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# Weak chaos in the disordered nonlinear Schroedinger chain: destruction of Anderson localization

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# Weak chaos in the disordered nonlinear Schrödinger chain: destruction of Anderson localization by Arnold diffusion


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# Discrete nonlinear Schrödinger equation with disorder

$$i\frac{d\psi_n}{dt} = \omega_n\psi_n - \Omega(\psi_{n+1} + \psi_{n-1}) + g\psi_n^*\psi_n^2$$

Anderson localization

nonlinearity

$$H(i\psi^*,\psi) = \sum_{n=-\infty}^{\infty}\omega_n\psi_n^*\psi_n + \sum_{n=-\infty}^{\infty}\frac{g}{2}\psi_n^*\psi_n^*\psi_n\psi_n - \sum_{n=-\infty}^{\infty}\Omega\left(\psi_n^*\psi_{n+1} + \psi_{n+1}^*\psi_n\right)$$

anharmonic oscillators

nearest-neighbor coupling

action-angle variables: $\psi_n = \sqrt{I_n}\,e^{-i\phi_n}$

$$H(I,\phi) = \sum_{n=-\infty}^{\infty}\omega_n I_n + \sum_{n=-\infty}^{\infty}\frac{g}{2}I_n^2 - \sum_{n=-\infty}^{\infty}\Omega\sqrt{I_nI_{n+1}}\,2\cos(\phi_n - \phi_{n+1})$$

# Disordered nonlinear 1D systems

Stationary solutions of NLSE: Iomin, Fishman (2007); Fishman *et al.* (2008); Bodyfelt *et al.* (2010)

Transmission of a finite sample: Gredeskul, Kivshar (1992); Paul *et al*. (2005); Paul *et al.* (2007); Tietsche, Pikovski (2008); Paul *et al.* (2009)

Dipole oscillations in a trap: Albert *et al.* (2008)

Wave packet spreading in discrete NLSE: Shepelyansky (1993); Molina, Tsironis (1994); Kopidakis *et al.* (2008); Pikovsky, Shepelyansky (2008); Fishman *et al.* (2008); Bourgain, Wang (2008); Wang, Zhang (2009); Flach *et al.* (2009); Skokos *et al.* (2009); Fishman *et al.* (2009); Veksler *et al.* (2009); Krivolapov *et al.* (2009); Veksler *et al.* (2010); Iomin (2010); Skokos, Flach (2010); Flach (2010); Mulansky, Pikovsky (2010); Laptyeva *et al.* (2010)

Wave packet spreading in other nonlinear disordered 1D systems: Fröhlich *et al.* (1986); Kopidakis *et al.* (2008); Flach *et al.* (2009); Skokos *et al.* (2009); Garcia-Mata, Shepelyansky (2009); Krimer *et al.* (2009); Flach (2010); Laptyeva *et al.* (2010)

Thermalization in NLSE and other nonlinear disordered 1D systems: Dhar, Lebowitz (2008); Dhar, Saito (2008); Oganesyan *et al.* (2009); Mulansky *et al.* (2009); Pikovsky, Fishman (2010)

**Numerics:** wave packet spreads as a power law

**Wang, Zhang + Fishman *et al*.:** slower than any power law

**Given**

1. Strong localization
2. Weak nonlinearity

} $\rightarrow$ worst conditions for transport

3. Arbitrary initial condition with extensive norm and energy

**Question:** will the system equilibrate at long distances, and how?

**Answer:** yes, by normal nonlinear diffusion:

$$\frac{\partial \rho}{\partial t} = \frac{\partial}{\partial x} D(\rho) \frac{\partial \rho}{\partial x}$$

**Mechanism:** CHAOS

- Arnold diffusion in the space of actions
- driven by rare local chaotic spots
- which migrate along the chain

(as seen by Oganesyan, Pal, Huse, 2009)

# Assumptions

$$i\frac{d\psi_n}{dt} = \omega_n\psi_n - \Omega(\psi_{n+1} + \psi_{n-1}) + g\psi_n^*\psi_n^2$$

disorder

$$-\frac{\Delta}{2} \le \omega_n \le \frac{\Delta}{2}$$

"tunnelling"

nonlinearity

$$g > 0$$

**1. Strong localization:** $\frac{\Omega}{\Delta} \equiv \tau \ll 1$

assumption about the Hamiltonian

**2. Weak nonlinearity:** $\frac{g|\psi_n|^2}{\Delta} \sim \rho \ll 1$

(nonlinear frequency shift << disorder)

note the invariance under $\psi_n \to C\psi_n, \quad g \to C^{-2}g$

**3. Single action scale:** $-\Delta\sum_n |\psi_n|^2 \ll H < 0$

(all oscillators are excited more or less equally; thermodynamic relations have a simple form)

assumptions about the initial conditions

# Thermalization and transport

Two conserved quantities: total energy $H$, total action $I = \sum |\psi_n|^2$

**Local equilibration** in a **finite** time → $\mathcal{P}(\{\psi_n\}) \propto e^{-\beta(H-\mu I)}, \quad \beta \equiv 1/T$

