The Abdus Salam International Centre for Theoretical Physics

2162-19

Advanced Workshop on Anderson Localization, Nonlinearity and Turbulence: a Cross-Fertilization

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Scaling of Weak Chaos in Disordered Nonlinear Lattices

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Scaling of weak chaos in disordered nonlinear lattices

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Anderson localization

- electron transport in disordered solids
- wave propagation in a random medium
- quantum chaos
- recent progress: cold bosons in disordered optical traps

Discrete Anderson model

$$i\frac{\mathrm{d}\psi_n}{\mathrm{d}t} = E_n\psi_n + \psi_{n+1} + \psi_{n-1}$$

Here E_n is a ranadom on-site potential, one often takes E_n as independent random variables distributed uniformly in a range $-W/2 < E_n < W/2$

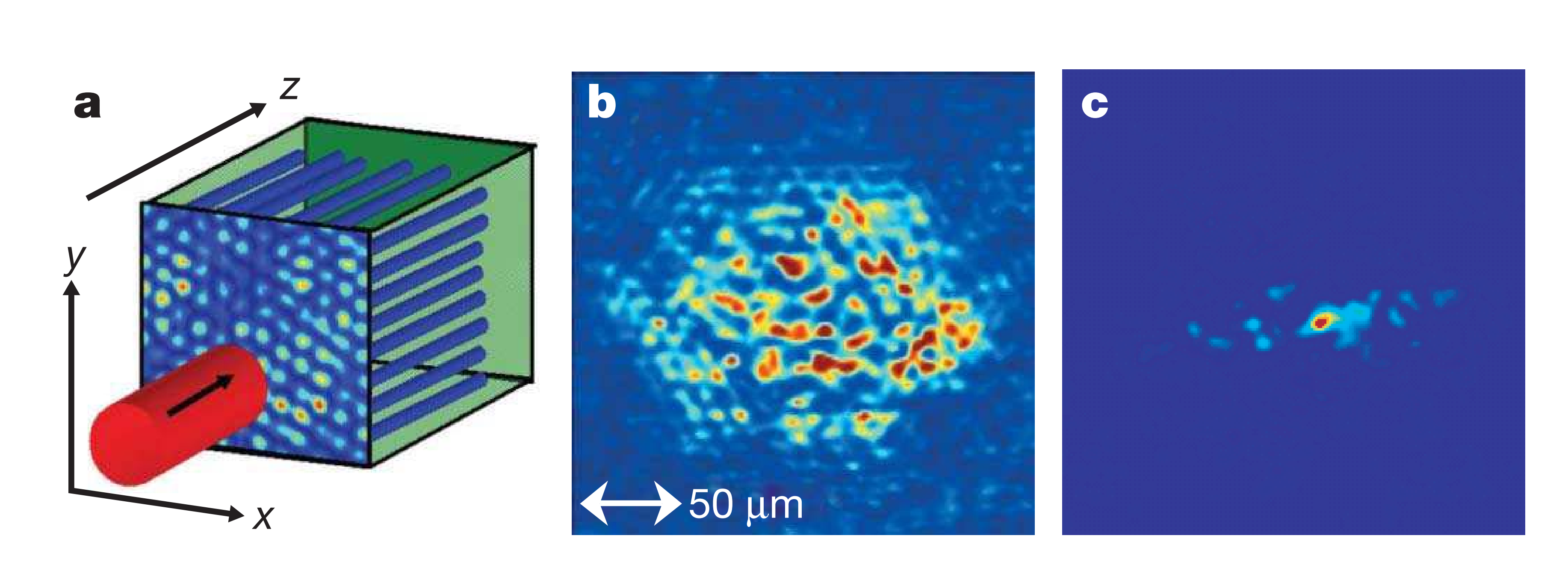
All eigenstates are exponentially localized $|\psi_n| \sim \exp(-\frac{|n-n_0|}{\lambda})$

Nonlinear effects

- Bose-Einstein condensate is described by a **nonlinear** Gross-Pitaevskii equation
- wave propagation in a **nonlinear** disordered medium
- disordered chains of **nonlinear** oscillators

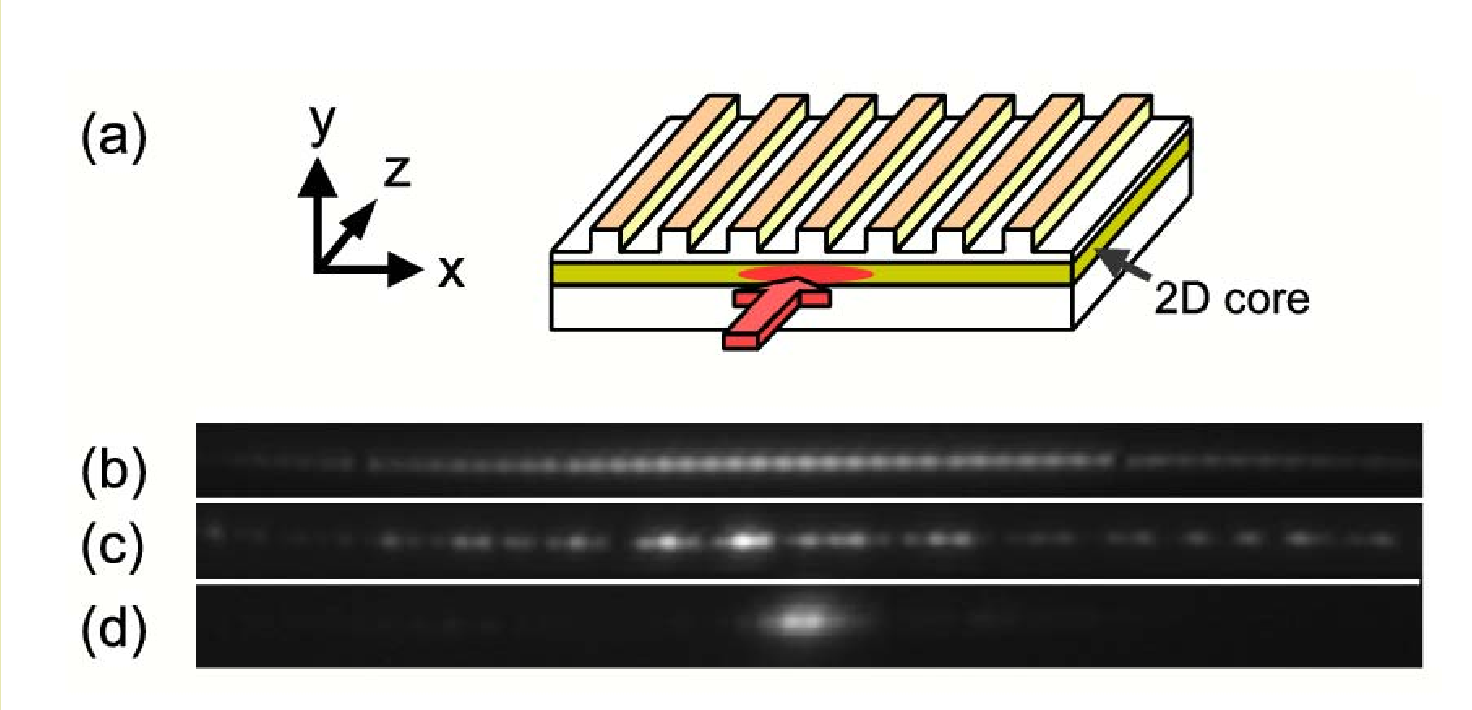
Q: Does nonlinearity enhance or destroy localization?

