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Arnold Diffusion in a Chain of Coupled Pendula

Mark LEVI

The Pennsylvania State University
Mathematics Department
University Park
U.S.A.

Arnold diffusion in a chain of pendula.

Mark Levi

With Vadim Kaloshin and Maria Saprykina

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Arnold diffusion.

3-tori *don't* separate $\mathbb{S}^5$

Can the action change by a ``large" amount for arbitrarily small perturbation of a completely integrable system?

Arnold's example (1964):

$$H = \frac{1}{2}(I_1^2 + I_2^2) + \varepsilon(\cos\varphi_1 - 1)(1 + \mu(\sin\varphi_2 + \cos t))$$

There exist arbitrarily small ε and μ for which I_2 changes by $O(1)$.

More precisely: given any ε, for all μ small the action changes by $O(1)$

"The details of the proof must be formidable, although the ideas of the proof are clearly outlined" (Moser's review of Arnold's article).

Much work has been done since Arnold's original example, among others, by:

Bourgain, Bessie, Bernard, R. Douady, Delschams, de la Llave,Kaloshin, LeCalvez, Mather, Nekhoroshev, Sausin, Seare, Treschev, Giftankin, Xia

Our goal.

Arnold diffusion in mechanics/geometry/optics.

For a particle in a weak potential:

$$\ddot{\theta} = -\varepsilon \nabla U(\theta), \quad \theta \in \mathbb{R}^3$$

the phenomenon becomes physicall transparent. Fix the energy, e.g.

$$\frac{\dot{\theta}^2}{2} + \varepsilon U(\theta) = \frac{1}{2}$$

Trajectories are geodesics in the Maupertuis metric

$$v \; ds = (1 + O(\varepsilon))ds$$

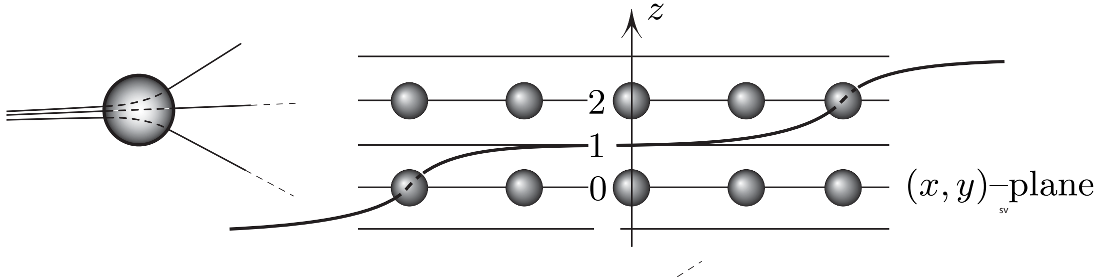
close to the Euclidean metric; these geodesics are nearly straight lines.

Arnold diffusion: *the existence of rays that change direction by* $O(1)$.

