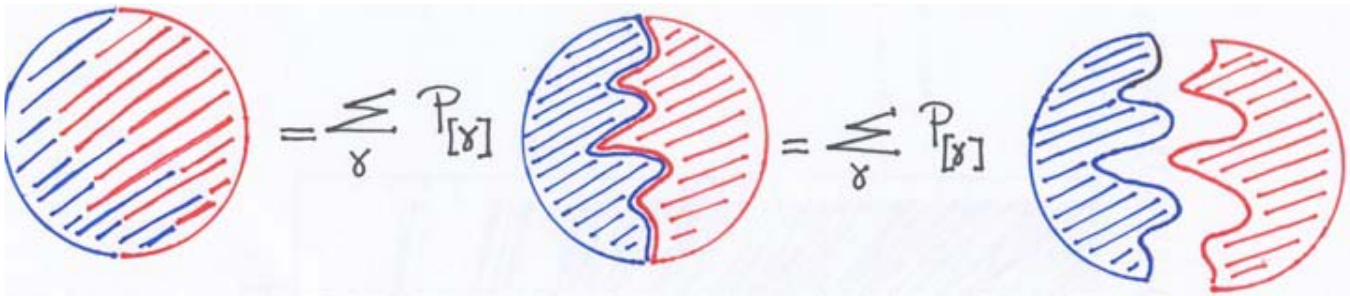
Different phases:



Relation with statistical mechanics

How is the SLE one-parameter of family of curves related to in 2D stat. models?

What do we learn from the math? Reorganize the statistical Boltzmann sums...



2D critical systems are classified by CFT (central charge c, etc...) ...a long list...

«Correspondence» (DB, MB)

Thm: Correlation functions of CFT with appropriate central charge and boundary operators in the cut domain are martingales for SLE(K).

$$M_t \equiv t \to < \psi_\infty \mathcal{O} \psi_{\gamma(t)} >_{D \backslash \gamma_{[0,t]}}$$

$$\mathbf{E}[< \psi_\infty \mathcal{O} \psi_{\gamma(t)} >_{D \backslash \gamma_{[0,t]}}] =< \psi_\infty \mathcal{O} \psi_o >_D$$

⟹ SLE probabilities $\longleftrightarrow$ CFT data

A few samples of results

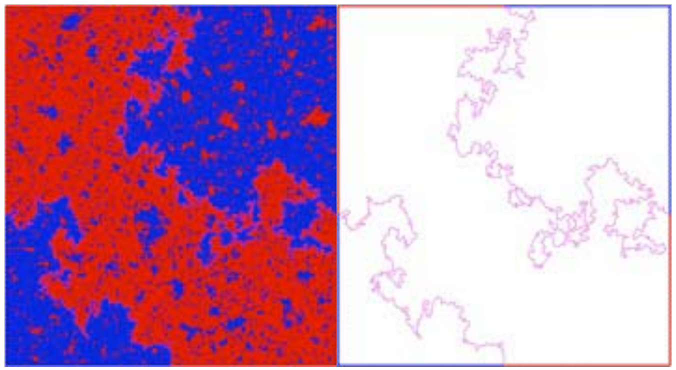
Schramm, Lawler, Werner, and others...

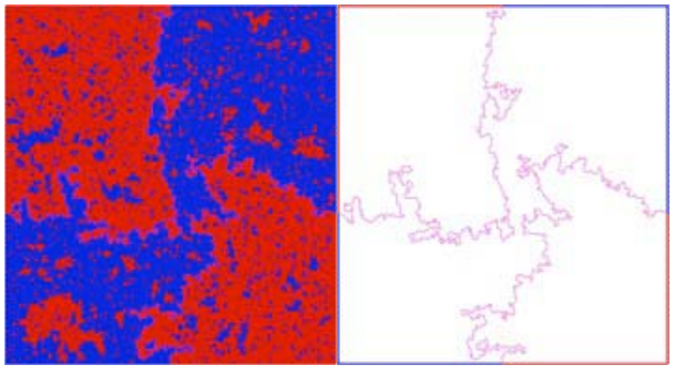
Basics properties of the curves
fractal dimension
continuum scaling limits
specific properties
natural parametrization
etc....

Brownian exponants
Proof of CFT prediction
Proof of Mandelbrot's conjecture:
dim(ext. perimeter)=4/3
link with Gaussian free field
etc...

A few open questions:

-- more on the relation with random fluctuating surfaces
-- limits of discrete models and off-criticality.
-- (multi) loop measure...