**Global equilibration: transport of the conserved quantities**

**Macroscopic action density:** $\rho(x) = \frac{1}{L^*} \sum_{n=x-L^*/2}^{x+L^*/2} \frac{g|\psi_n|^2}{\Delta} \approx \frac{gT}{|\mu|\Delta}$

get rid of the energy density thanks to $|H| \ll I\Delta$

$$\frac{\partial \rho}{\partial t} = \frac{\partial}{\partial x} D(\rho) \frac{\partial \rho}{\partial x}$$

**finite!**

# Diffusion coefficient

$$D(\rho) \sim \exp\left(-\mathcal{C}\ \ln^2 \frac{1}{\tau^p \rho} \ln \frac{1}{\rho}\right) \qquad \frac{1}{2} \le p \le 3$$

stronger than any
power law

$$\frac{1}{3}\left[1 + \ln\left(1 + \frac{\ln(1/\rho)}{\ln(1/\tau)}\right)\right]^{-2} \le \mathcal{C} \le 8\left[1 + \ln\left(1 + \frac{\ln(1/\rho)}{\ln(1/\tau)}\right)\right]^{-2}$$

double logarithm $\sim$ constant

$D^{-1}$ is self-averaging at distances $L^{*} \gg \exp\left(\mathcal{C}\ \ln^2 \frac{1}{\tau^p \rho}\right)$

A finite-norm wave packet spreads as size $\sim \exp\left[(\ln t)^{1/3}\right]$

# Off-resonant coupling

two-oscillator Hamiltonian:

$$H = \omega_1|\psi_1|^2 + \frac{g}{2}|\psi_1|^4 + \omega_2|\psi_2|^2 + \frac{g}{2}|\psi_2|^4 - \tau\Delta(\psi_1^*\psi_2 + \psi_2^*\psi_1)$$

nonlinear frequency shifts: $\phi_n = (\omega_n + g|\psi_n^0|^2)t$

perturbative correction from tunnelling:

$$\psi_1(t) = \psi_1^0 e^{-i(\omega_1 + g|\psi_1^0|^2)t} - \frac{\tau\Delta\, \psi_2^0 e^{-i(\omega_2 + g|\psi_2^0|^2)t}}{\omega_2 + g|\psi_2^0|^2 - \omega_1 - g|\psi_1^0|^2}$$

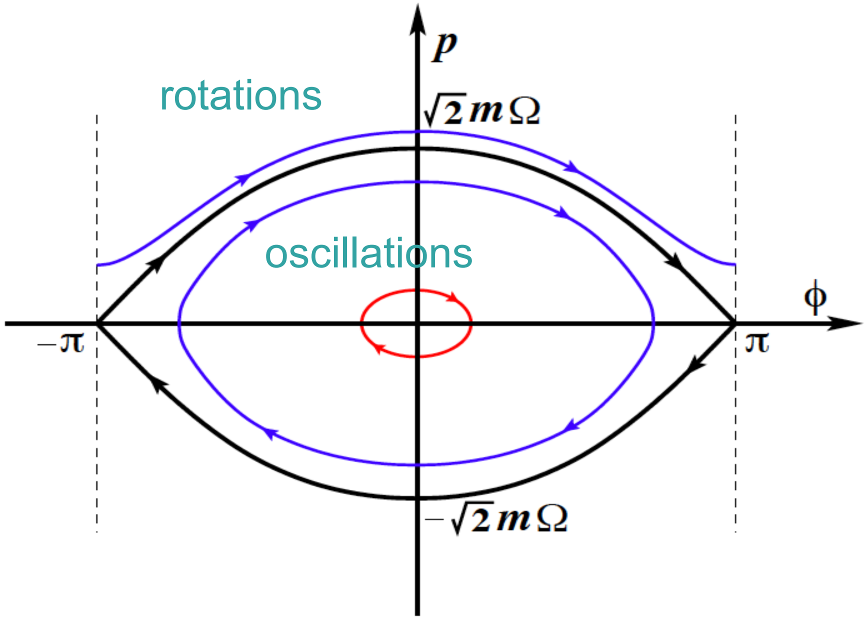
The correction to $|\psi_1|^2$ is small at all times unless the denominator → 0