Optical experiments I



Schwartz, Bartal, Fishman and Segev, Nature 446 (2007)

Optical experiments II



Lahini, Avidan, Pozzi, Sorel, Morandotti, Christodoulides and Silberberg, PRL 013906 (2008)

Basic model: DANSE

We study Discrete Anderson Nonlinear Schrödinger Equation

$$i\frac{\mathrm{d}\psi_n}{\mathrm{d}t} = E_n\psi_n + \beta|\psi_n|^2\psi_n + \psi_{n+1} + \psi_{n-1}$$

β characterizes nonlinearity

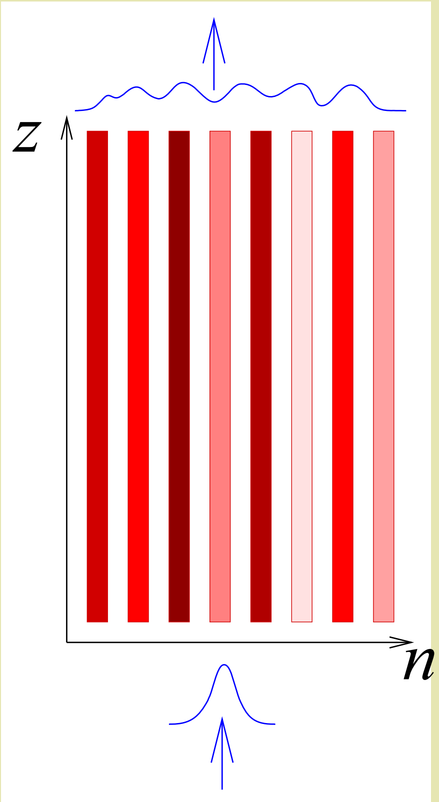
In the context of optical experiments: propagation direction z plays a role of time

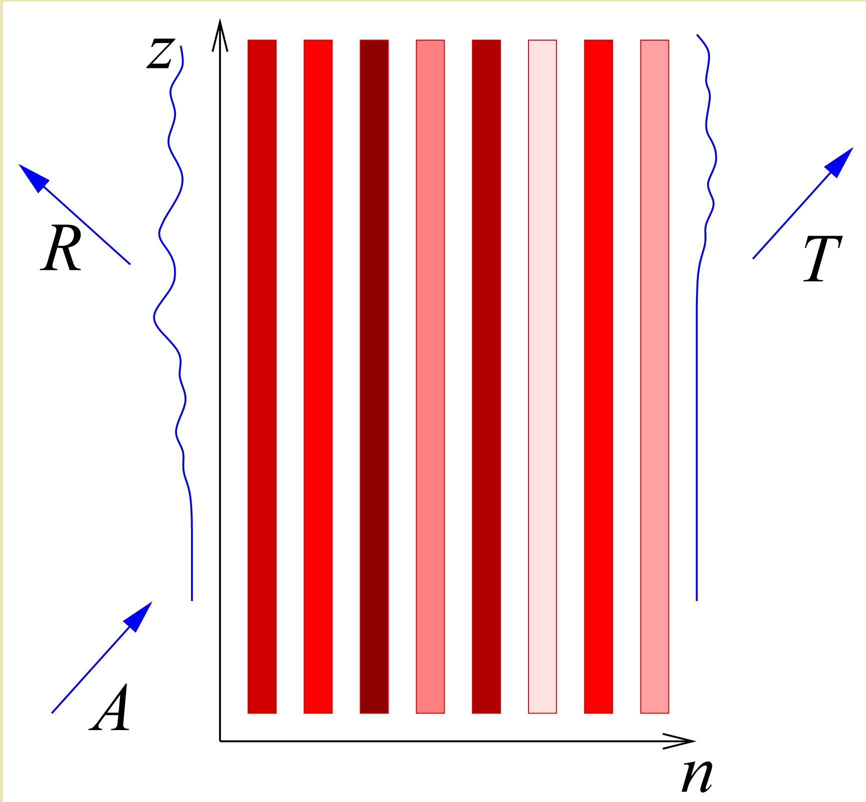
$$i\frac{\mathrm{d}\mathcal{E}_n}{\mathrm{d}z} = E_n\mathcal{E}_n + \beta|\mathcal{E}_n|^2\mathcal{E}_n + \mathcal{E}_{n+1} + \mathcal{E}_{n-1}$$

For optical experiments

Initial value problem

Scattering problem (see Tietsche & Pikovsky, EPL 84, 10006 (2008))





Evolution of a local initial state

How an initially localized field $|\psi_n(0)|^2 = \delta_{n,0}$ is spreading?

One characterizes this with the averaged squared width, i.e. the second moment $\langle(\Delta n)^2\rangle = \sigma(t) = \sum_n (n - \langle n \rangle)^2 |\psi_n(t)|^2$.

Wave packet spreading I

Early calculations by M. I. Molina (Phys. Rev. B, 58, 12547 (1998)) gave

$\sigma \sim t^{0.27}$

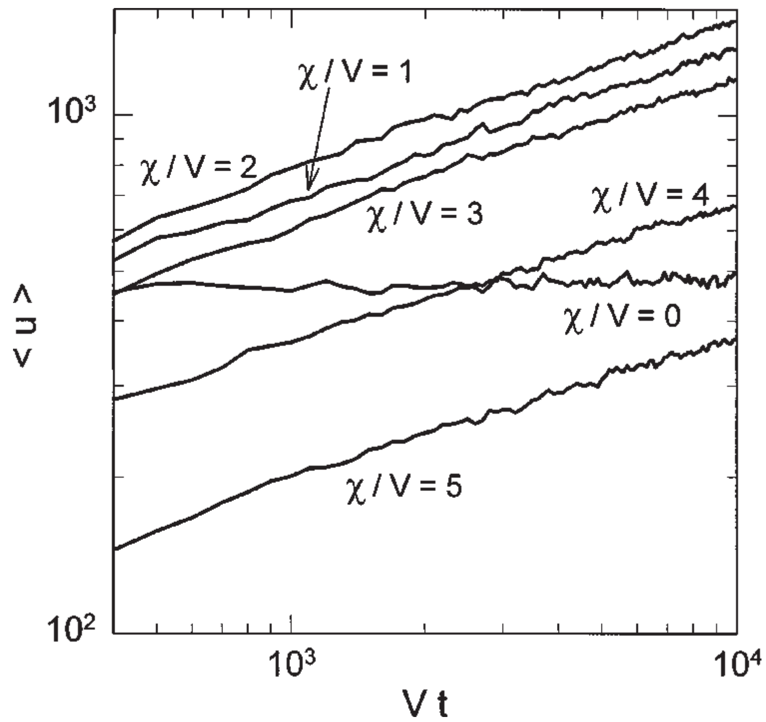
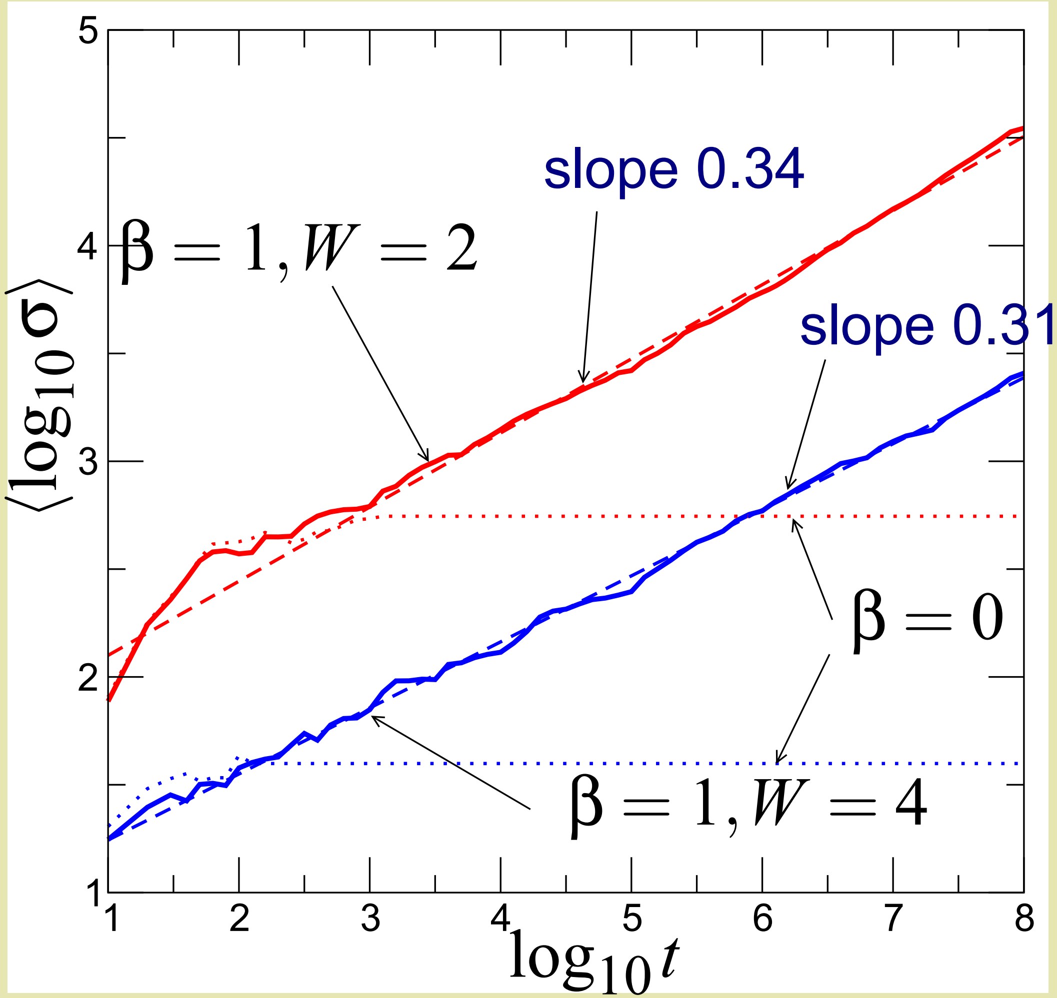