Interfaces in the scaling limit near criticality

-- How to describe the off-critical measures?
-- What are the Loewner driving processes?
(a.s.) known for LERW and GFF (Smirnov-Makarov, DB-MB)

Ising: $T - T_c \simeq a$
Percolation: $p - p_c \simeq a^{3/4}$

-- Is the off-critical measure singular w.r.t to the critical one?
Yes for percolation (Nolin-Werner),
No for LERW or GFF (description of the off critical drift, DB-MB)

Asymmetry >> Fluctuations

$$a^{3/4}\, a^{-d_\kappa} \gg a^{-d_\kappa/2}$$

Length of SLE curves (natural parametrization)

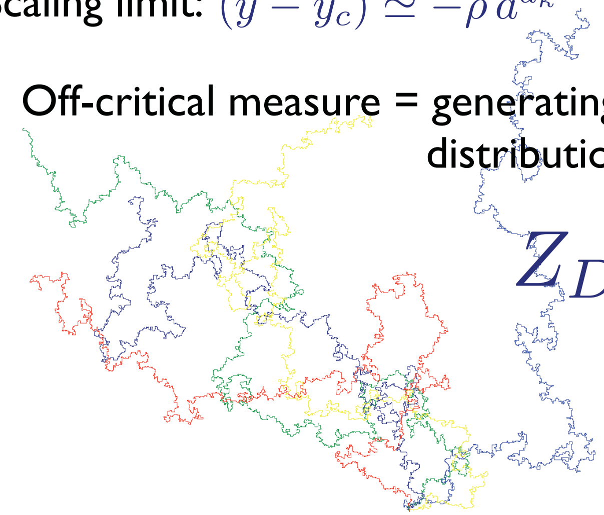
S.A.W. $W_\gamma = y^{||\gamma||}$ with fugacity y

Number of steps: $||\gamma|| \simeq a^{-d_\kappa}\, \ell_D(\gamma)$

Renormalised length

Scaling limit: $(y - y_c) \simeq -\rho\, a^{d_\kappa}$

Off-critical measure = generating function for the probability distribution of the renormalised length



$$Z_D = \sum_\gamma W_\gamma = E_{SLE}\left[\, e^{-\rho\, \ell_D(\gamma)}\, \right]$$

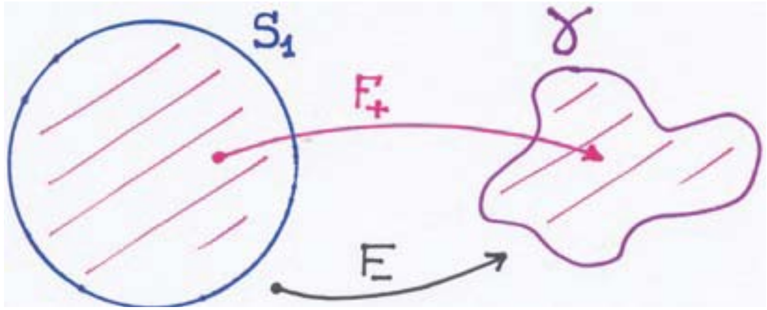
Conformal welding, Liouville and SLE (?)

- Diffeomorphism of the circle

$$h : S_1 \to S_1$$

factorized into holomorphic maps

$$h = F_-^{-1} \circ F_+ \quad \text{on } S_1$$



Similarly for half lines and curves drawn on the sphere

Conjectured measure:
(Jones)

$$h(\theta) \propto \int_0^{e^{i\theta}} du\, e^{\sqrt{\kappa}\varphi(u)/2}$$

with $\varphi(u)$ Gaussian free field

Interpretation: identify points at equal Euclidean and Liouville lengths from 0

- Is a simple «regular» curve in Liouville geometry is an SLE like curve in Euclidean geometry ??

KPZ formula (relating fractal dim. measure with Euclidean or Liouville balls) (KPZ ... DS)

$$d_H = 2 - 2x, \quad d_H^Q = 2 - 2\Delta$$

$$x = \Delta + \tfrac{\kappa}{4}\Delta(\Delta - 1) \qquad \text{(David)}$$

$$\text{for } d_H^Q = 1, \quad d_H = 1 + \kappa/8$$

Is a Liouville geodesic a SLE-like curve?

Is there conformal invariance in 2D Turbulence?

an example of conformally invariant curves out-of-equilibrium
(non-locality and strongly interacting systems....)

