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Advanced Workshop on Anderson Localization, Nonlinearity and Turbulence: a Cross-Fertilization

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High-gradient Operators and Anderson Localization

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S. Ryu, A. Furusaki, A. W. W. Ludwig, and C. Mudry, Nucl. Phys. **B780**, 105 (2007)

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Outline

Goal (elusive)

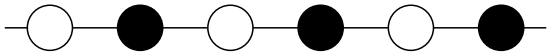
What are the conductance fluctuations at an Anderson metal-insulator transition when dimensionality of space d is larger than or equal to 2?

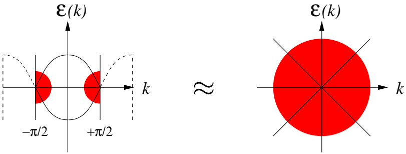
Pandora box was opened with the prediction of universal conductance fluctuations (UCF): Altshuler (1985); Stone and Lee (1985).

Key words that need to be explained:

- Symmetry classes (of Anderson localization)
- Universality classes (of Anderson localization)
- NLσM
- High-gradient operators

Definition of the (weakly) random Hamiltonian





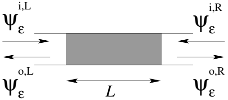
Hamiltonian for non-interacting and spinless fermions hopping along an open chain at half-filling with weak static disorder:

$$H := -\tau_3 i \frac{d}{dx} - \sum_{\mu=0}^{2} \tau_\mu v_\mu(x), \qquad v_{\mathrm{F}} = \hbar = 1;$$

$$\left\langle v_\mu(x) \right\rangle = 0, \qquad \left\langle v_\mu(x) v_\nu(x') \right\rangle = \ell_\mu^{-1} \delta_{\mu\nu} \delta(x - x').$$

For any realization of the random potential, the symmetries are:
(AI) $\sigma_1 H^* \sigma_1 = \;\; H$, i.e., time-reversal symmetry holds generically,
(BDI) $\sigma_1 H \; \sigma_1 = -H$ if $\ell_0 = \ell_1 = \infty$, i.e., chiral symmetry holds if $\ell_0 = \ell_1 = \infty$.

The scattering matrix $\mathcal{S}_\varepsilon$, is defined by



$$\Longleftrightarrow \begin{pmatrix} \psi^{\rm o,L} \\ \psi^{\rm o,R} \end{pmatrix}_\varepsilon = \mathcal{S}_\varepsilon \begin{pmatrix} \psi^{\rm i,L} \\ \psi^{\rm i,R} \end{pmatrix}_\varepsilon \equiv \begin{pmatrix} r & t' \\ t & r' \end{pmatrix}_\varepsilon \begin{pmatrix} \psi^{\rm i,L} \\ \psi^{\rm i,R} \end{pmatrix}_\varepsilon .$$

It is unitary from which follows the (non-unique) polar decomposition

$$\mathcal{S}_\varepsilon = \begin{pmatrix} v'^* & 0 \\ 0 & u \end{pmatrix}_\varepsilon \begin{pmatrix} -\tanh x & \operatorname{sech} x \\ \operatorname{sech} x & \tanh x \end{pmatrix}_\varepsilon \begin{pmatrix} v & 0 \\ 0 & u'^* \end{pmatrix}_\varepsilon .$$

The constraints on the scattering matrix are:

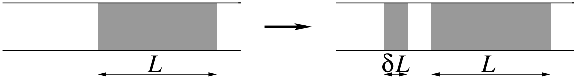
$\mathcal{S}_\varepsilon = (\mathcal{S}_\varepsilon)^T$ due to time-reversal symmetry,

$\mathcal{S}_{+\varepsilon} = \left(\mathcal{S}_{-\varepsilon}\right)^\dagger$ due to the chiral symmetry if $\ell_0^{-1} = \ell_1^{-1} = 0$.