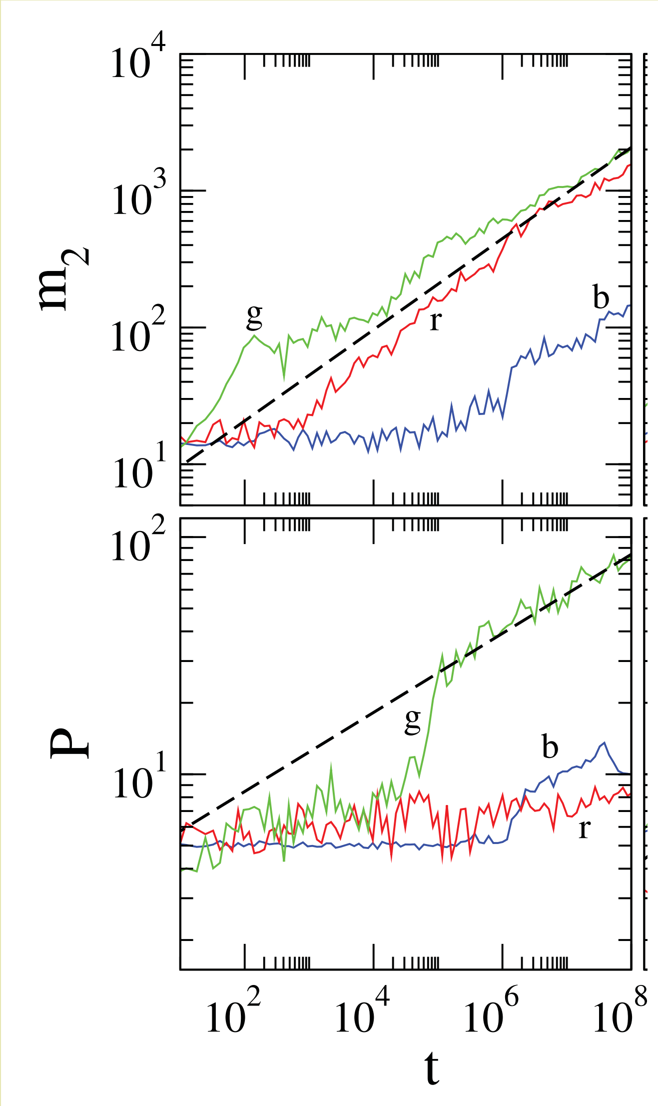
FIG. 1. Disorder-averaged (100 realizations) mean square displacement of an initially localized excitation, for different values of the nonlinearity parameter $(-1<\epsilon_n<1)$.

Wave packet spreading II

In our calculations (Pikovsky & Shepelyansky, Phys. Rev. Lett 100, 094101 (2008)) the averaging over disorder realizations was performed for the logarithm of this quantity, i.e. for $\log\sigma$.



Wavepacket spreading III



Flach, Krimer, & Skokos (Phys. Rev. Lett. 102, 024101 (2009)) found $\sigma \sim t^{0.33}$

Does spreading persist at very large times?

Numerics suggests:

- Initial wavepacket spreads seemingly unboundedly
- Subdiffusion spreading with exponent ≈ 0.33

But to answer questions

- Does it last forever?
- Does it depend on the nonlinearity constant?