Fluid dynamics: Navier-Stokes equations (at high Re)

Kraichnan (1967)

The double cascade picture

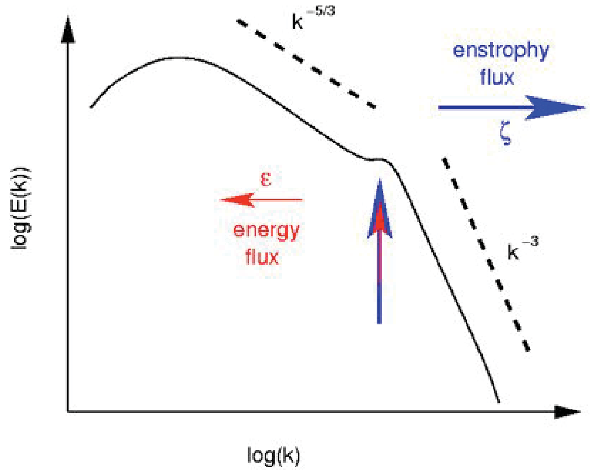
Direct cascade: enstrophy flux

Inertial range: $L_{inj} \gg \ell \gg l_{diss}$
scaling $u_\ell \simeq \zeta^{1/3}\ell$
possibly anomalous

Inverse cascade: energy flux

Inertial range $L_{IR} \gg \ell \gg l_{inj}$
scaling $u_\ell \simeq \epsilon^{1/3}\ell^{1/3}$
folklore: non anomalous

$$\partial_t \omega + \mathbf{u} \cdot \nabla \omega - \nu \nabla^2 \omega + \omega/\tau = F$$



$$E(k) \simeq k^{-3} \ , \ E(k) \simeq k^{-5/3}$$

Soap film/electrolyte cell turbulence
Geophysical flows

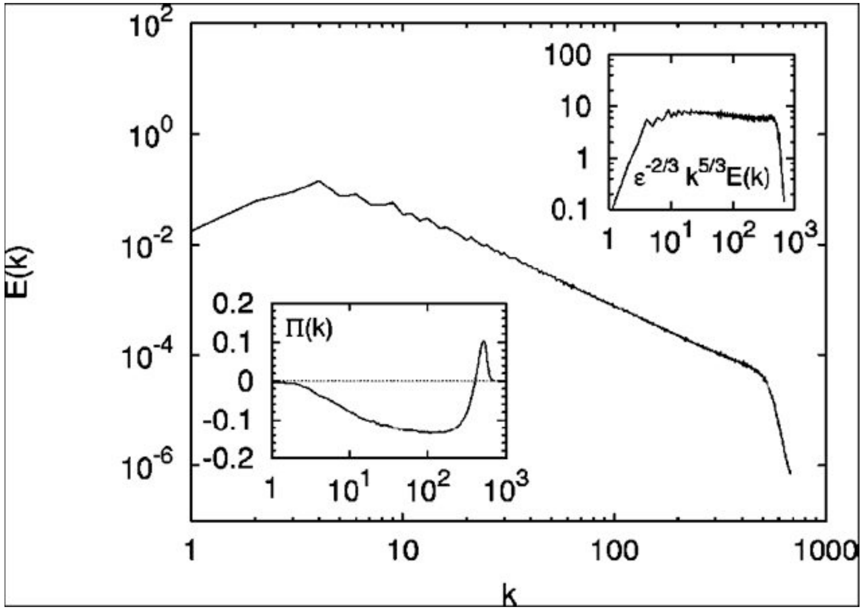
Energy spectrum in the inverse energy cascade.

$$E(k) \simeq k^{-5/3}$$

Kolmogorov spectrum

N: spatial resolution, dx: grid spacing, v: viscosity; a: friction,....