The dimensionless (Landauer) conductance is:

$$g_\varepsilon := 1 - |r_\varepsilon|^2 = 1 - \tanh^2 x_\varepsilon = \frac{1}{\cosh^2 x_\varepsilon}.$$

Add a slice of disordered region with the thickness δL, $\mathfrak{a} \ll \delta L \ll \ell$:



$r_{\varepsilon,L} = \tanh|x_{\varepsilon,L}| e^{\mathrm{i}\phi_{\varepsilon,L}}$ obeys the continuous Langevin process

$$\frac{dx_{\varepsilon,L}}{dL} = +v_1 \sin\phi_{\varepsilon,L} - v_2 \cos\phi_{\varepsilon,L} + \mathcal{O}\left(v_\mu^2\right),$$

$$\frac{d\phi_{\varepsilon,L}}{dL} = 2\left(\varepsilon + v_0\right) + \frac{2}{\tanh 2x_{\varepsilon,L}}\left(v_1 \cos\phi_{\varepsilon,L} + v_2 \sin\phi_{\varepsilon,L}\right) + \mathcal{O}\left(v_\mu^2\right).$$

This follows from the composition law (Ryu, Mudry, and Furusaki 2004)

$$r_{\varepsilon,L+\delta L} = r_{\varepsilon,\delta L} + t'_{\varepsilon,\delta L}\left(1 - r_{\varepsilon,L} r'_{\varepsilon,\delta L}\right)^{-1} r_{\varepsilon,L} t_{\varepsilon,\delta L},$$

when the width δL of the slice is much larger than the lattice spacing $\mathfrak{a} \sim k_{\mathrm{F}}^{-1}$ but much smaller than the mean free path ℓ, here defined by

$$1 \gg \left\langle r_{\varepsilon,\delta L} r^{*}_{\varepsilon,\delta L}\right\rangle =: \frac{\delta L}{\ell}, \qquad \ell^{-1} = \ell_1^{-1} + \ell_2^{-1}.$$

One-parameter scaling wrt $\mathsf{l} := \frac{L}{\ell}$ is, here, not the rule. One-parameter scaling only holds under the non-generic assumptions:

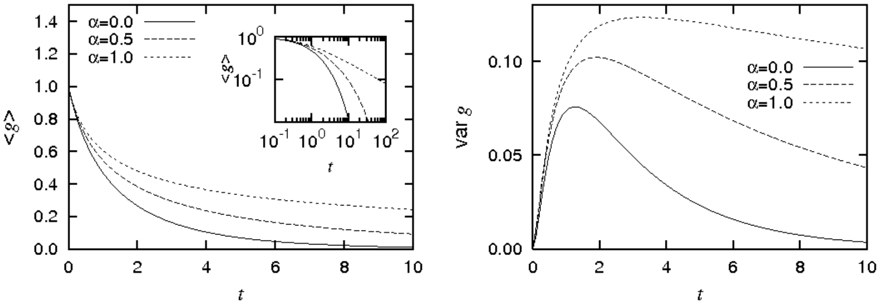
- **AI** (orthogonal symmetry class): ϕ_ε is initially uniformly distributed in $[0, 2\pi[$ and independent of x_ε in which case the probability distribution for the dimensionless conductance is (Abrikosov 1981)

$$\mathcal{G}(g_\varepsilon; \mathsf{l}) = \sqrt{\frac{4}{\pi \mathsf{l}^3 g_\varepsilon^3}} \int_{y_{g_\varepsilon}}^{+\infty} dy \, \frac{y \, e^{-(\mathsf{l}/4)-(y^2/\mathsf{l})}}{\left(g_\varepsilon \cosh^2 y - 1\right)^{1/2}},$$

$$y_{g_\varepsilon} := -\frac{1}{2} \ln \left[2 g_\varepsilon^{-1} \left(1 - \sqrt{1 - g_\varepsilon}\right) - 1\right].$$

- **BDI** (chiral-orthogonal symmetry class): $\varepsilon = \ell_0^{-1} = \ell_1^{-1} = 0$ while $\phi_{\varepsilon=0}$ is initially equally likely to be 0 or π and independent of $x_{\varepsilon=0}$ in which case the probability distribution for the dimensionless conductance is (Stone+Joannopoulos 1981)

$$\mathcal{G}(g_{\varepsilon=0}; \mathsf{l}) = \frac{1}{\sqrt{2\pi \mathsf{l}(1 - g_{\varepsilon=0})}\, g_{\varepsilon=0}} e^{-\frac{\left(\mathrm{arccosh}\, 1/\sqrt{g_{\varepsilon=0}}\right)^2}{2\mathsf{l}}}.$$

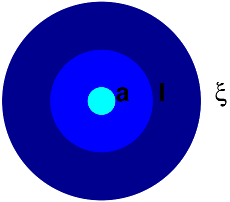


Approximate crossover of the mean and variance of the conductance at the band center $\varepsilon = 0$ from the AI symmetry class $\alpha = 0$ to the BDI symmetry class $\alpha = 1$ as a function of $t = L/\ell$ after Ryu, Mudry, and Furusaki 2004.