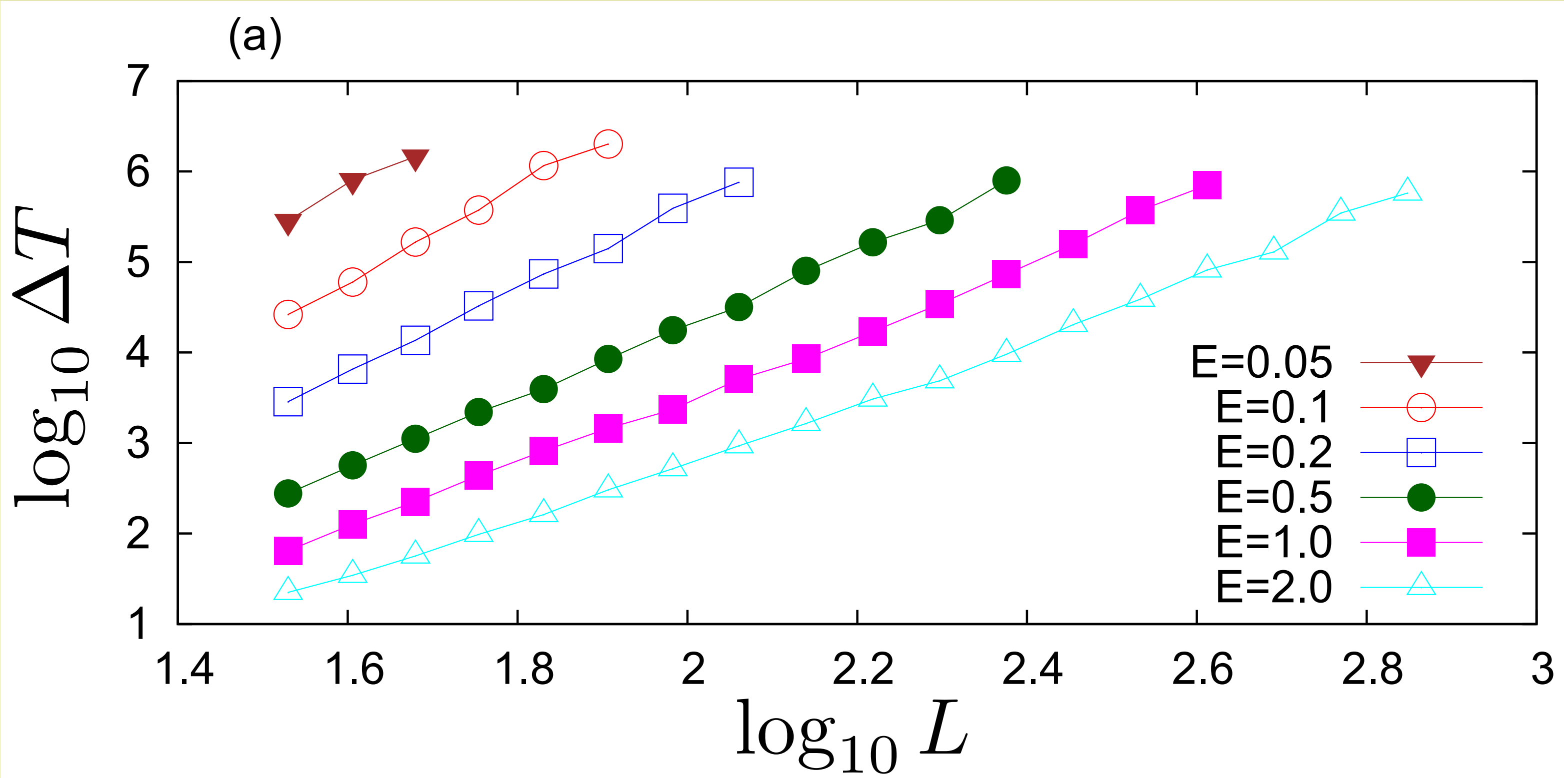
We need to

- Study very large lattices
- At very large times

Scaling approach to spreading

(Talk by Mario Mulansky, Thursday at 10)

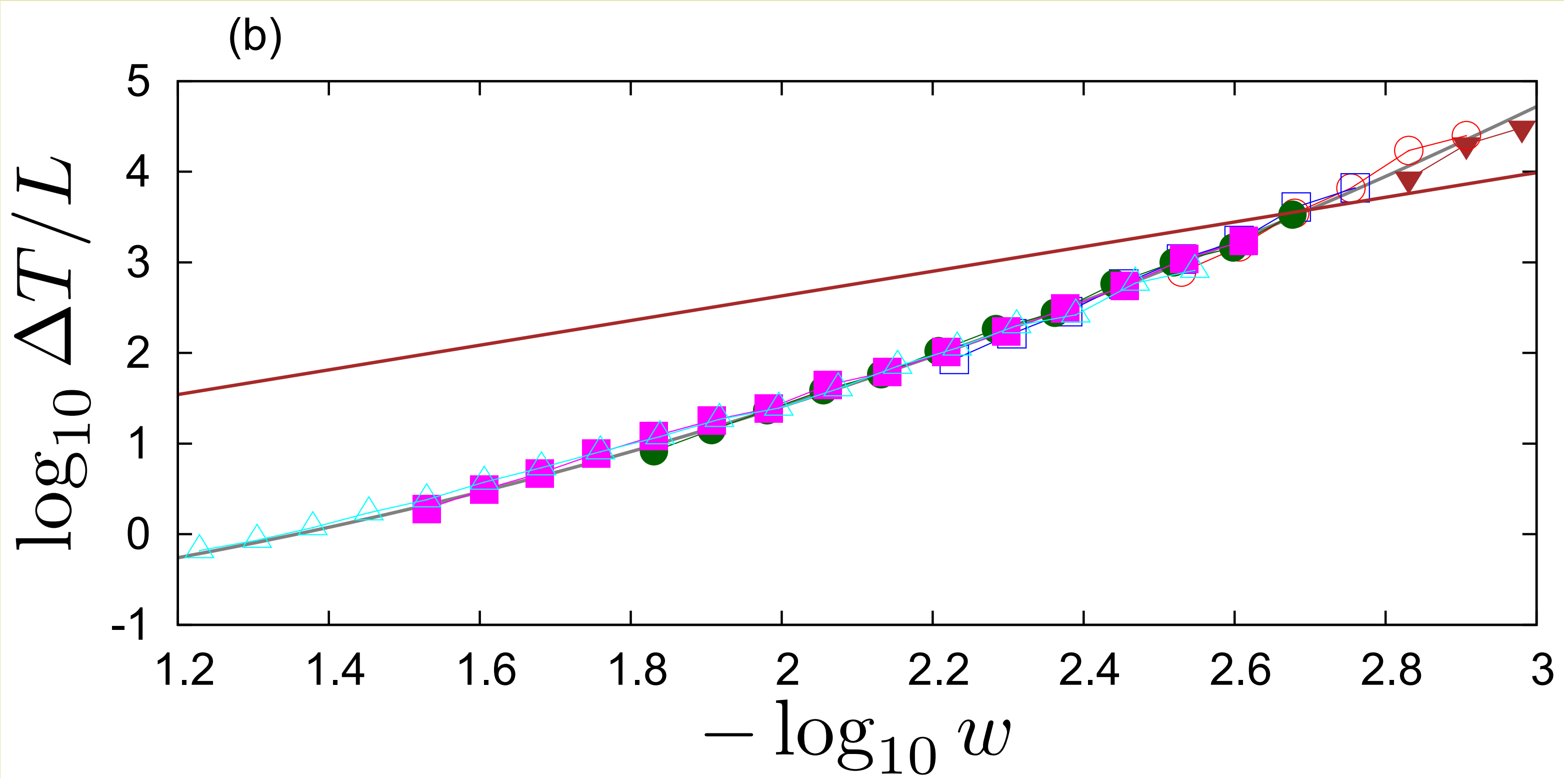
Calculate spreading for different parameters



Scaling approach to spreading

(Talk by Mario Mulansky, Thursday at 10)

Rescale coordinates to achieve collapse on one curve



Scaling approach to weak chaos in DANSE

We try to extrapolate from small to large lattices using scaling relations

First rescale the DANSE model

$$i\frac{\mathrm{d}\psi_n}{\mathrm{d}t} = E_n\psi_n + \psi_{n+1} + \psi_{n-1} + \beta |\psi_n|^2\psi_n$$

to keep only the relevant parameters

- by rescaling $|\psi|$ we set $\beta = 1$
- by rescaling time t we set the width of the linear band to be disorder-independent

Rescaled DANSE I

$$i\frac{\mathrm{d}\psi_n}{\mathrm{d}t} = \frac{W\varepsilon_n}{1+W}\psi_n + \frac{\psi_{n+1}+\psi_{n-1}}{1+W} + |\,\psi_n\,|^2\psi_n$$

with $-1 < \varepsilon_n < 1$

Three relevant parameters:

Disorder strength W

Field norm $N = \sum_n |\psi_n|^2$

Lattice length L: $1 \leq n \leq L$ with periodic boundary conditions

Rescaled DANSE II

Intensive parameter W governs localization length:

inverse participation number $\mu^{-1} = \sum_n |\psi_n|^4$ scales as $\mu \approx 1 + W^{-1}$