**Theorem of Kolmogorov, Arnold, & Moser:**

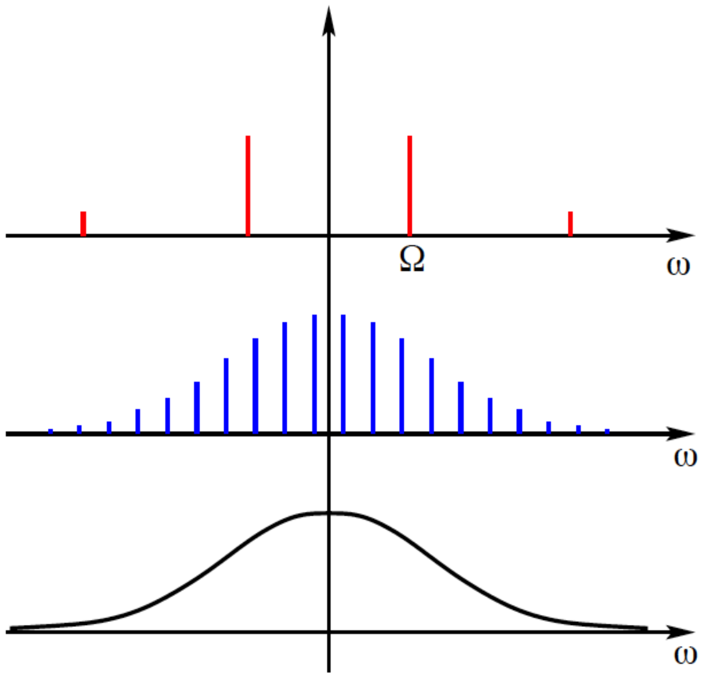
in most of the phase space the perturbed trajectories are small deformations of the unperturbed trajectories

# Pendulum: $H(p,\phi) = \frac{p^2}{2m} - m\Omega^2 \cos\phi$

**Phase space:**




**Spectrum** $\int \phi(t)\, e^{i\omega t}\, dt$



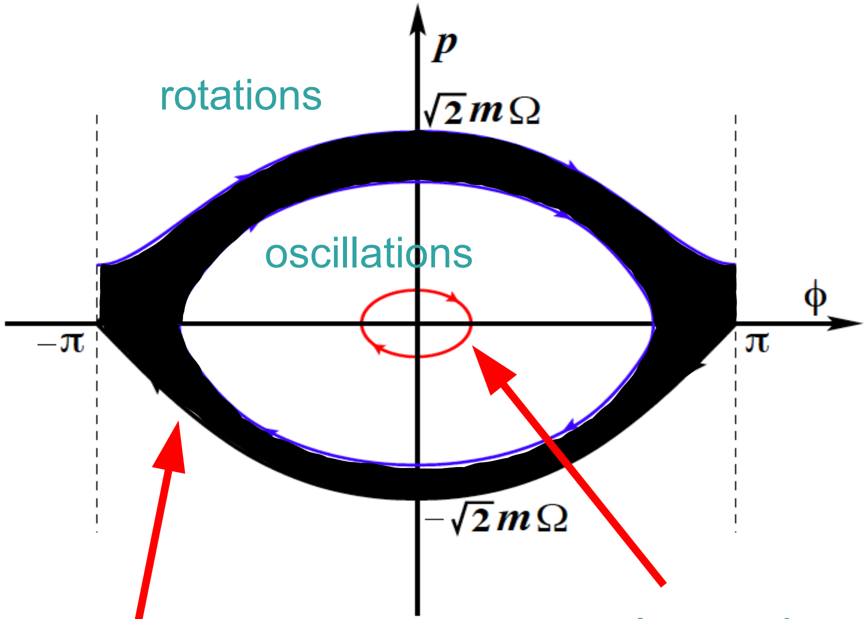

the period diverges
at the separatrix

↔

the separatrix motion
has a continuous spectrum

# Perturbed pendulum:

$$H(p,\phi,t) = \frac{p^2}{2m} - m\Omega^2 \cos\phi - V\cos(\phi - \omega t)$$




ergodic trajectories
within
the stochastic layer

regular motion
survives

Stochastic layer area:

$$W_s \equiv \int_{\text{layer}} \frac{dp\,d\phi}{2\pi} \sim \frac{V}{\Omega} e^{-|\omega|/\Omega}$$

Melnikov-Arnold integral

$|\omega| \gg \Omega$

Continuous spectrum
of the chaotic motion:

$$\left\langle e^{i\phi(t)} e^{-i\phi(t')} \right\rangle_\omega \sim \frac{1}{\Omega} e^{-|\omega|/\Omega}$$

review: B. Chirikov (1979)

# Making a pendulum out of oscillators

two-oscillator Hamiltonian:

$$H = \omega_1 I_1 + \frac{g I_1^2}{2} + \omega_2 I_2 + \frac{g I_2^2}{2} - 2\tau\Delta\sqrt{I_1 I_2}\cos(\phi_1 - \phi_2)$$

canonical transformation:

$$I = I_1 + I_2, \quad \phi = \frac{\phi_1 + \phi_2}{2}, \quad \tilde{I} = \frac{I_1 - I_2}{2}, \quad \tilde{\phi} = \phi_1 - \phi_2$$

$$H = H_0(I) + (\omega_1 - \omega_2)\tilde{I} + g\tilde{I}^2 - 2\tau\Delta\sqrt{I^2/4 - \tilde{I}^2}\cos\tilde{\phi}$$