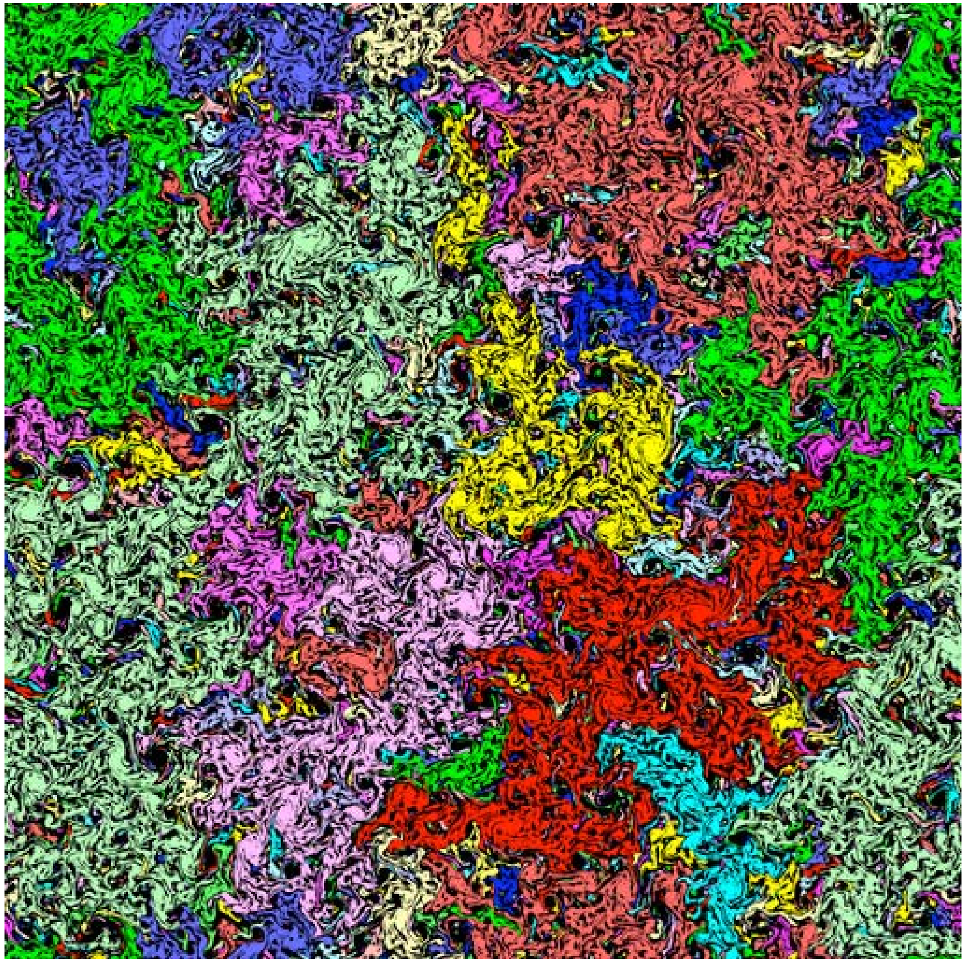
G. Boffetta, A. Celani



N	dx	ν	α	u_{rms}	ℓ_f	ℓ_d	ε_I	ε_ν	ε_α
2048	4.9×10^{-4}	2×10^{-5}	0.015	0.26	0.01	2.4×10^{-3}	3.9×10^{-3}	1.8×10^{-3}	2.1×10^{-3}
4096	2.4×10^{-4}	5×10^{-6}	0.024	0.26	0.01	1.2×10^{-3}	3.9×10^{-3}	0.7×10^{-3}	3.2×10^{-3}
8192	1.2×10^{-4}	2×10^{-6}	0.025	0.27	0.01	7.8×10^{-4}	3.9×10^{-3}	0.3×10^{-3}	3.6×10^{-3}
16384	0.6×10^{-4}	1×10^{-6}	0.0	0.24	0.01	5.5×10^{-4}	3.8×10^{-3}	0.2×10^{-3}	3.6×10^{-3}

Aim: analyze vorticity clusters in the scale domain of the energy cascade.

Numerical simulations

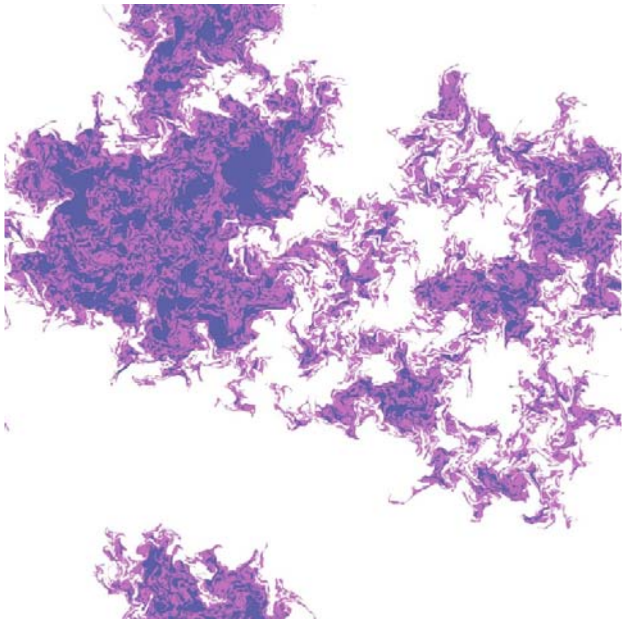


Fluid clusters (IR... Inverse cascade)