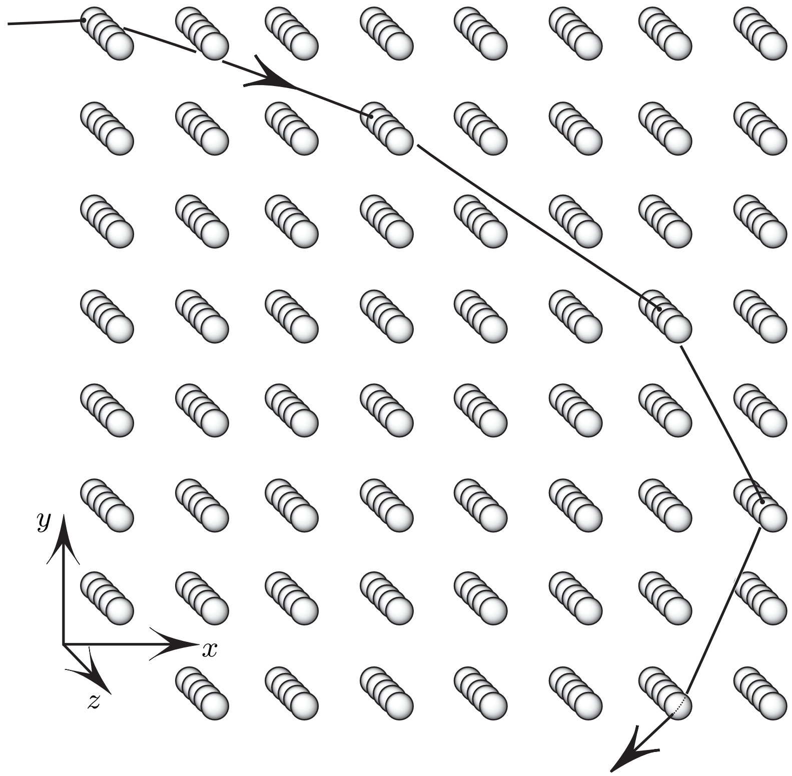
Diffusion in a nearly-flat metric.

A simple example (Kaloshin-L): Riemannian metric ρds,

$$\rho = 1 + \varepsilon \cos z + \varepsilon^3 \beta(x, y, z, \varepsilon)$$



Diffusion viewed in the configuration (as opposed to the phase) space:

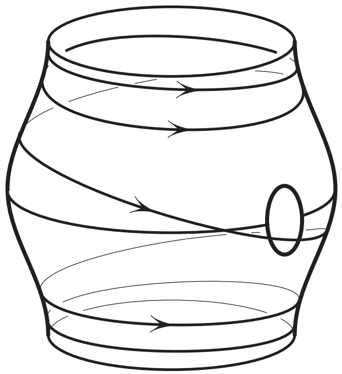


This geometrical representation shows a transparent view of the whiskered tori, of the intersection of invariant (stable and unstable) manifolds, and of the resonances.

Another mechanical example:

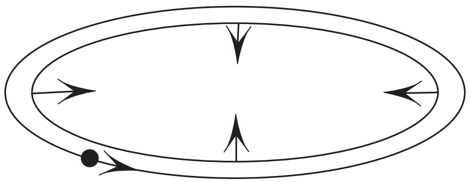
A bead on a periodically oscillating hoop: *velocity is bounded forever.*

A geodesic on a vibrating surface: some geodesics have *unbounded* velocity.



The curvature in disk oscillates periodically in time.

An aside: Adiabatic invariants via the force of constraint.



Coupled pendula.

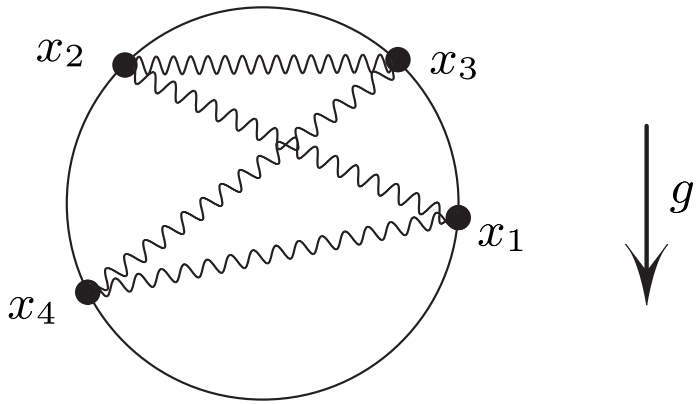
Torsionally coupled pendula:

$$\ddot{x}_j + \gamma\dot{x}_j + \sin x_j = k((x_{j-1} - 2x_j + x_{j+1}) + I$$

The conservative case (no forcing or friction):

$$\ddot{x}_j + \sin x_j = \varepsilon\beta(x_{j-1}, x_j, x_{j+1})$$

An example:



(Assume a periodic lattice from now on). (Josephson junction...)

KAM tori.