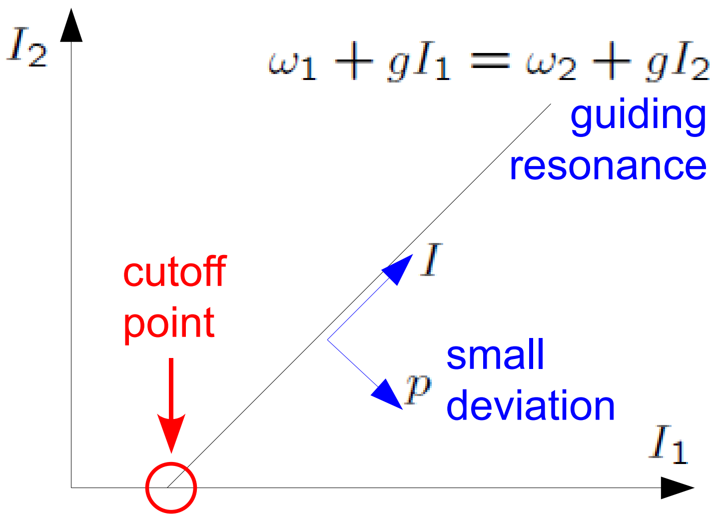
constant

shift

$$\tilde{I} = \frac{\omega_2 - \omega_1}{2g} + p$$

almost constant

$$\tau \ll 1$$




A third oscillator:

$$2\tau\Delta\sqrt{I_2 I_3}\cos(\phi_2 - \phi_3)$$

perturbation of the pendulum

**Three oscillators are sufficient to generate chaos**

# The price of making a pendulum

**To find a separatrix:**

the shift is possible only if $I_1, I_2 > 0$

$\omega_1 + gI_1 = \omega_2 + gI_2$ cutoff point

$$\frac{H - \mu I}{T} > \frac{|\mu||\omega_1 - \omega_2|}{gT} \sim \frac{1}{\rho}$$

unless $|\omega_1 - \omega_2| \ll \Delta$

Look for a resonance
or
pay the thermal exponential

**(guiding resonance)**

**To create the stochastic layer:**

the pendulum frequency $\Omega \sim \sqrt{\tau \Delta g I}$

$$\frac{|\omega_2 - \omega_3|}{\Omega} \sim \frac{1}{\sqrt{\tau \rho}}$$

unless $|\omega_2 - \omega_3| \ll \Delta$

Look for another resonance
or
pay the Melnikov-Arnold exponential

**(layer resonance)**

**Chaotic oscillators are rare:** density $\sim \min\{\tau\rho, \rho^2\}$

# Making a pendulum out of more oscillators

$$-\tau\Delta(\psi_1^*\psi_2 + \psi_2^*\psi_1) + \omega_2\psi_2^*\psi_2 - \tau\Delta(\psi_2^*\psi_3 + \psi_3^*\psi_2)$$

effective coupling 1 ↔ 3: $\dfrac{(\tau\Delta)^2}{\omega_1 - \omega_2}(\psi_1^*\psi_3 + \psi_3^*\psi_1)$

works when $\omega_1 \approx \omega_3 \neq \omega_2$

Tunnelling + nonlinearity → effective couplings of the form

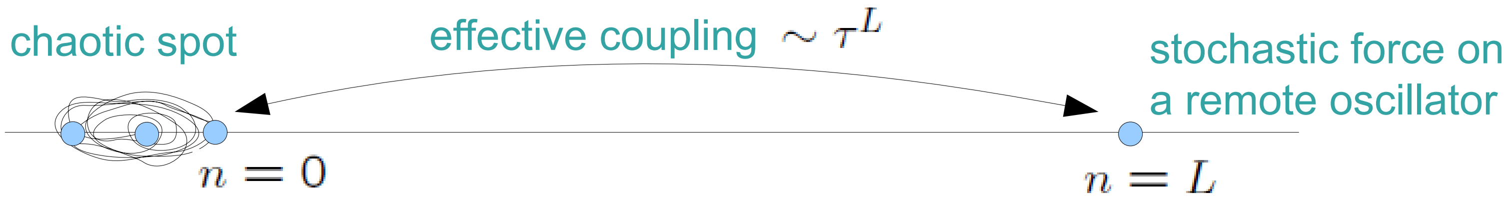
$$\psi_1^*\psi_2^*\psi_3^*\psi_4\psi_5\psi_6 \rightarrow \cos(\phi_1+\phi_2+\phi_3-\phi_4-\phi_5-\phi_6)$$

**Guiding and layer resonances can be generated in high orders of the perturbation theory**

**Competition: <u>number of combinations ↔ power of the coupling constants</u>**

**size of a chaotic spot << distance between chaotic spots**

# Arnold diffusion




Effective coupling $2V_{m_1 \dots m_N} \cos(m_1\phi_1 + \dots + m_N\phi_N)$

action conservation: $m_1 + \dots + m_N = 0$

Change in actions due to the stochastic force after time $t$

$$\langle \delta I_n \delta I_{n'} \rangle \sim t\, V^2_{m_1 \dots m_N} \frac{m_n m_{n'}}{\Omega} \exp\left(-\frac{|m_1\omega_1 + \dots + m_N\omega_N|}{\Omega}\right)$$

probability = $f(\{I_n\})\, W_s\, g|\vec{m}^{\mathrm{g}}|^2\, \delta\left(m_1^{\mathrm{g}}(\omega_{n_1^{\mathrm{g}}} + gI_{n_1^{\mathrm{g}}}) + \dots + m_N^{\mathrm{g}}(\omega_{n_N^{\mathrm{g}}} + gI_{n_N^{\mathrm{g}}})\right) \prod_n dI_n$

distribution function — $f(\{I_n\})$

stochastic layer area — $W_s$

guiding resonance — $\delta(\dots)$

$$W_s \frac{\partial f}{\partial t} = \sum_{n,n'} \frac{\partial}{\partial I_n} W_s D_{nn'} \frac{\partial f}{\partial I_{n'}}$$