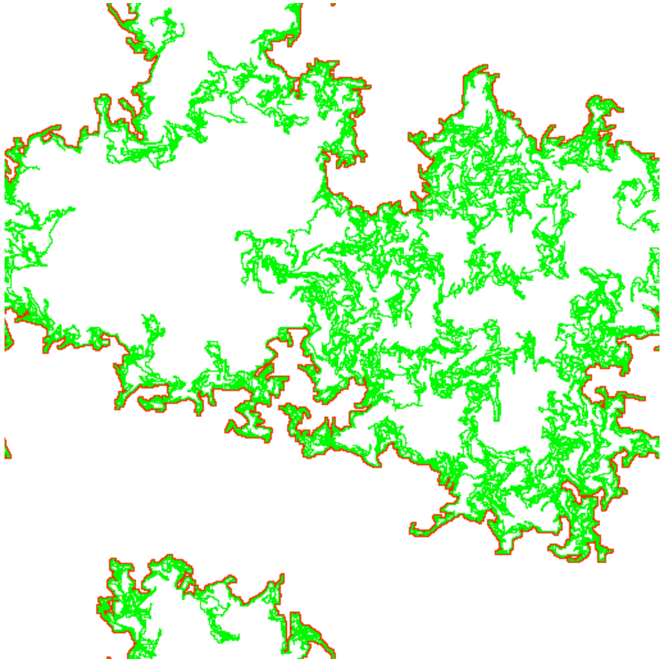
Vorticity clusters and their perimeters

Vorticity cluster: connected component of set of points with positive vorticity

Cluster boundary: macroscopic zero-isovorticity lines



A large vorticity cluster
(with filled holes)



Frontier and external perimeter
(without fjords) of a cluster

Fractal dimension of vorticity clusters

A naive argument:

using KK inverse cascade scaling

$$u_\ell \simeq \epsilon^{1/3}\ell^{1/3}$$

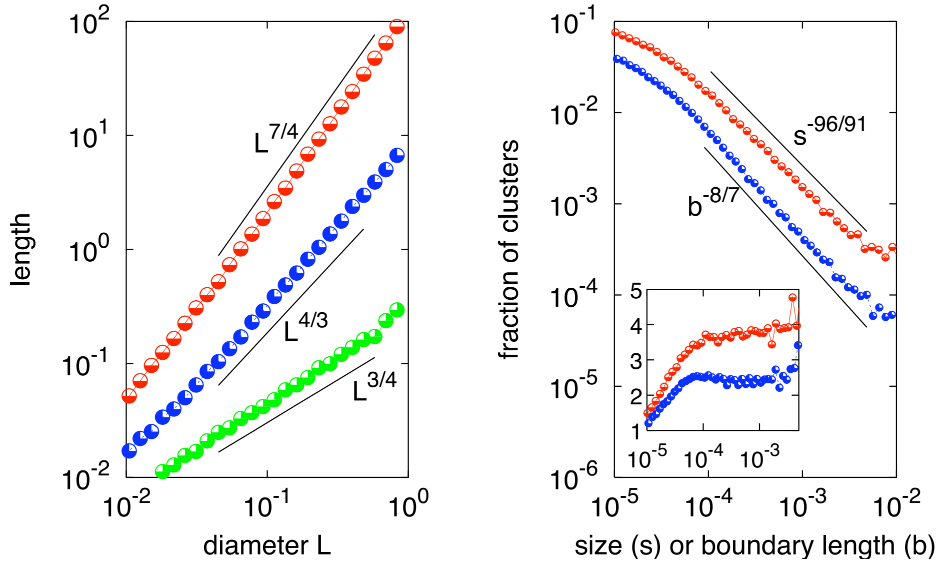
Macroscopic cluster size L

$$\Gamma \equiv \int d^2x\omega \simeq \omega_L L^2 \propto L^{4/3}$$

$$\Gamma \equiv \oint u \cdot dl \simeq N_{L_{inj}} u_{L_{inj}} L_{inj} \propto P$$

$$P \propto L^{4/3}$$

Dim of the frontier = 7/4 (as SLE(6))
Dim ext. perimeter= 4/3 (as SLE(8/3))
Dim of fjords = 3/4 (as percolation)



→ Traces of conformal sym in the 2D inverse cascade

Idem as conformal predictions for percolating clusters

Reconstructing (discrete) SLE in turbulence

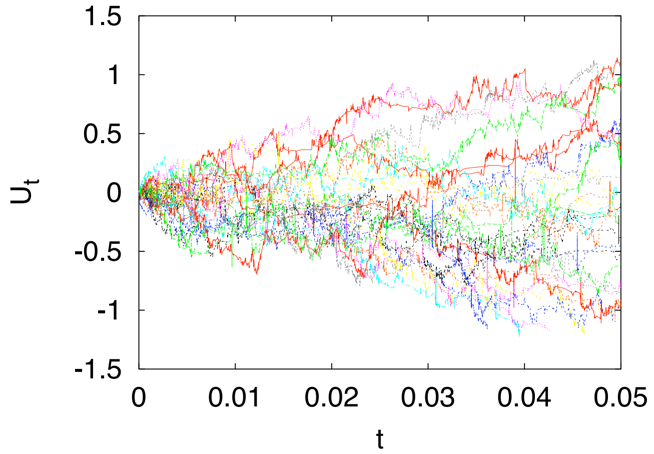
How to: i) Extract curves from sample of turbulent flows