If the coupling perturbation in

$$\ddot{x}_j + \sin x_j = \varepsilon\beta(x_{j-1}, x_j, x_{j+1})$$

is small, then the tori fill ``most'' of the phase space.

A typical torus looks like a decoupled motion.

Arnold--diffusing orbits, if they exist, ``wind'' their way among these tori.

The diffusion means the change of the action vector. This, in turn, means the exchange of energy between the pendula.

The main result.

$$\ddot{x}_j + \sin x_j = \varepsilon\beta(x_{j-1}, x_j, x_{j+1})$$

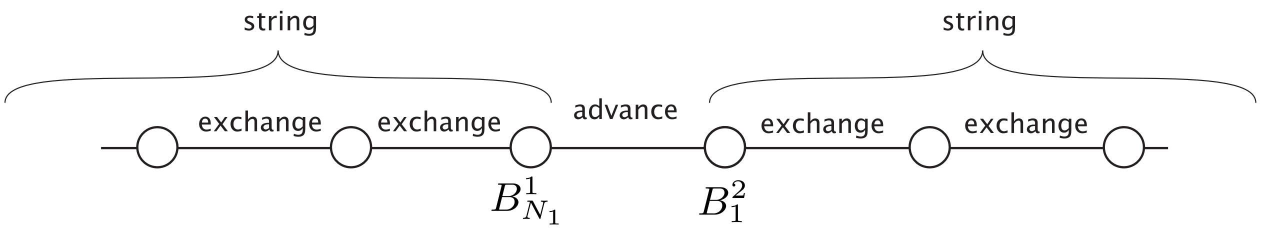
Assume that $\beta(x, y, z)$ is supported on small neighborhoods of integer points in $\mathbb{R}^3$.

From now on, fix the energy, and to a value sufficient to put at least one pendulum over the top.

THM (Kaloshin, Saprykina, L). *Given any infinite sequence of integers* j_k *with* $|j_{k+1} - j_k| = 1$, *there exists a sequence of times* t_k *such* j_k*th pendulum has energy* $1 + O(\varepsilon)$, *while all others have* $O(\varepsilon)$ *at the time* t_k .

(zero energy is the top equilibrium)

A schematic description of itinerary:



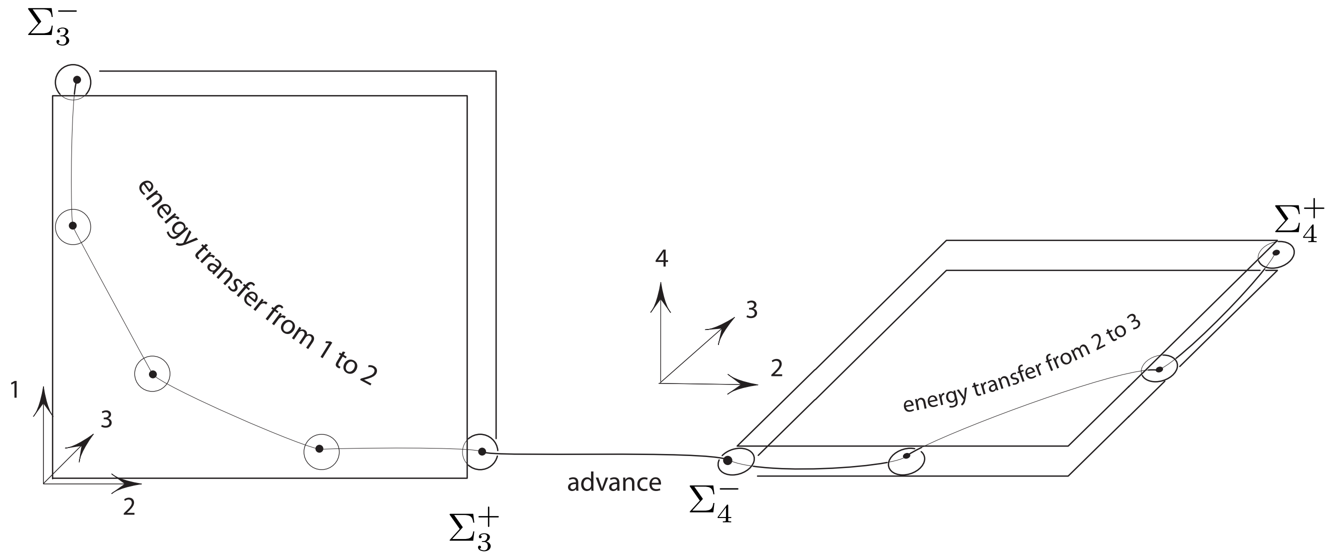
$$\dots\ (B^1_1 B^1_2 \dots B^1_{N_1})\ (B^2_1, B^2_2, \dots, B^2_{N_2}) \dots\ (B^j_1 B^j_2 \dots B^j_{N_j}) \dots$$