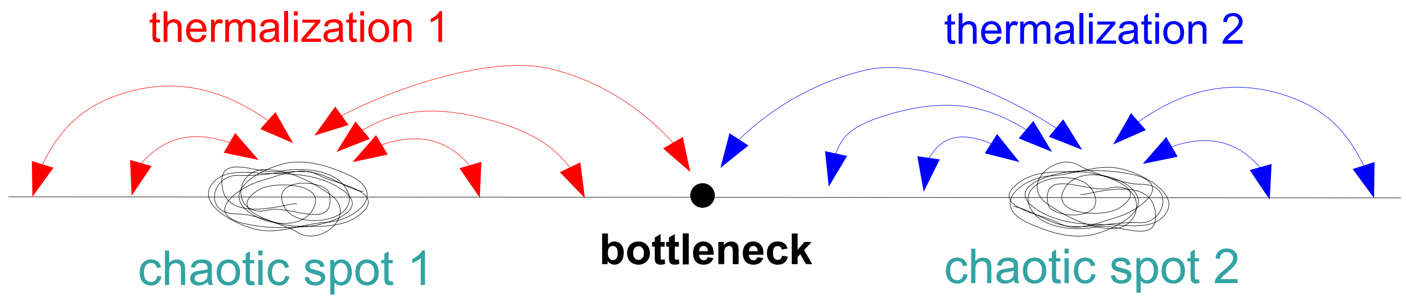
**Constraints on the diffusion:**

**1. Total action is conserved**

**2. Total energy is conserved**

**3. The system stays on the guiding resonance**

# Long-distance relaxation




Typical density of chaotic spots $\sim \rho^2$

Coupling between the chaotic spots and the bottleneck $\sim \tau^{1/\rho^2}$

worse than activation ($\rho \propto T$)

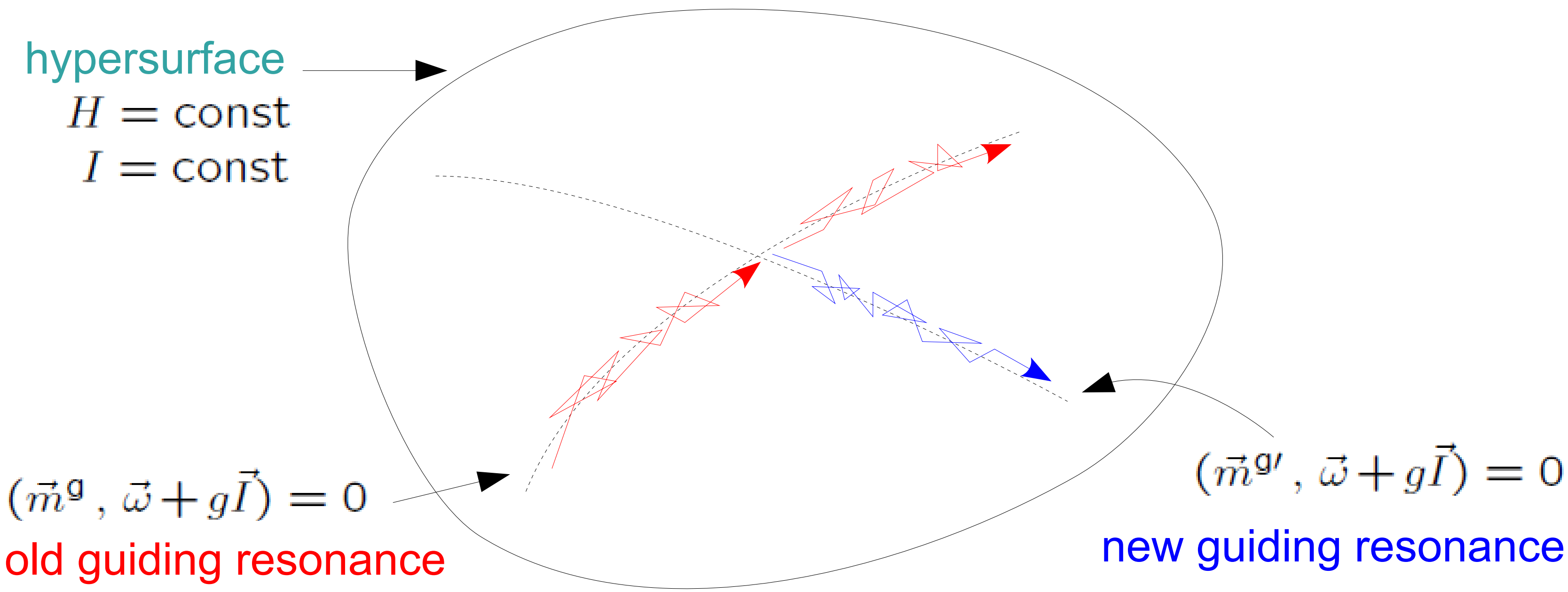
Look for a better mechanism!

