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Scaling of Energy Spreading in Strongly Nonlinear Disordered Lattices

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Scaling of energy spreading in strongly nonlinear, disordered lattices

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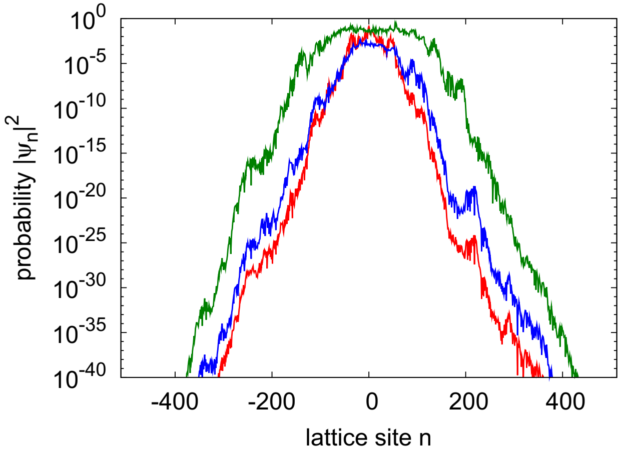
Structure of this Talk

Motivation

- Study spreading in disordered systems due to nonlinearity
- How does an initially localized excitation evolve in disordered nonlinear systems?
- Will it spread at all or stay localized? Will it spread forever?

Popular example: Discrete Anderson Nonlinear Schrödinger equation (DANSE): $i\frac{\partial \psi_n}{\partial t} = V_n\psi_n + \psi_{n+1} + \psi_{n-1} + \beta|\psi_n|^2\psi_n$



initial single site exciation at times 10^5, 10^7, 10^9

Numerical Observation:

- Subdiffusive Spreading
- Width of excitation $\sim t^\mu$ with $\mu \approx 0.15 \ldots 0.25$
- Possible description with the Nonlinear Diffusion Equation?

Our Approach

- Find simplistic model to study the interplay of disorder and nonlinearity
- Use Nonlinear Diffusion Equation (NDE) as phenomenologic description for the spreading
- Find spreading and scaling laws from this equation
- Compare with numerical results and verify scaling predictions from the NDE

Nonlinear Diffusion Equation

We propose the Nonlinear Diffusion Equation (NDE) as phenomenological description of spreading in nonlinear disordered lattices:

$$\frac{\partial \rho}{\partial t} = D\frac{\partial}{\partial x}\left(\rho^a \frac{\partial \rho}{\partial x}\right), \qquad \text{with} \qquad \int \rho \mathrm{d}x = E.$$

a is the nonlinearity index of the NDE.

We identify E as the energy and ρ as the energy density.

This equation has a self-similar solution:

$$\rho(x,t) = \begin{cases} (D(t-t_0))^{-1/(2+a)} f(x/(D(t-t_0))^{1/(2+a)}) & \text{for} \quad |x| < X \\ 0 & \text{for} \quad |x| > X \end{cases}$$

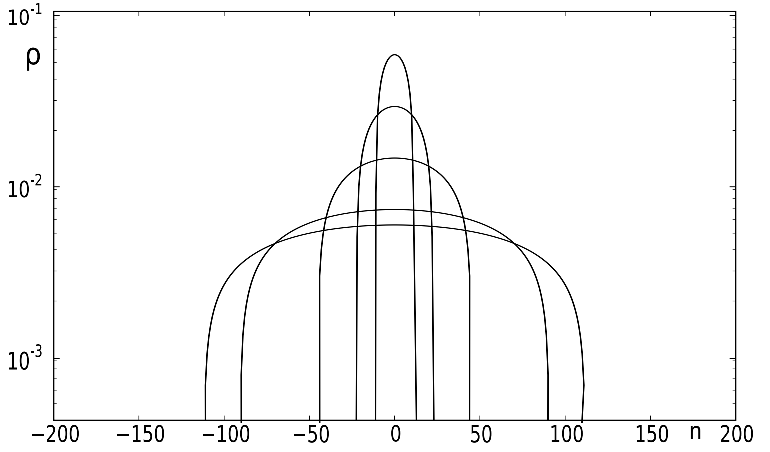
$$f(y) = \left(E^{a/(2+a)} - \frac{ay^2}{2(a+2)}\right)^{1/a},$$

Where the edge X propagates according to:

$$X = \sqrt{2\frac{2+a}{a}} E^{a/(2+a)} (D(t-t_0))^{1/(2+a)}$$

Nonlinear Diffusion Equation

Self similar solution:



- Self similar solution at times $t = 10^4 \dots 10^8$
- Very sharp tails
- Width $\sim X \sim t^{1/(a+2)}$ (subdiffusive)

Scaling of the NDE

Aim: Find scaling relation that can be checked numerically.
Starting from edge propagation:

$$X = \sqrt{2\frac{2+a}{a}E^{a/(2+a)}(D(t-t_0))^{1/(2+a)}} \ .$$

The dependence on energy and time is described with one parameter: a.

$$\frac{X}{E} \sim \left(\frac{t-t_0}{E^2}\right)^{1/(2+a)} \ , \quad \frac{1}{X}\frac{\mathrm{d}t}{\mathrm{d}X} \sim \left(\frac{E}{X}\right)^{-a}$$

Note, that for the inverse spreading velocity no explicit time dependence appears. Using $w = E/X$ (energy density) we can write a one-parameter scaling relation:

$$a(w) = -\frac{\mathrm{d}\log\frac{1}{X}\frac{\mathrm{d}t}{\mathrm{d}X}}{\mathrm{d}\log w}$$

Strongly Nonlinear Lattice

- Hamiltonian system, as simple as possible
- Should include nonlinearity and disorder
- Leads to coupled oscillators with local disorder and nonlinear coupling

$$H = \sum_k \frac{p_k^2}{2} + \underbrace{W\omega_k^2 \frac{q_k^\kappa}{\kappa}}_{\text{local disorder}} + \underbrace{\frac{\beta}{\lambda}(q_{k+1} - q_k)^\lambda}_{\text{nonlinear coupling}} \tag{1}$$

- $\kappa \geq 2$, $\lambda > 2$ describe the nonlinearity
- ω_k is random, chosen iid from $[0, 1]$
- Similar to DANSE model in eigenmode representation with strong localization (effective nearest neighbour couling)

Parameter Reduction for $\kappa \neq \lambda$

$$H = \sum_k \frac{p_k^2}{2} + W\omega_k^2 \frac{q_k^\kappa}{\kappa} + \frac{\beta}{\lambda}(q_{k+1} - q_k)^\lambda$$

For $\kappa \neq \lambda$ apply transformations with $\alpha = 1/(\lambda - \kappa)$:

$$q_k \to (W/\beta)^\alpha \, q_k, \;\; p_k \to \left(W^\lambda/\beta^\kappa\right)^{\alpha/2} p_k, \;\; H \to \left(W^\lambda/\beta^\kappa\right)^\alpha H$$

This leads to:

$$H = \sum_k \frac{p_k^2}{2} + \omega_k^2 \frac{q_k^\kappa}{\kappa} + \frac{1}{\lambda}(q_{k+1} - q_k)^\lambda$$

with the energy $E := H$ as the only remaining parameter.

Scaling Invariance for $\kappa = \lambda$

$$H = \sum_k \frac{p_k^2}{2} + W\omega_k^2 \frac{q_k^\kappa}{\kappa} + \frac{\beta}{\lambda}(q_{k+1} - q_k)^\lambda$$

For $\kappa = \lambda$ the Hamiltonian is invariant under the scaling:

$$q \rightarrow \gamma q, \quad p \rightarrow \gamma^{\kappa/2} p, \quad t \rightarrow t/\gamma^{\kappa/2-1}, \quad H \rightarrow \gamma^\kappa H.$$

After a trivial transformation $q_k \rightarrow q_k/W^{1/\kappa}$ and $\beta/W \rightarrow \beta$:

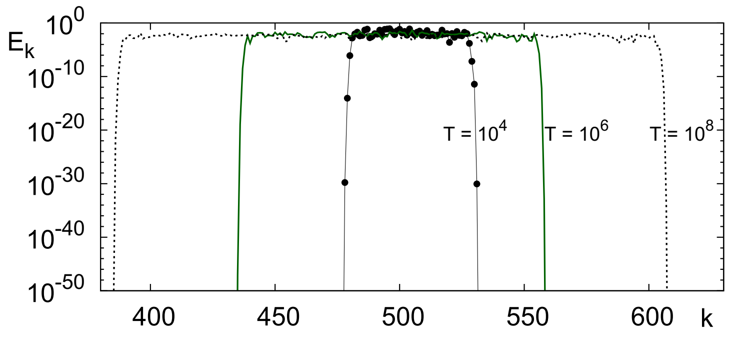
$$H = \sum_k \frac{p_k^2}{2} + \omega_k^2 \frac{q_k^\kappa}{\kappa} + \frac{\beta}{\kappa}(q_{k+1} - q_k)^\kappa$$

with β as the only relevant parameter.
The scaling invariance (time and energy interrelated) allows us to compute the parameter of the NDE: $a = 1/2 - 1/\kappa$ **!**