$$B^1_1 = B_{2\pi(0,0,0,\frac{1}{2})}(\sqrt{\varepsilon})$$

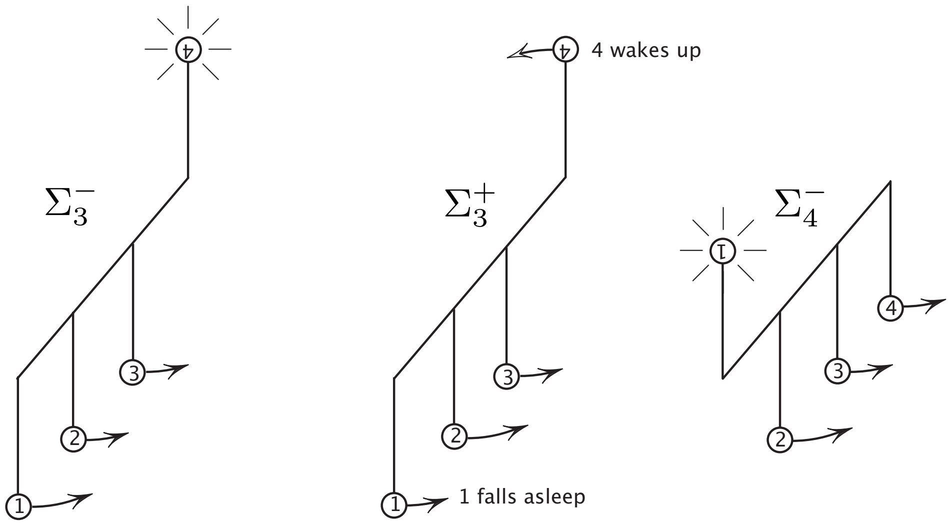
$$B^j_{k+1} = B^j_k + 2\pi v^j_k, \quad |v^j_k| \geq \varepsilon^{-a}$$

$$|e^j_{k+1} - e^j_k| \leq \varepsilon^b, \quad |e^{j+1}_1 - e^j_{N_{j-1}}| \leq \varepsilon^b.$$

A view in the configuation space:



A view in the physical space:

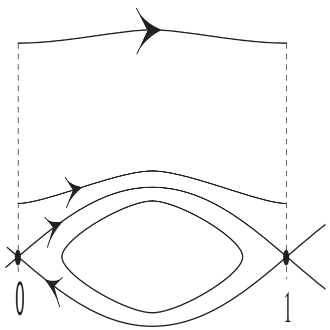


Energy transfers from 1 to 2.