for $W > 5$

Intensive parameter density $\rho = \frac{N}{L}$ governs nonlinearity level

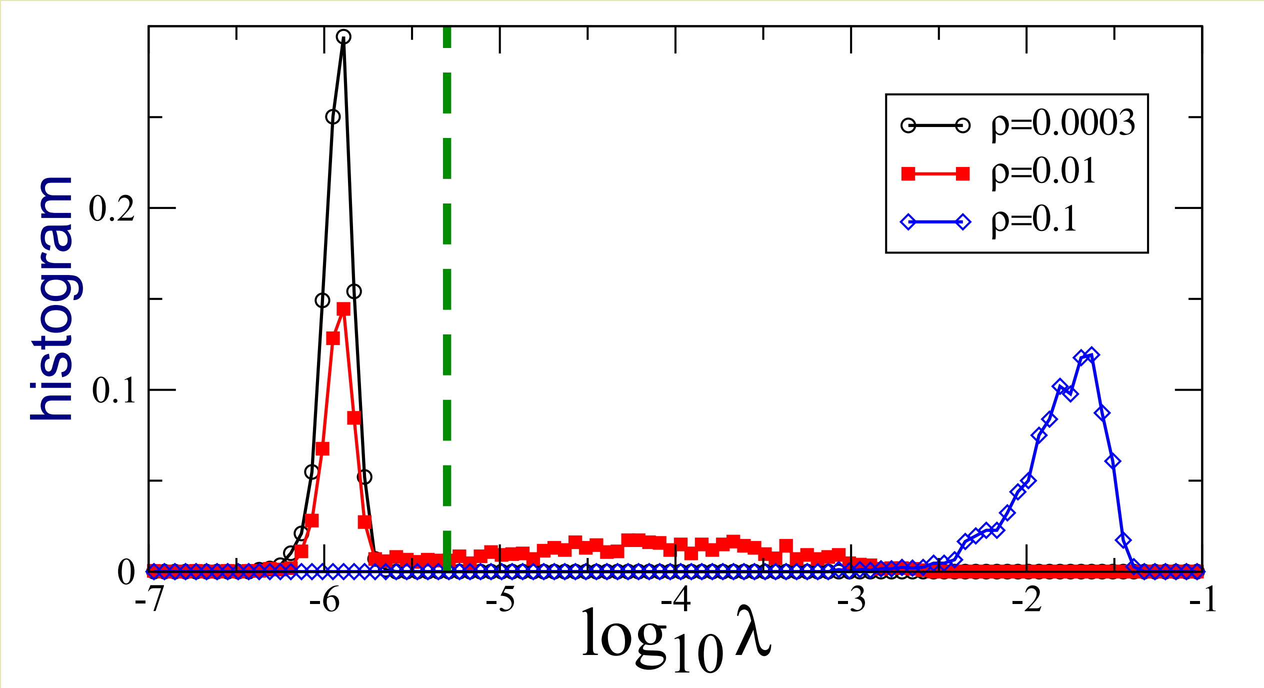
(another integral of motion – energy – is close to zero and irrelevant)

Extensive parameter lattice length L

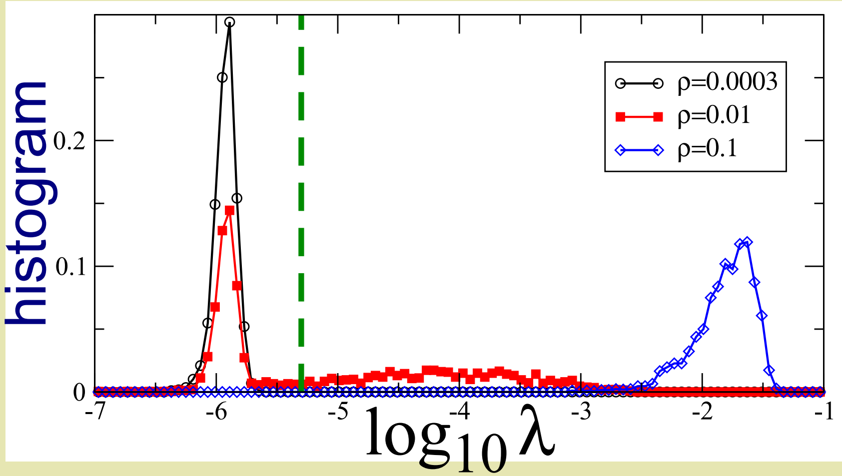
Characterizing regularity and chaos I

For different W, ρ, L we calculated the largest Lyapunov exponent λ for many realizations of disorder, starting from uniform in space initial conditions

Here the histograms for $W = 10$ and $L = 16$:



Characterizing regularity and chaos II



Close to zero values of largest Lyapunov exponent $\lambda \approx 10^{-6}$ indicate regular (quasiperiodic) dynamics

Positive values of λ indicate chaos

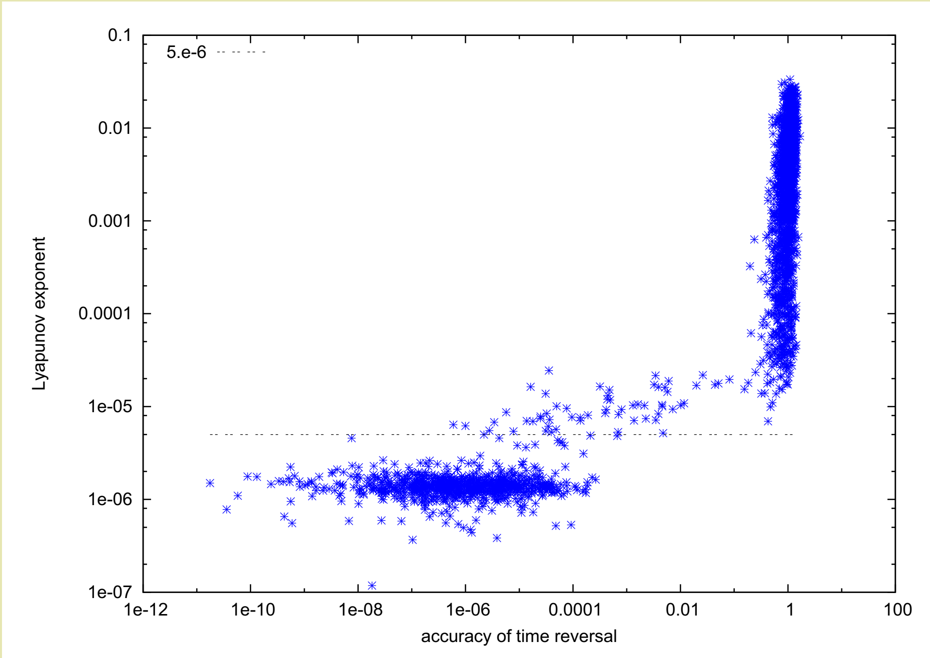
Small density: only regularity

Large density: only chaos

Inremediate density: chaos in some realizations of disorder, regularity in other ones

Poor man's characterization of regularity and chaos

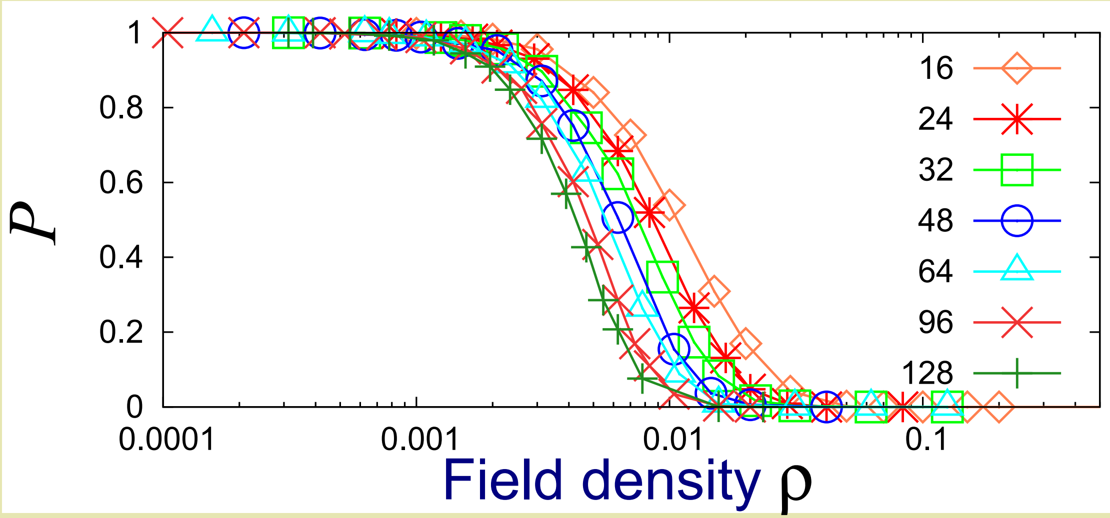
Instead of calculating Lyapunov exponent one can check time-reversibility of trajectory



Possible experimental implementation in optics using phase conjugation?