Measures of Spreading

- Based on local energy: E_k
- Classical method: excitation width $L(t)$, measured with Participation Number $P^{-1} = \sum(E_k/E)^2$
- Nonlinear coupling: $\rightarrow$ **nearly compact spreading states**

- Creates possibility to measure excitation times ΔT required to excite a new lattice site ("First passage time").
- "Adjoint" method to $P(t)$

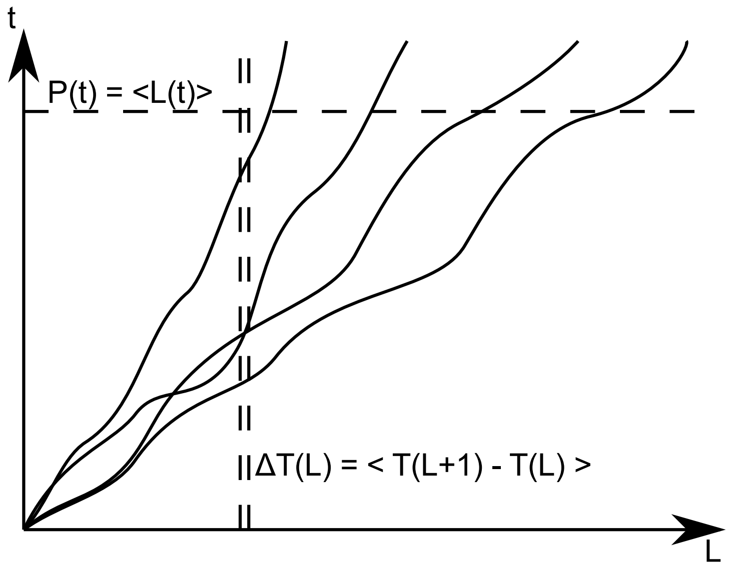


time evolution of initial single site excitation

- Assume L sites with indices $j \dots k$ are excited.
- Measure the time required to excite a new site $j-1$ or $k+1$ as a function of the current excitation length L: $\Delta T(L)$.

Comparison of Averaging

We are interested in the average spreading behavior – averaging over disorder realizations has to be performed.
There is a crucial difference between the two methods $P(t)$ and $\Delta T(L)$.



Comparism of averaging procedures

- L at given t: averaging over excitations with different energy densities $w = E/L$.
- $T(L)$ at given L: averaging over excitations with same energy densities.
  $\longrightarrow$ better, because NDE predicts w as crucial value for spreading.

NDE vs SNL

Nonlinear Diffusion Equation $\Longleftrightarrow$ Strongly Nonlinear Lattice

$$\frac{1}{X}\frac{\mathrm{d}t}{\mathrm{d}X} \sim \left(\frac{E}{X}\right)^{-a} \quad \Longleftrightarrow \quad \frac{1}{L}\frac{\Delta T}{1} \sim \left(\frac{E}{L}\right)^{-a}$$

$$\frac{X}{E} \sim \left(\frac{t-t_0}{E^2}\right)^{1/(2+a)} \quad \Longleftrightarrow \quad \frac{P}{E} \sim \left(\frac{t-t_0}{E^2}\right)^{1/(2+a)}$$

Numerical Results for $\kappa = \lambda = 4$

$$H = \sum_k \frac{p_k^2}{2} + \omega_k^2 \frac{q_k^4}{4} + \frac{\beta}{4}(q_{k+1} - q_k)^4$$