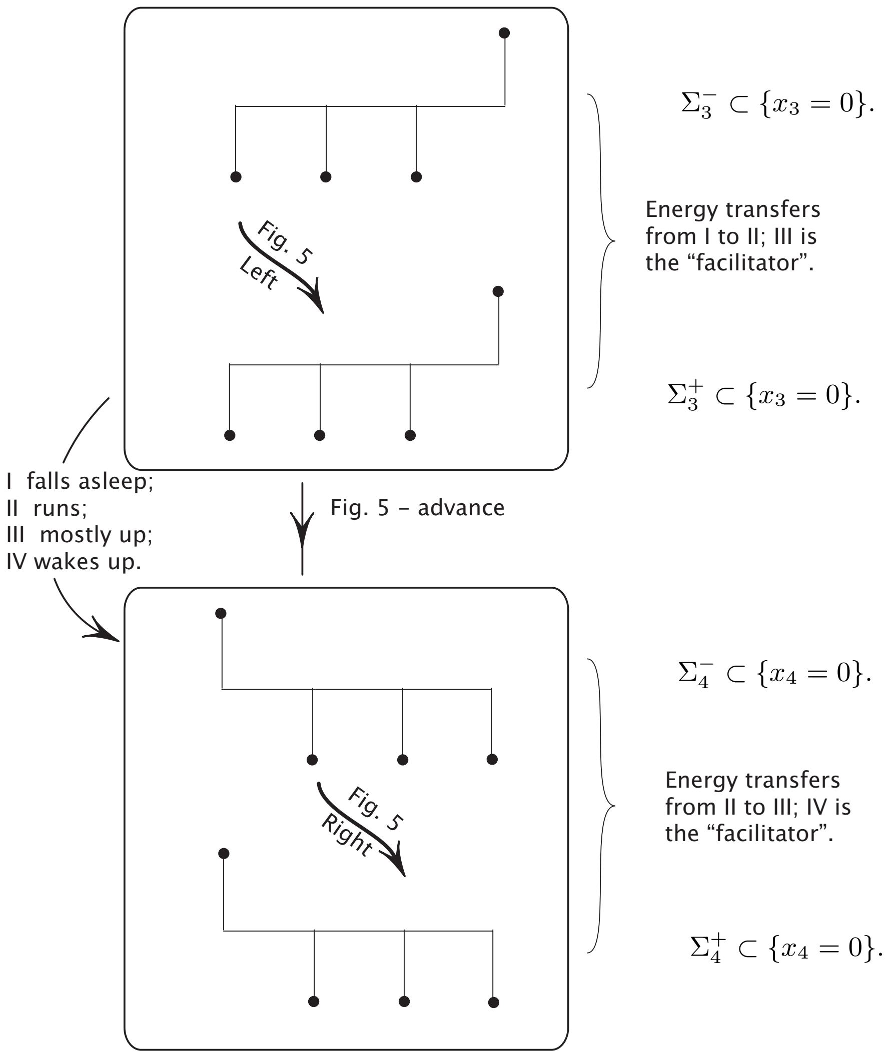
Advance from 123 to 234.

➢ : exponentially slow

➤ : speed = O(1)



One more view in the physical space:



A sketch of proof.

0. Construct sections.
1. geodesic segments between sections exist.
2. energy distribution is ``continuous” as a function of displacement.
3. variational matching of geodesic segments.

The ``Hamilton-Jacobi" lemma

Here is a convenient tool for finding normals to the wave front:

Gradient of the geodesic distance $L(O,p) = \int_{Op} \lambda ds$ is given by

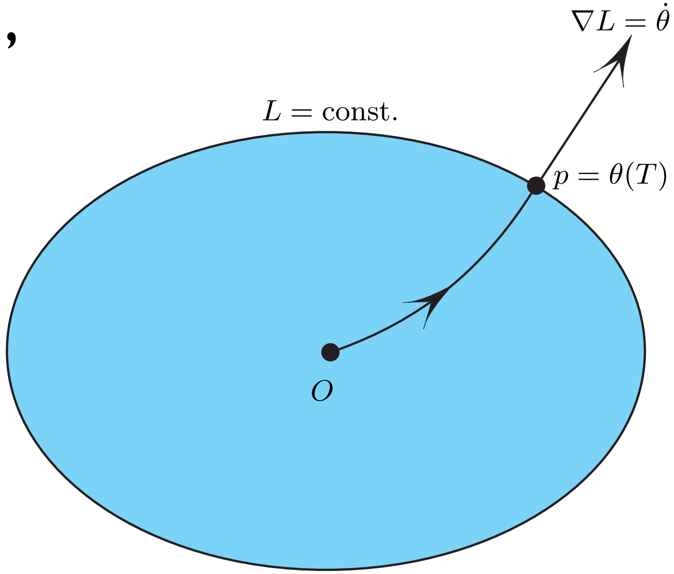
$$\nabla_p L(O,p) = \dot{\theta}(t),$$

the velocity of the solution of

$$\ddot{\theta} = \nabla\left(\frac{\lambda^2(\theta)}{2}\right)$$

with

$$|\dot{\theta}| = \lambda(\theta)$$



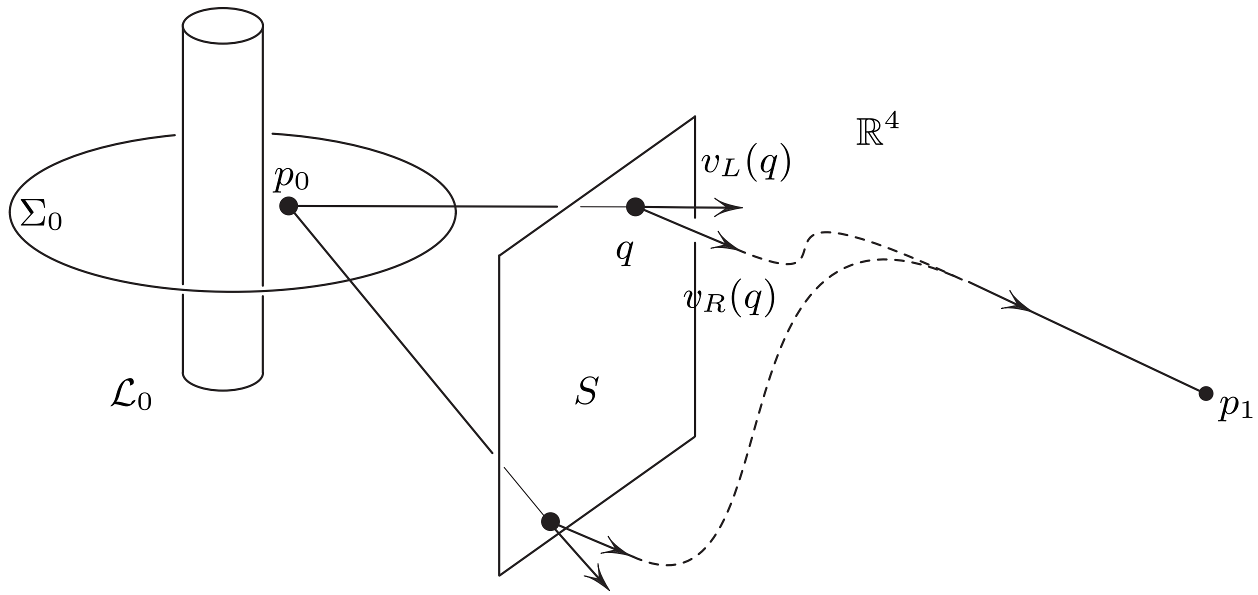
Proof.

(a) Both ∇L and $\dot{\theta}$ are normal to the wave front (Huygens' principle).

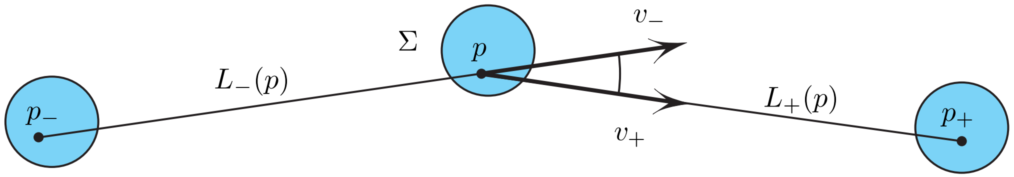
(b) $|\nabla L| = \lambda = |\dot{\theta}|$.