Diffusive regime and universality classes

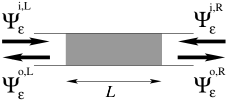
When one-parameter scaling in a symmetry class (of Anderson localization) becomes generic at sufficiently long length scales, then one gets a universality class (of Anderson localization).

A sufficient condition for a universality class (of Anderson Localization) in a quantum wire is the emergence of the diffusive regime in the thick quantum wire limit:



$$\underbrace{\mathfrak{a} \sim k_{\mathrm{F}}^{-1} \ll \ell}_{\text{weak disorder}}, \qquad \underbrace{\ell \ll L \ll \xi}_{\text{diffusive regime}}$$

The thick quantum wire limit



$$\Longleftrightarrow \begin{pmatrix} \Psi^{\mathrm{o,L}} \\ \Psi^{\mathrm{o,R}} \end{pmatrix}_{\varepsilon} = \mathcal{S}_{\varepsilon} \begin{pmatrix} \Psi^{\mathrm{i,L}} \\ \Psi^{\mathrm{i,R}} \end{pmatrix}_{\varepsilon} \equiv \begin{pmatrix} r & t' \\ t & r' \end{pmatrix}_{\varepsilon} \begin{pmatrix} \Psi^{\mathrm{i,L}} \\ \Psi^{\mathrm{i,R}} \end{pmatrix}_{\varepsilon},$$

In a quantum wire, the diffusive regime is defined by

$$\ell \ll L \ll N\ell$$

with N the number of transverse channels. For thin wires, N of order 1, there is no diffusive regime.

The thick quantum wire limit is the scaling limit

$$L \to \infty, \qquad N \to \infty, \qquad \frac{L}{N\ell} \text{ fixed.}$$

The limit $N \to \infty$ arises as a semiclassical limit, when λ_{F} is much smaller than the diameter of the wire, or by increasing the diameter of the wire. In the latter case, the diameter should not exceed the transverse localization length.

Example 1: Class D superconducting quantum wire

Define the static random quasi-one-dimensional Hamiltonian

$$H := K + V, \quad K := \sigma_0 \otimes \gamma_0 \otimes \tau_3 \otimes I_N \mathrm{i}\partial_x, \quad V := \begin{pmatrix} v & \Delta \\ -\Delta^* & -v^{\mathrm{T}} \end{pmatrix}$$

with the static disorder of vanishing means and covariances

$$\langle v_{ij}(x) v^*_{kl}(x') \rangle = \frac{1}{8N\ell_v} \delta_{ik}\delta_{jl}\delta(x-x'),$$
$$\langle \Delta_{ij}(x) \Delta^*_{kl}(x') \rangle = \frac{1}{8N\ell_\Delta} \left(\delta_{ik}\delta_{jl} - \delta_{il}\delta_{jk}\right) \delta(x-x').$$

Claim: The difference $\delta\ell \equiv |\ell_\Delta - \ell_v|$ is irrelevant in the thick quantum wire limit

$$L \to \infty, \qquad N \to \infty, \qquad \frac{L}{N\ell} \text{ fixed with } \ell^{-1} := \ell_v^{-1} + \ell_\Delta^{-1}.$$

Justification: Let (Brouwer, Mudry, Furusaki 2003)

$$\ell^{-1} := \ell_v^{-1} + \ell_\Delta^{-1}, \qquad \xi_\Delta := \sqrt{\frac{\ell \ell_\Delta}{8}}.$$

In the symmetry class A defined by $0 = \ell_\Delta^{-1}$,

$$\langle g \rangle = \frac{4N\ell}{L} + 0 + \mathcal{O}\left(\frac{L}{N\ell}\right), \quad \langle g \rangle \propto e^{-L/(8N\ell)}, \quad \langle \ln g \rangle \propto -\frac{L}{2N\ell} + \mathcal{O}(1).$$