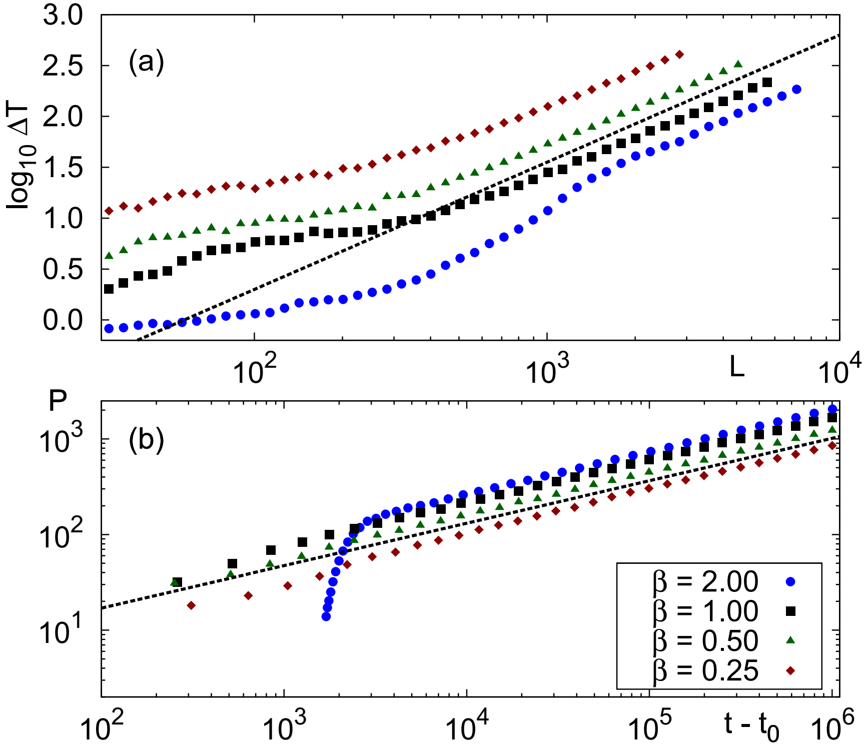
From the scaling invariance of the Hamiltonian for $\kappa = \lambda$, the parameter a can be calculated to be $a = 1/2 - 1/\kappa = 1/4$.

Numerical results compared with analytic estimates:

$$\Delta T \sim L^{a+1} \sim L^{5/4}$$

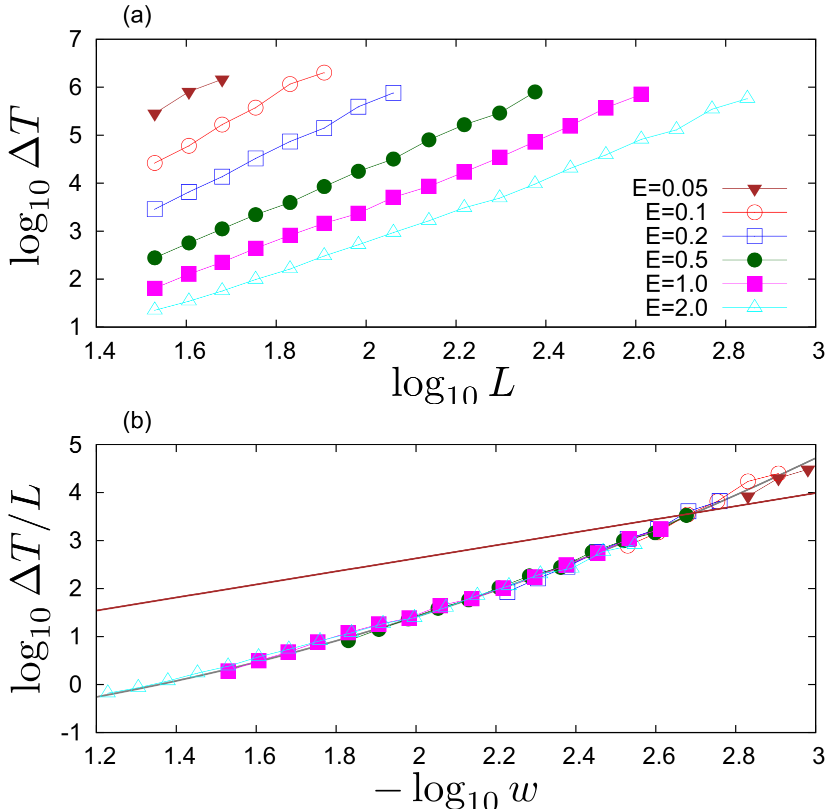
$$P \sim (t - t_0)^{1/(2+a)} \sim (t - t_0)^{4/9}$$

NDE gives a very good description of the spreading



Numerical Results for $\kappa = 2$, $\lambda = 4$

$$H = \sum_k \frac{p_k^2}{2} + \omega_k^2 \frac{q_k^2}{2} + \frac{1}{4}(q_{k+1} - q_k)^4$$