Characterizing regularity and chaos III

We calculate probability to observe regular dynamics $P(\rho, W, L)$ as a function of relevant parameters, by attributing all realizations with $\lambda > 5 \cdot 10^{-6}$ to chaos



Here we show $P(\rho, W = 10, L)$ as a function of ρ

Scaling with lattice length L

For fixed intensive parameters ρ and W, how $P(\rho, W, L)$ depends on lattice length L? Suppose we divide a long lattice of length L into sufficiently long subsystems of length L_0. If L_0 is much larger than localization length, interaction between subsystems is relatively small, and we assume that

regularity in the whole lattice requires regularity in ALL subsystems

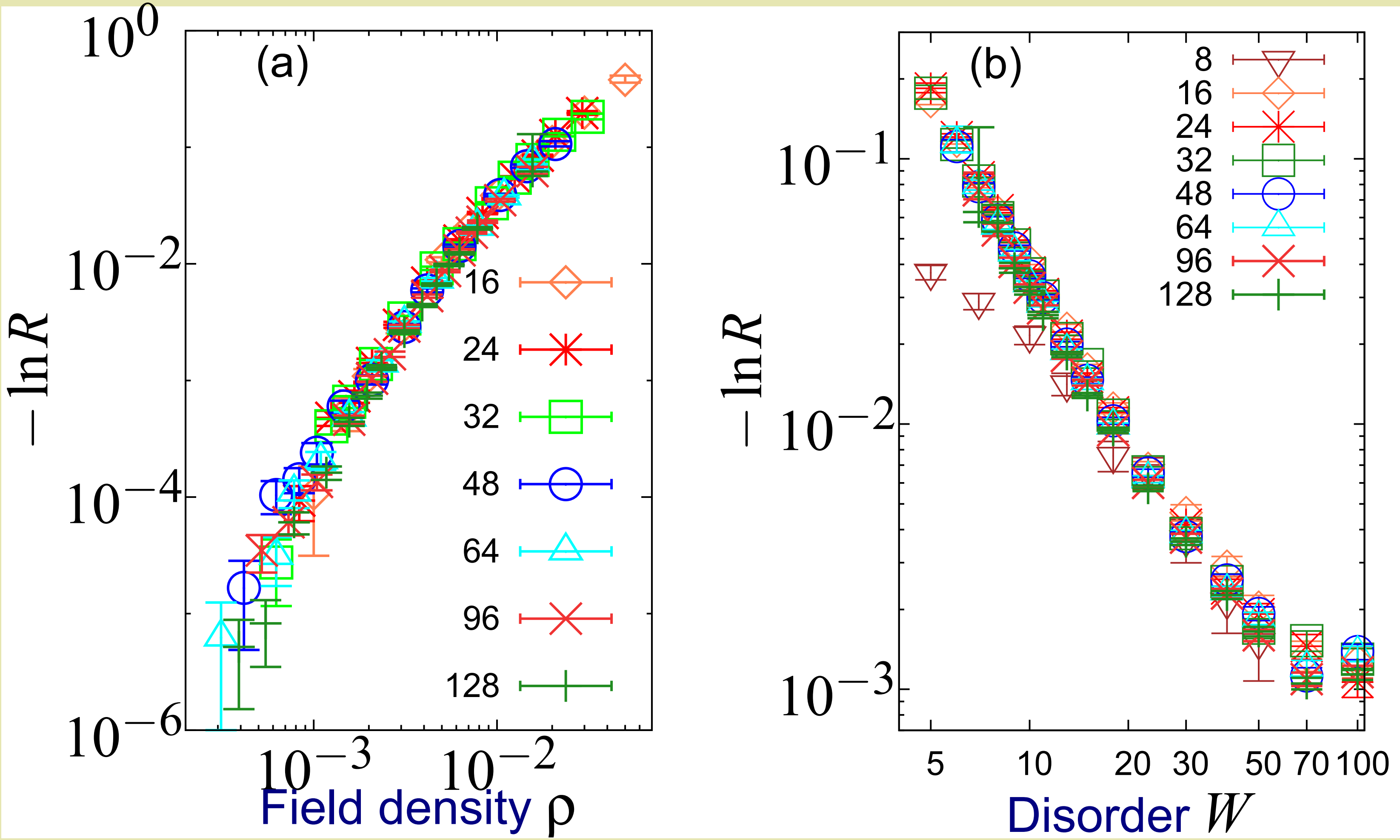
i.e. any chaotic subsystem spoils regularity. Therefore

$$P(\rho, W, L) = [P(\rho, W, L_0)]^{L/L_0}$$

Equivalently, one can introduce a length-independent, intensive quantity

$$R(\rho, W) = [P(\rho, W, L)]^{1/L}$$

Checking scaling with lattice length L

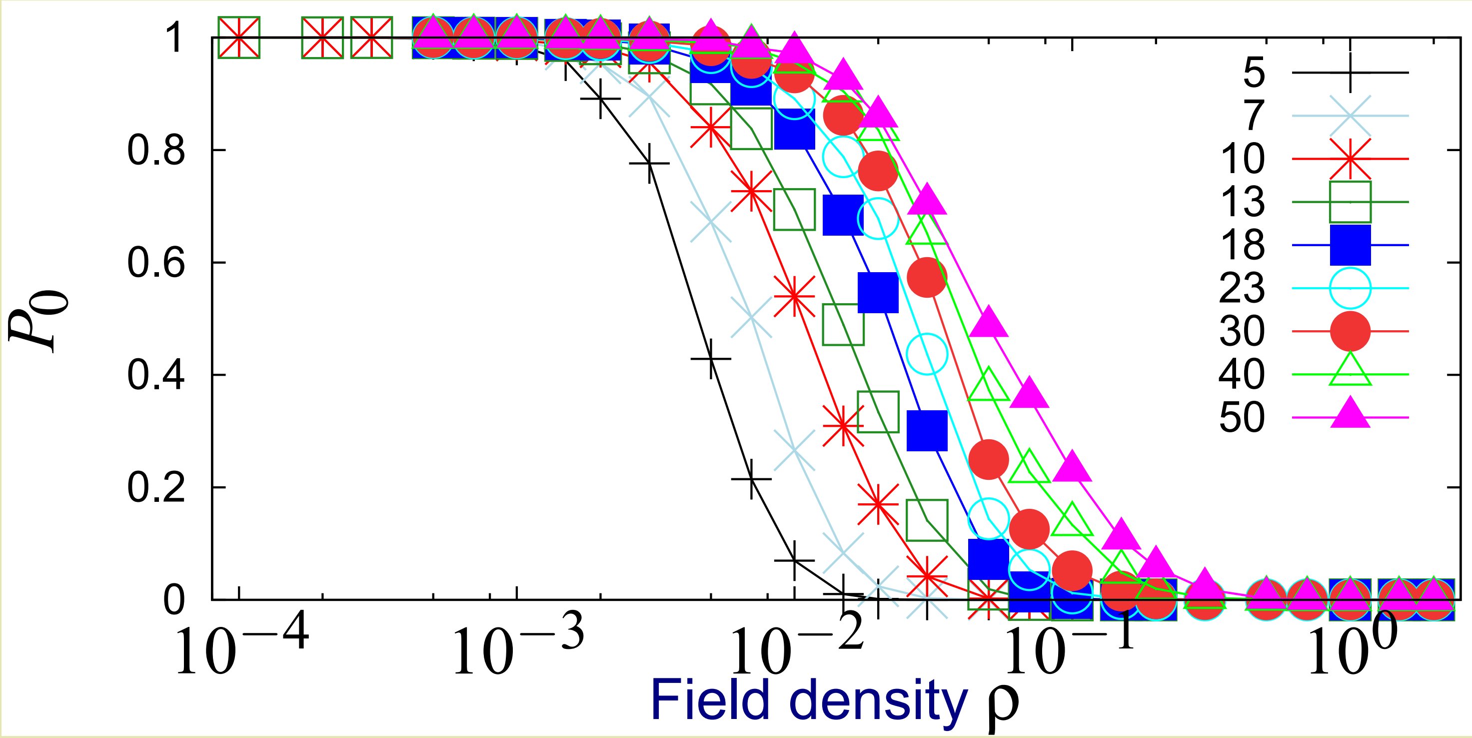


(a): Fixed disorder $W = 10$

(b): Fixed density $\rho = 0.01$

Scaling with intensive parameters

We fix $L = L_0 = 16$ and consider dependence of $P_0(\rho, W) = P(\rho, W, 16)$ on intensive parameters: density ρ and disorder W