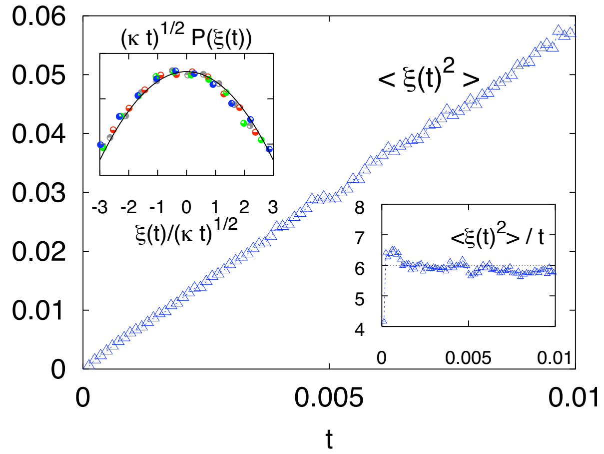
ii) code the discretized curves into discrete Loewner equations via iteration of maps

iii) Extract and analyze the Loewner source

$$G^{(n)} = g_n \circ G^{(n-1)}$$

$$\text{eg}: \quad g_n(z) = \sqrt{(z-a_n)^2 + b_n^2} + a_n$$

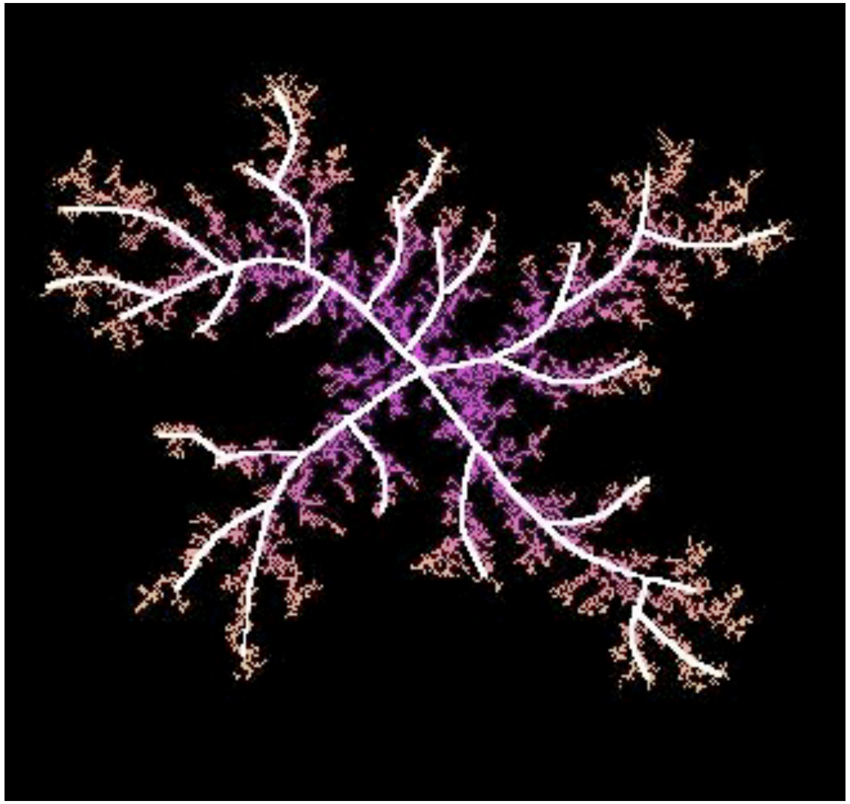




Statistics of the Loewner source is close to that of 1D brownian motion ..... SLE(6)

Zero iso-vorticity lines are (probably) conformally invariant

More test (ok), other models (ok)... but no analytical understanding



....