# Changing the guiding resonance

1. **Time needed to create another chaotic spot at a distance** $L$

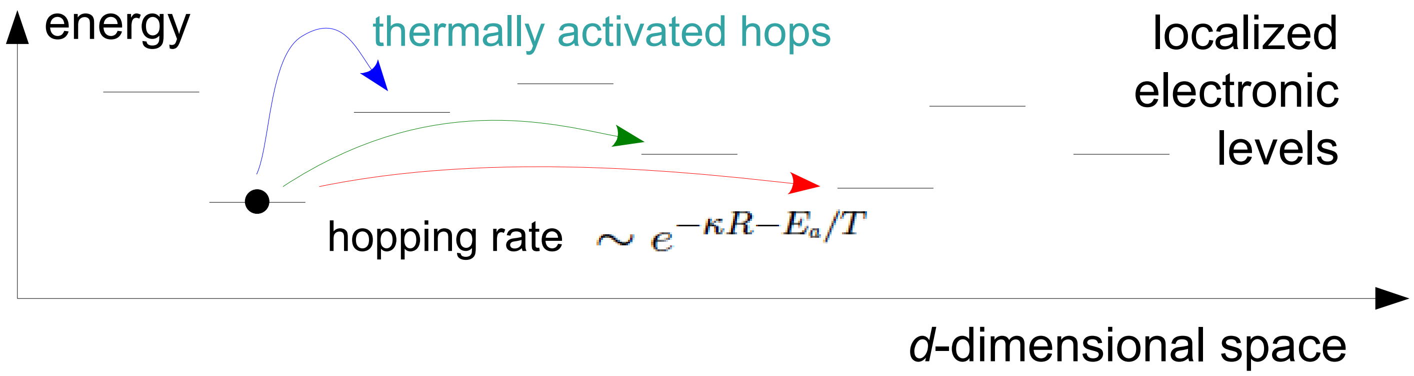
   **~ (thermalization time at the distance** $L$**)** $\times\, e^{\text{activation}}$

2. **One of the two chaotic spots is quickly quenched**




**Chaotic spots can randomly migrate along the chain**

# Variable-range hopping of electrons




To find a low level one should explore large distances $E_a^{min} \sim \frac{1}{\nu R^d}$

Competition between $e^{-\kappa R}$ and $e^{-E_a/T}$

$$\sigma(T) \propto \max_R e^{-\kappa R - (\nu R^d)^{-1}/T} = \exp\left[-\frac{d+1}{d}\left(\frac{\kappa^d}{\nu T}\right)^{1/(d+1)}\right]$$

Mott (1969)

stretched exponential after optimization

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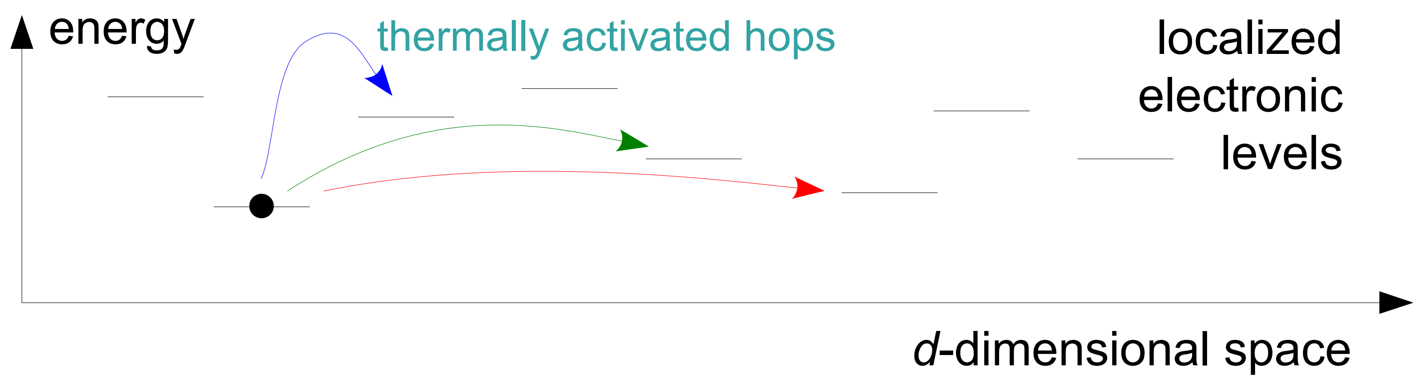


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Mott (1969)

stretched exponential after optimization

One dimension:

$$\sigma(T) \sim \exp\left(-\frac{\kappa}{2\nu T}\right)$$

Kurkijarvi (1973)