Dependence on ρ for different values of W

Extended scaling function I

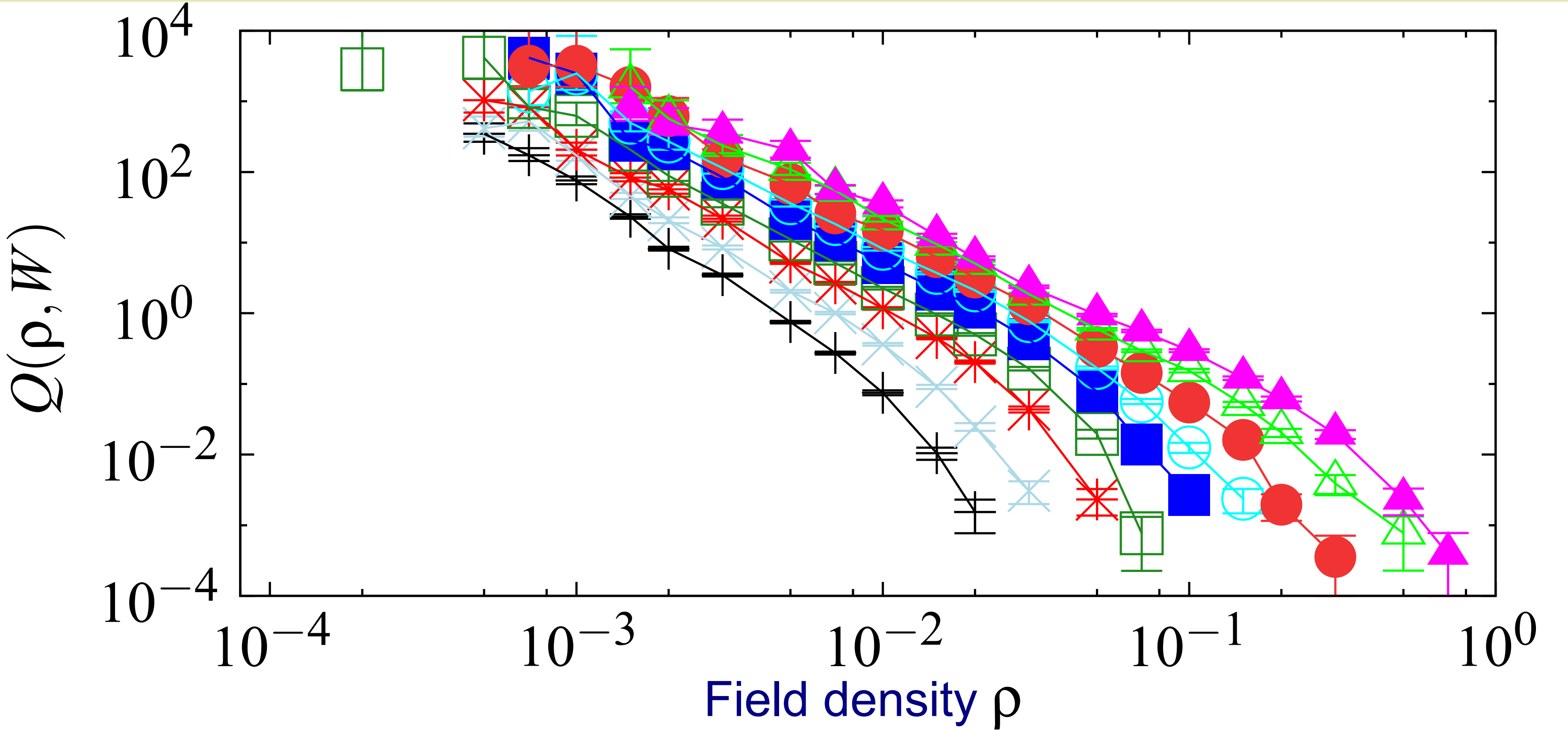
Because P_0 has sigmoidal form, to resolve the tails it is convenient to introduce a new function

$$Q(\rho, W) = \frac{P_0}{1 - P_0} \qquad P_0 = \frac{Q(\rho, W)}{1 + Q(\rho, W)} = \frac{1}{1 + Q^{-1}(\rho, W)}$$

In the regularity limit $P_0 \to 1$ and $Q \to \infty$

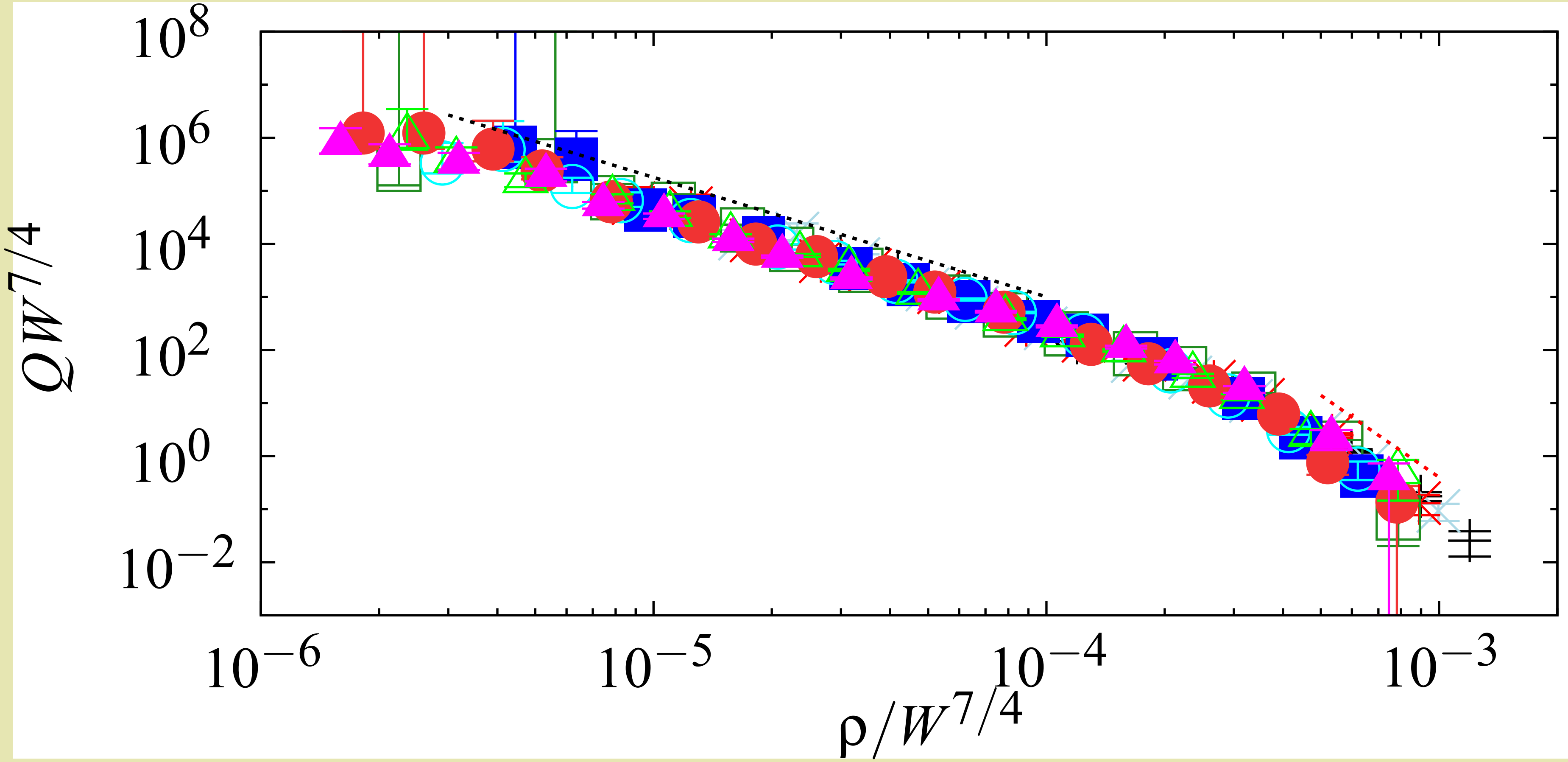
In the chaotic limit $P_0 \to 0$ and $Q \to 0$