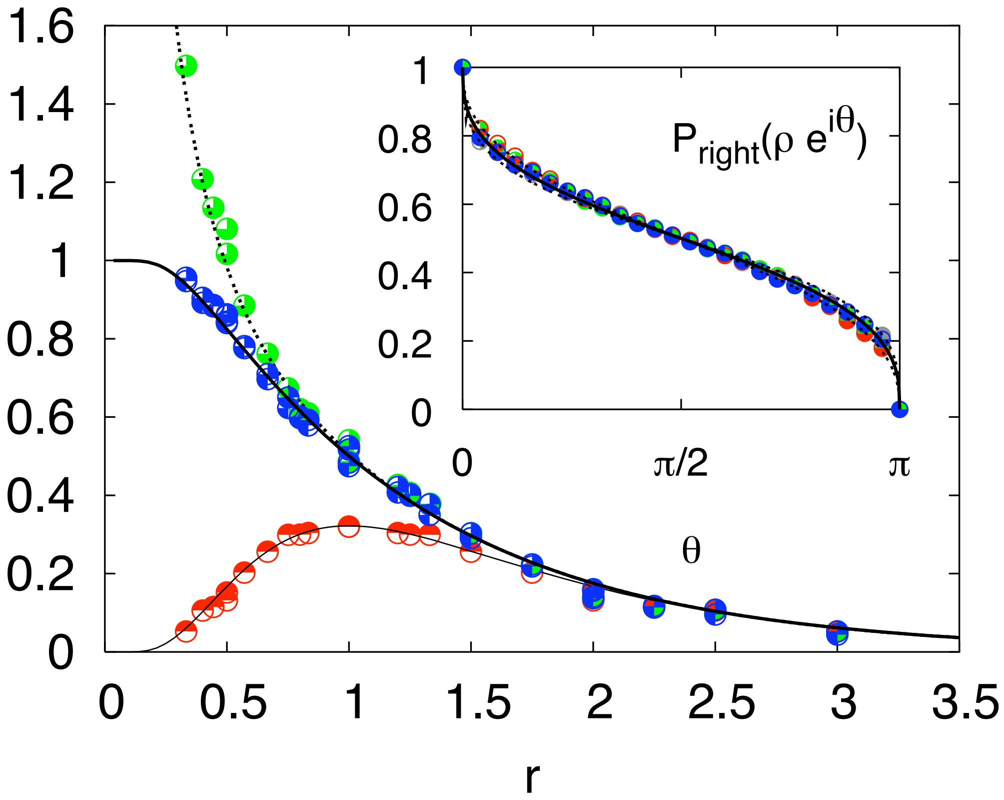
MERCI !

More SLE test in 2D turbulence

-- Cardy formula (blue): probability for a cluster to cross a rectangle

-- Watts formula (red): probability of a 4-legged cluster

-- Schramm formula (insert): probability of left passage



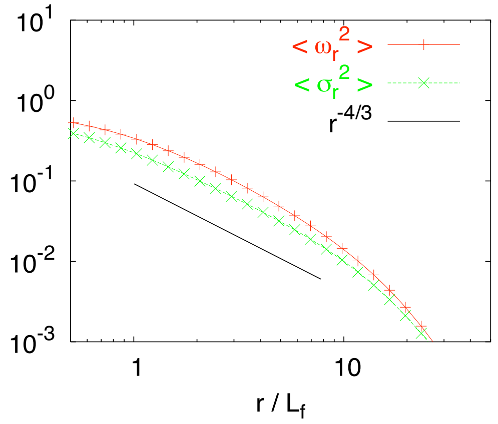
More SLE test in 2D turbulence

-- Non Gaussianity: randomized phase

$$\hat{\omega}(r) = \int d^2k \; e^{ikr} \omega_k \; e^{i\phi_k}$$

Identical 2-point function (but not pdf):
Loewner source in not a 1D Brownian
No conformal invariance

-- Harris criteria (Long range correlations)



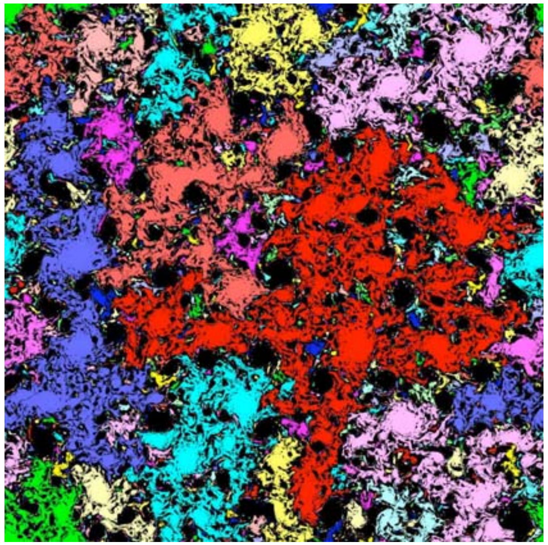
Long range correlation for the vorticity
Long range correlation for sin(vorticity)