In the symmetry class D defined by $\ell_v^{-1} = \ell_\Delta^{-1}$,

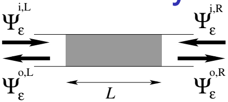
$$\langle g \rangle = \frac{4N\ell}{L} + \frac{1}{3} + \mathcal{O}\left(\frac{L}{N\ell}\right), \quad \langle g \rangle = \sqrt{\frac{8N\ell}{\pi L}}, \quad \langle \ln g \rangle = -\sqrt{\frac{2L}{\pi N\ell}}.$$

In the diffusive regime, the crossover is

$$\langle g \rangle = \frac{4N\ell}{L} + \frac{1}{3} + \frac{\xi_\Delta^2}{L^2} - \frac{\xi_\Delta}{L} \coth \frac{L}{\xi_\Delta} + \mathcal{O}\left(\frac{L}{N\ell}\right).$$

In the thick quantum wire limit, $\xi_\Delta \ll L \ll N\ell$ is always permissible so that the crossover to the diffusive regime of class D follows.

Universality classes in the thick quantum wire limit



$$\Longleftrightarrow \begin{pmatrix} \Psi^{o,L} \\ \Psi^{i,L} \end{pmatrix}_{\varepsilon} = \mathcal{M}(\varepsilon, L) \begin{pmatrix} \Psi^{i,R} \\ \Psi^{o,R} \end{pmatrix}_{\varepsilon}.$$

The eigenvalues of $\mathcal{M}(\varepsilon, L)\mathcal{M}^{\dagger}(\varepsilon, L)$ come in D-degenerate pairs $e^{\pm 2x_j(\varepsilon, L)}$. The dimensionless (Landauer) conductance is

$$g(\varepsilon, L) = \sum_{j=1}^{N^*} \frac{1}{\cosh^2 x_j(\varepsilon, L)}, \qquad N^* = \frac{2N}{D}.$$

The transformation law

$$\mathcal{M}(\varepsilon, L + \delta L) = \mathcal{M}(\varepsilon, \delta L)\mathcal{M}(\varepsilon, L)$$

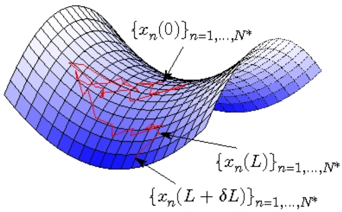
implies that the Lyapunov exponents $x_j(\varepsilon, L)$ undergo the following Brownian motion on a symmetric space (unique in the thick quantum wire limit):

$$\frac{\partial P}{\partial L} = \frac{1}{2\gamma\ell} \sum_{j=1}^{N^*} \frac{\partial}{\partial x_j} J \frac{\partial}{\partial x_j} J^{-1} P,$$

[known as the DMPK (82-88) equation in the physics literature]

$$J = \prod_{j} \sinh^{m_l}(2x_j) \prod_{k<j} \prod_{\pm} \sinh^{m_{o\pm}}(x_j \pm x_k),$$

$$\gamma = (m_{o+} + m_{o-})(N^* - 1)/2 + 1 + m_l.$$



The table lists the multiplicities of the ordinary and long roots $m_{o\pm}$ and m_l of the symmetric spaces associated with the transfer matrix. Except for the three chiral classes, one has $m_{o+} = m_{o-} = m_o$. For the chiral classes, one has $m_{o+} = 0$, $m_{o-} = m_o$. The table also lists the degeneracy D of the transfer matrix eigenvalues, as well as the symbols for the symmetric spaces associated to the transfer matrix $\mathcal{M}$ and the Hamiltonian $\mathcal{H}$. The last three columns list theoretical results for the weak-localization correction δg for $\ell \ll L \ll N\ell$, the average of $\ln g$ at $L \gg N\ell$. The results for $\langle \ln g \rangle$ in the chiral classes refer to the case of N even. For odd N, $\langle \ln g \rangle$ are the same as in class D.