The connection lemma

1. There exists a connecting geodesic for any pair of sections.
2. If the displacement vectors are closely aligned and long enough, then the energy distribution for the two solutions are close to each other.



Mending a broken geodesic via defocusing lenses: a variational approach.



Consider the length of the broken geodesic:

$$S_\varepsilon(p) = L_-(p) + L_+(p)$$

Goal: show that $S_\varepsilon(p)$ achieves minimum in the *interior* of Σ.
Remove the lens at Σ; denote the modified action by S_0. We have

$$\nabla S_0(p) = \nabla L_-(p) + \nabla L_+(p) = v_- - v_+.$$

If $|v_- - v_+|$ is small, then S_0 is nearly constant. Adding the lens back in will create a desired "dip". More preciesly:

$$S_\varepsilon(O) \ < \ S_0(O) - \varepsilon^b < (S_0(\partial\Sigma) + |\nabla S_0| \cdot D) - \varepsilon^b =$$
$$S_\varepsilon(\partial\Sigma) + (|\nabla S_0| \cdot D - \varepsilon^b) \ < \ S_\varepsilon(\partial\Sigma)$$

Q.E.D.

The extra consideration required in the case of the pendula:

S is no longer near--constant in some variables,
but fortunately ``convex'' in that direction,
thus having an internal minimum.

Computational shadowing:

Variational construction of diffusing orbits
overcomes computational instability
caused by sensitive dependence on initial data
(hyperbolicity).

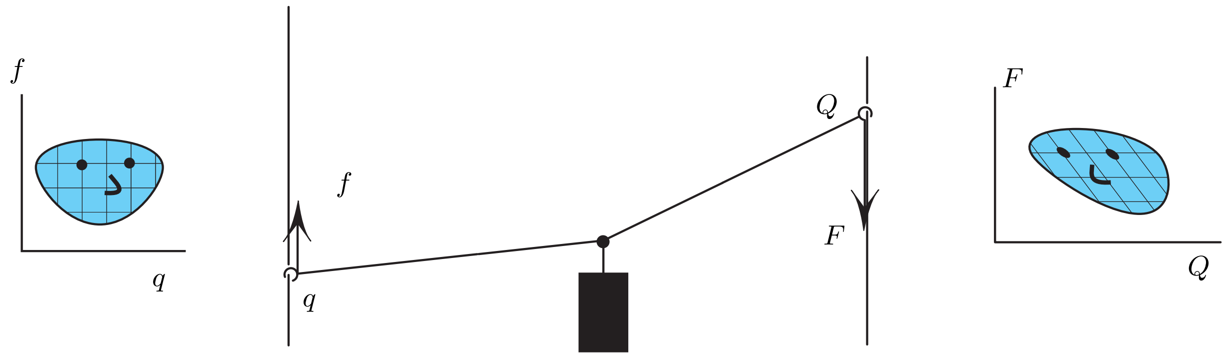
An open problem:
prove diffusion in Hedlund's metric that is arbitrarily close to flat.

An open problem:
Other motions?

A bigger open problem:
prove diffusion in a typical *near-flat metric in R^3.*
(Mather announced a solution for an essentially equivalent problem.)

The end

An aside: Symplectic Maps via Mechanics.



$f, \quad F$: forces with which the system ``wants'' to go.

$E(q, Q)$: potential energy

$$f = -\frac{E(q,Q)}{\partial q}, \quad F = -\frac{E(q,Q)}{\partial Q}$$

total work I do over a cycle:

$$0 = \int_\gamma (-f)\, dq + \int_\gamma (-F)\, dQ = -\int_\gamma p\, dq + \int_\gamma P\, dQ$$