Rare “bad” regions block the transport in 1D

**“Breaks”**

# Chaotic fraction $w_n$

$$w_n = \frac{1}{Z} \int_{\mathrm{chaotic}(n)} e^{-(H-\mu I)/T} \prod_n \frac{dI_n\, d\phi_n}{2\pi}, \qquad Z \equiv \int e^{-(H-\mu I)/T} \prod_n \frac{dI_n\, d\phi_n}{2\pi}$$

chaotic($n$): all guiding resonances whose leftmost oscillator is *n*

$w = e^{-\lambda}$ for chaotic spots ⟺ $e^{-E_a/T}$ for electrons

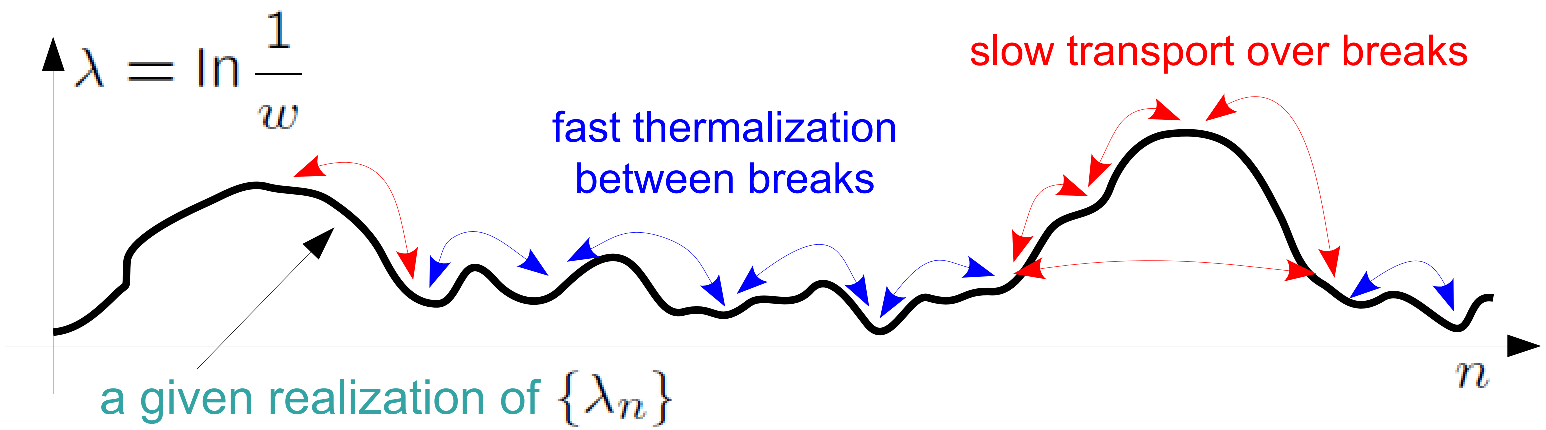
$\lambda$:
- thermal exponential
- Melnikov-Arnold exponential
- coupling constants

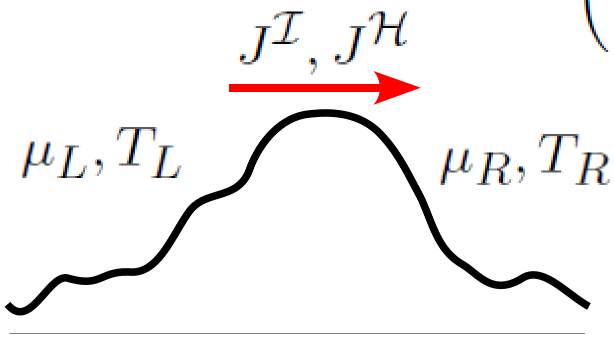
$w$ is a random quantity determined by the disorder

**Probability distribution:**

$$\mathcal{P}\,\{w < w_0\} = \exp\left(-\mathcal{C}_1 \rho\, w_0^{1/[\mathcal{C}\ln^2(1/\tau^p \rho)]}\right)$$

# From $\lambda$ to $\sigma$: break resistance




1. Definition of the current

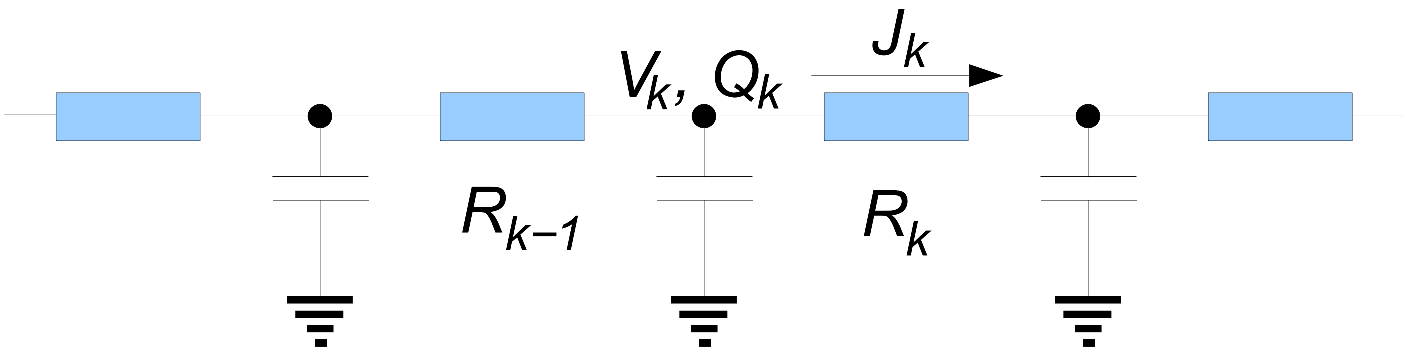
$$\begin{pmatrix} J^{\mathcal{I}} \\ J^{\mathcal{H}} \end{pmatrix} = \frac{d}{dt} \begin{pmatrix} \langle I \rangle_R \\ \langle H \rangle_R \end{pmatrix}$$