Disorder	Class	TRS	SRS	m_o	m_l	D	$\mathcal{M}$	$\mathcal{H}$	δg	$\langle -\ln g \rangle$
	O	Y	Y	1	1	2	CI	AI	$-2/3$	$2L/(\gamma\ell)$
generic	S	Y	N	4	1	2	DIII	AII	$+1/3$	$2L/(\gamma\ell)$
	U	N	Y(N)	2	1	2(1)	AIII	A	0	$2L/(\gamma\ell)$
	chO	Y	Y	1	0	2	AI	BDI	0	$2m_oL/(\gamma\ell)$
sublattice	chS	Y	N	4	0	2	AII	CII	0	$2m_oL/(\gamma\ell)$
	chU	N	Y(N)	2	0	2(1)	A	AIII	0	$2m_oL/(\gamma\ell)$
	CI	Y	Y	2	2	4	C	CI	$-4/3$	$2m_lL/(\gamma\ell)$
particle-hole	C	N	Y	4	3	4	CII	C	$-2/3$	$2m_lL/(\gamma\ell)$
	DIII	Y	N	2	0	2	D	DIII	$+2/3$	$4\sqrt{L/(2\pi\gamma\ell)}$
	D	N	N	1	0	1	BDI	D	$+1/3$	$4\sqrt{L/(2\pi\gamma\ell)}$

Why NLσM?

Fluctuations play a key role in sufficiently low-dimensional systems, whether classical or quantum, as they can preempt spontaneous symmetry breaking.

When the symmetry is both global and continuous, the tool of choice to address the role of fluctuations in low-dimensional systems is the non-linear sigma model (NLσM).

However, the usefulness of NLσMs has come to transcend situations in which a pattern of symmetry breaking is immediately obvious; as is the case in the context of Anderson localization to access the transition from a metallic to an insulating phase induced by weak disorder or to compute probability distributions of spectral, wavefunction, and transport characteristics in chaotic metallic grains.

Let x^μ ($\mu = 1, \cdots, d$) be the coordinate of a point in d-dimensional Euclidean space and let a point on a connected Riemannian manifold $\mathfrak{M}$ of finite dimension $\mathfrak{n}$ have the coordinates $\phi^i(x)$ ($i = 1, \cdots, \mathfrak{n}$). The action of the NL$\sigma$M is

$$S := \frac{1}{4\pi t} \int \frac{d^d x}{\mathfrak{a}^{d-2}} \left(\partial_\mu \phi^i\right)(x)\, G_{ij}\left[\phi(x)\right]\, \left(\partial_\mu \phi^j\right)(x)$$

where t is the dimensionless coupling constant, $\mathfrak{a}$ is the short-distance cutoff, and $G_{ij}\left[\phi\right]$ is a component of the metric tensor on $\mathfrak{M}$,
Example 2: The O(3)/O(2) NLσM has the action

$$\begin{aligned} S =& \frac{1}{4\pi t} \int \frac{d^d x}{\mathfrak{a}^{d-2}} (\partial_\mu \boldsymbol{n})^2 \qquad \text{with } 1 = \boldsymbol{n}^2 \equiv \sigma^2 + \phi_1^2 + \phi_2^2 \\ =& \frac{1}{4\pi t} \int \frac{d^d x}{\mathfrak{a}^{d-2}} (\partial_\mu \phi^i) \left(\frac{\phi_i \phi_j}{1 - \phi_1^2 - \phi_2^2} + \delta_{ij} \right) (\partial_\mu \phi^j). \end{aligned}$$

It encodes the fate of the classical ferromagnetic phase under thermal fluctuations (t is the bare dimensionless temperature) (interacting spin waves $\phi \equiv (\phi_1, \phi_2)$).

Example 3: Anderson localization with the help of fermionic replicas

RMT	Class	TRS	SRS	ChS	BdG	$\mathfrak{M}$ with M, N $\to$ 0	Topology of $\mathfrak{M}$ could matter
O	AI	✓	✓	×	×	Sp(M + N)/Sp(M)×Sp(N)	×
S	AII	✓	×	×	×	O(M + N)/O(M)×O(N)	✓ $\pi = \theta$ term(Ryu et al)
U	A	×	-	×	×	U(M + N)/U(M)×U(N)	✓ u(1)∋ θ term (Pruisken)
chO	BDI	✓	✓	✓	×	U(2N)/Sp(N)	×
chS	CII	✓	×	✓	×	U(N)/O(N)	✓ $\pi = \theta$ term (Ryu et al.)
chU	AIII	×	-	✓	×	U(N)×U(N)/U(N)	✓ WZW term (Guruswamy et al.)
-	CI	✓	✓	×	✓	Sp(N)×Sp(N)/Sp(N)	✓ WZW term (Nersesyan et al.)
-	C	×	✓	×	✓	Sp(N)/U(N)	✓ u(1)∋ θ term (Senthil et al.)
-	DIII	✓	×	×	✓	O(N)×O(N)/O(N)	✓ WZW term (Fendley)
-	D	×	×	×	✓	O(2N)/U(N)	✓ u(1)∋ θ term (Bocquet et al.)