Conformal invariance in SQG turbulence

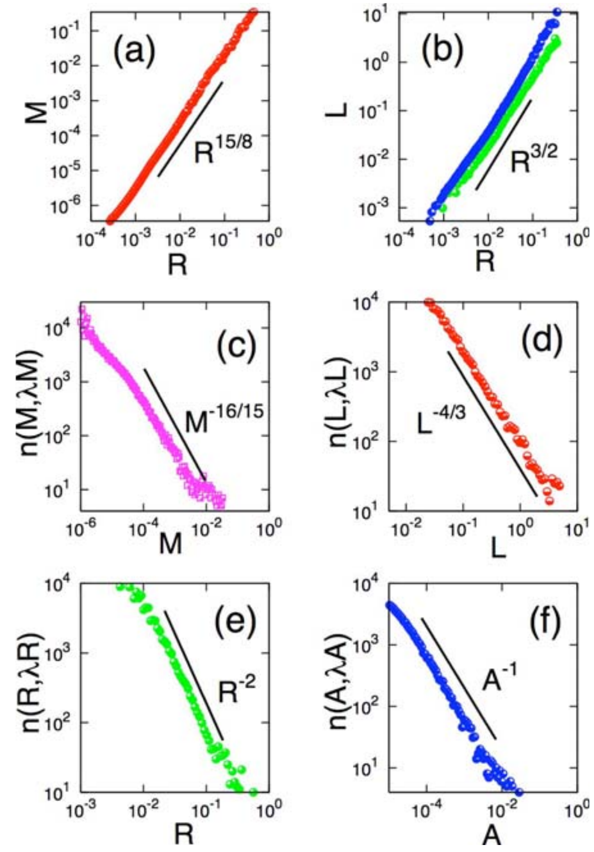
$$\partial_t \theta + u \cdot \nabla \theta - \nu \nabla^2 \theta = F$$

with $u^j = \epsilon^{ji} \partial_i \psi, \quad \psi_k = |k|^{-\alpha} \, \theta_k$

(here alpha=1)

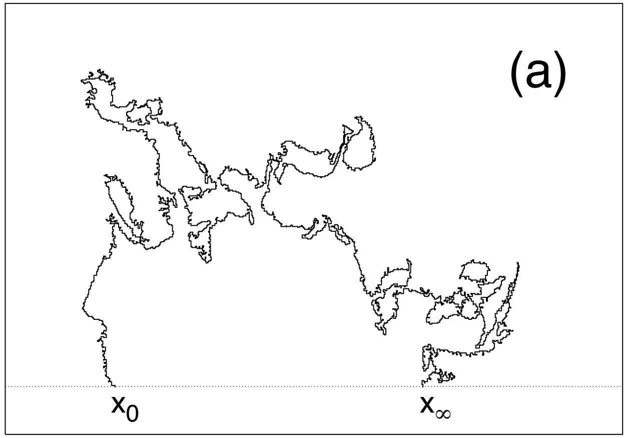


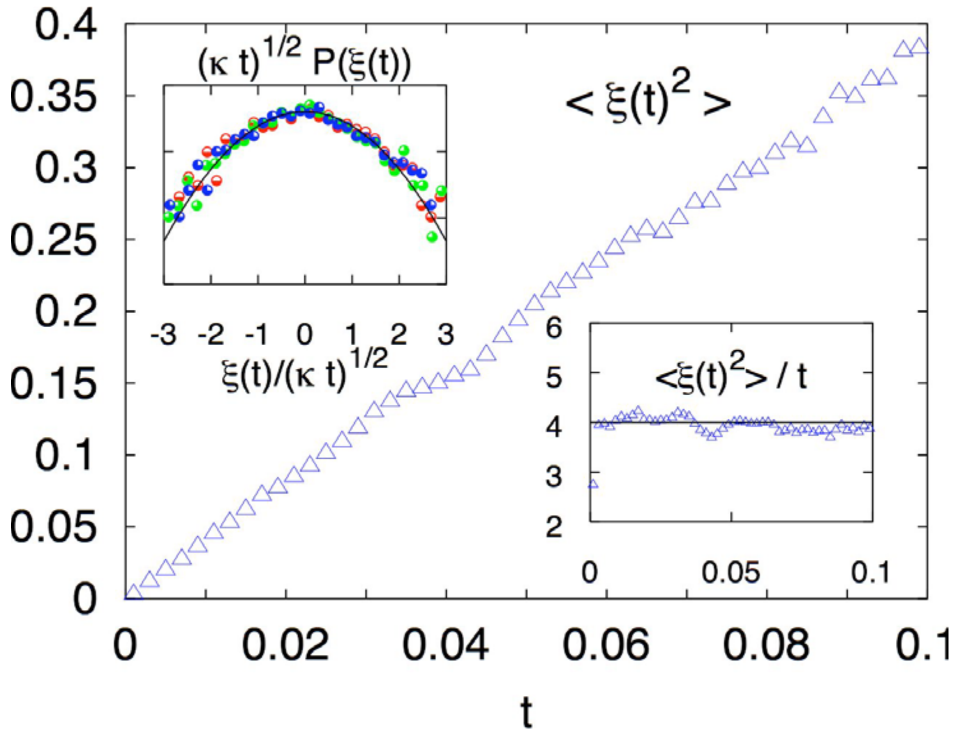
Scalar zero isolines and clusters



(a) mass versus radius; (b) length versus radius [D=3/2]; (c) number clusters versus radius; (d) number of loops versus length; (e) number of loops versus radius

Conformal invariance in SQG turbulence





The scalar isolines are (potentially) conformally invariant... SLE(4)

Comparing with 2D direct cascade