Extended scaling function II



Range of disorder $5 \leq W \leq 50$

Scaling ansatz I



We adopt powers $7/4$ to achieve the best collapse of curves

Scaling ansatz II

In the considered range of parameters function Q fulfills scaling ansatz:

$$Q = \frac{1}{W^{\alpha}} q\left(\frac{\rho}{W^{\beta}}\right)$$

with $\alpha = \beta = \frac{7}{4} = 1.75$

$q(x)$ is a singular function at its limits:

$$q(x) \sim c_1 x^{-\zeta} \text{ for } x \to 0, \quad c_1 \approx 10^{-6},\ \zeta \approx 9/4 = 2.25$$

$$q(x) \sim c_2 x^{-\eta} \text{ for } x \to \infty, \qquad c_2 \approx 10^{-12},\ \eta \approx 5.2$$

Consequences of scaling

$$P(\rho, W, L) = \left[1 + W^{\alpha} q^{-1}\left(\frac{\rho}{W^{\alpha}}\right)\right]^{-\frac{L}{L_0}}$$

In the regular limit $q \to \infty$

$$-\ln P(\rho, W, L) \approx \text{Prob(chaos)} \approx \frac{LW^{\alpha}}{L_0 q\left(\frac{\rho}{W^{\alpha}}\right)}$$

Using the scaling $q(x) \sim c_1 x^{-\zeta}$ we get

$$\text{Prob(chaos)} \approx \frac{LW^{\alpha(1-\zeta)}\rho^{\zeta}}{c_1 L_0}$$

Scaling for a constant norm N

Application to spreading problem:

Norm $N = \rho L$ is constant

Probability to observe regularity/chaos

$$\text{Prob(chaos)} \sim \frac{L^{1-\zeta} W^{\alpha(1-\zeta)} N^{\zeta}}{c_1 L_0} = \frac{L^{-5/4} N^{9/4}}{c_1 L_0 W^{35/16}}$$

As $L \to \infty$ regularity wins Prob(chaos) $\to 0$ and Prob(regularity) $\to 1$

Estimation of maximal spreading length

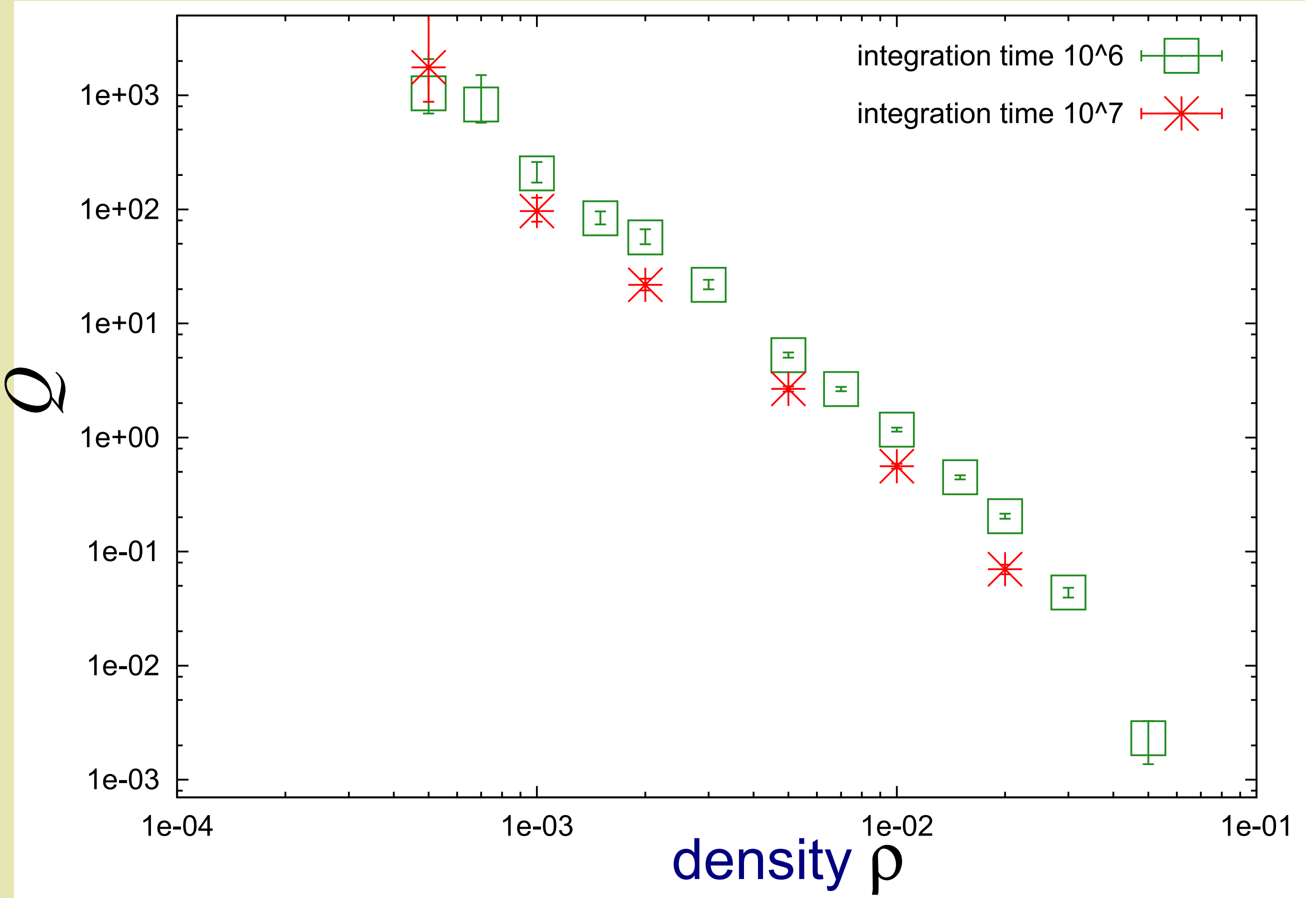
Fixing probability level at which chaos extincts as Prob(chaos) $= D$ we can estimate the length as

$$L_{max} = (c_1 D L_0)^{1/(1-\zeta)} N^{\zeta/(\zeta-1)} W^{-\alpha}$$

Because of the smallness of $c_1 \approx 10^{-6}$ we get $L_{max} \approx 10^5$ if other parameters W, N are of order one and $D \approx 1/L_0 \approx 0.06$

Sensitivity to averaging time

Changing averaging time from 10^6 to 10^7 slightly changes values of Q (data for $W = 10$)



Conclusion

- We studied statistics of regularity and chaos in disordered Anderson nonlinear Schrödinger model in dependence on disorder W, field density ρ and lattice length L
- Probability to obesrve chaos/regularity is a scaling function of these parameters
- For a fixed norm prob(chaos) tends to zero as $L \longrightarrow \infty$
- We predict saturation of spreading at large L
- Very weak chaos (e.g. Arnold-like “diffusion”) is beyond our numerics
- Extension to other disordered nonlinear lattices in progress