2. The diffusion equation for actions

$$\begin{pmatrix} J^{\mathcal{I}} \\ J^{\mathcal{H}} \end{pmatrix} = R_b^{-1} \begin{pmatrix} \mu_L/T_L - \mu_R/T_R \\ 1/T_R - 1/T_L \end{pmatrix}$$

**Each break can be characterized by its "resistance"**

# From $\lambda$ to $\sigma$: resistors in series




$$\begin{cases} \dfrac{dQ_k}{dt} = J_{k-1} - J_k \\ J_k = R_k^{-1}(V_k - V_{k+1}) \\ Q_k = Q_k(V_k) \end{cases}$$

definition of the current

"resistance" of the break

thermodynamics

$$\begin{cases} \dfrac{\partial Q}{\partial t} = \dfrac{\partial}{\partial x}\sigma(V)\dfrac{\partial V}{\partial x} \\ Q = Q(V) \end{cases}$$

macroscopic "charge density" $Q = \frac{1}{L}\sum Q_k$

macroscopic "conductivity" $\sigma = \left(\frac{1}{L}\sum R_k\right)^{-1}$

# Optimal breaks

$$\sigma^{-1} = \frac{1}{L} \sum_{\text{breaks} \in L} R_b$$

self-averaging at long distances

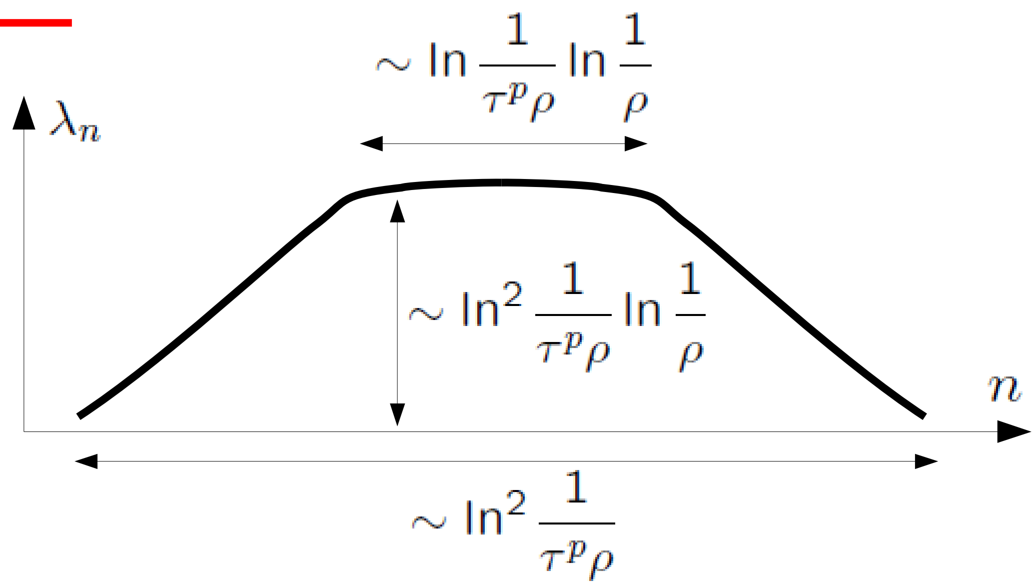
$$\sigma^{-1} = \int R_b(\{\lambda_n\})\, dP(\{\lambda_n\})$$

probability measure per unit length

increasing

decreasing

The integral is dominated by configurations close the optimal one



# Macroscopic diffusion coefficient: three logarithms

Pendulum frequency (guiding resonance) → Melnikov-Arnold exponential

Strength of the perturbation to destroy the separatrix

Activation

$$D \sim \sigma \sim \exp\left(-\mathcal{C}\ln^2\frac{1}{\tau^p\rho}\ln\frac{1}{\rho}\right)$$

# Macroscopic length scale

$$L^* \sim \exp\left(\mathcal{C}\ln^2\frac{1}{\tau^p\rho}\right)$$

distance between the optimal breaks

# Conclusions

1. Anderson localization + weak nonlinearity → weak chaos
2. Rare chaotic spots play the role of a bath
3. They induce relaxation by driving the Arnold diffusion
4. They migrate along the chain
5. In 1D the transport of conserved quantities is determined by rare breaks

D. M. Basko, arXiv:1005.5033