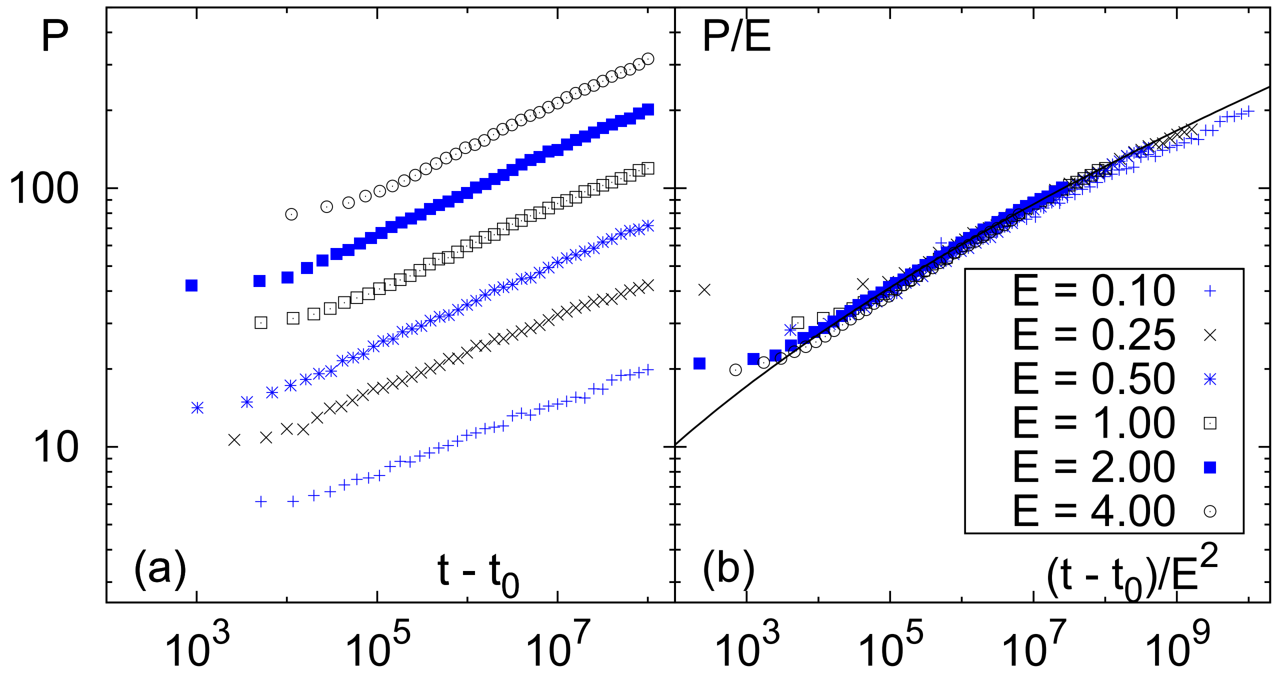


- Subdiffusive behavior for all energies $E = 0.05 \dots 4$.
- NDE suggests scaled variables: $\Delta T/L$ vs E/L

- Scaling works perfectly
- Scaling exponents appears to be density dependent $a(w) \approx -0.09 - 1.36 \log_{10} w$

Deviation from pure power law spreading.

Numerical Results for $\kappa = 2$, $\lambda = 4$



Results for $P(t)$ are consistent. Again, predicted scaling is found.

Conclusions

- Introduced a new model to study the interplay between disorder and nonlinearity
- New quantification of Spreading: $\Delta T(L)$
- Proposed the Nonlinear Diffusion Equation as phenomenologic description of the spreading
- For $\kappa = \lambda = 4$, predictions of the NDE were nicely reproduced by numerical simulations
- For $\kappa = 2, \lambda = 4$, energy scaling of the solution of the NDE was found and also successfully checked by numerics
- Spreading does not follow perfect power law, but has a density dependent index $a(w)$

Scaling of average spreading in disordered systems due to nonlinear interactions can be well described in the frame of the Nonlinear Diffusion Equation.

Comparison with DANSE

Discrete Anderson Nonlinear Schrödinger Equation:

$$H = \sum_n \psi_{n-1}\psi_n^* + \psi_n\psi_{n-1}^* + V_n|\psi_n|^2 + \frac{\beta}{2}|\psi_n|^4 \qquad (2)$$

- Local disorder (V_n random) and nonlinearity, linear coupling
- For $\beta = 0$, eigenmodes φ_k are exponentially localized
- Expansion into these eigenmodes $\psi = \sum_k C_k \varphi_k$ leads to:

$$H = \sum_k \epsilon_k |C_k|^2 + \frac{\beta}{2} \sum_{m,m',m''} V_{mm'm''k} C_m C_{m'}^* C_{m''} C_k^* \qquad (3)$$

- $V_{mm'm''k}$ is the 4-mode spatial overlap of eigenmodes
- Decreases exponentially with spatial distance of eigenmodes
- For highly localized modes (localization length 1), this means effectively a nonlinear nearest neighbour coupling.