Wegner (1979); Efetov, Larkin, and Khemlnitskii (1980); Gade and Wegner (1993); Senthil, Fisher, Balents, and Nayak (1998)

Sketch:

1. For any fixed static random potential, represent the product of M advanced and N retarded single-particle Green functions as a path integral over a Boltzman weight.
2. Perform the disorder average. For Gaussian white-noise correlated static disorder, this induces a quartic interaction.
3. Introduce a matrix-valued Hubbard-Stratonovich field to decouple the disorder-induced quartic interaction.
4. In the diffusive regime, do the saddle-point approximation (spontaneous symmetry breaking for finite M and N).
5. Do a gradient expansion of the fermion (boson) determinant (or superdeterminant).

The inverse $1/t$ of the coupling constant t in the NLσM represents the bare value of the (mean) conductance in the diffusive regime.

The NLσM encodes the fate of the diffusive metallic fixed point in the presence of disorder-induced fluctuations.

The signature of Anderson localization transition is to be found in the flow of t under the rescaling

$$\mathfrak{a} \to (1 + dl)\mathfrak{a}, \qquad dl > 0,$$

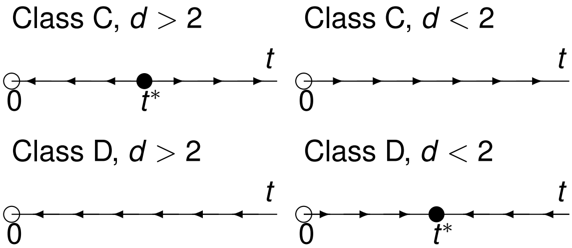
i.e., one absorbs all the changes induced to the partition function

$$Z = \int \mathcal{D}[\phi] e^{-\frac{1}{4\pi t} \int \frac{d^d x}{\mathfrak{a}^{d-2}} \left(\partial_\mu \phi^i\right)(x) \, G_{ij}\left[\phi(x)\right] \left(\partial_\mu \phi^j\right)(x)}$$

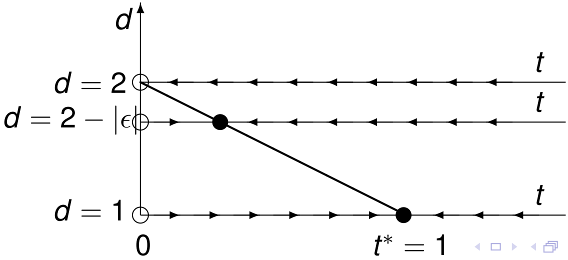
by $\mathfrak{a} \to (1 + dl)\mathfrak{a}$ into the one-parameter infrared flow

$$\frac{dt}{dl} = \beta(t) \approx \beta_1 t + \beta_2 t^2 + \cdots .$$

Example 4:The superconducting classes with TRS-breaking C and D



For class D, this is consistent with the DMPK equation in the thick quantum wire limit:



The DMPK equation yields informations about the conductance distribution. What does the $2-|\epsilon|$ expansion of the class D NLσM tells us about conductance fluctuations?

According to Altshuler, Kravtsov, and Lerner (mid 1980's), the scaling of high-gradient operators controls the scaling of the cumulants of the conductance distribution.

A high-gradient operator $\mathcal{O}_s(x)$ of order $2s$ ($s>1$) is any local product of the fields that contains $2s$ gradients and transforms like a scalar. Its scaling dimension x_s at the critical point

$$\langle \mathcal{O}_s(x)\mathcal{O}_s^\dagger(x')\rangle \sim \left(\frac{\mathfrak{a}}{|x-x'|}\right)^{2x_s}, \qquad x_s = x_s^{(0)} + \gamma_s,$$

is made of the engineering dimension $x_s^{(0)} = 2s$ (which is larger than 2 for $s>1$) and of the anomalous dimension

$$\gamma_s = -(s^2-s)|\epsilon| + \mathcal{O}(\epsilon^2) \text{ for class D.}$$

Ryu, Furusaki, Ludwig, and Mudry (2007)

If we apply the analysis of the conductance fluctuations of Altshuler, Kravtsov, and Lerner (mid 1980's) to the stable fixed point in symmetry class D in $d = 2 - |\epsilon| < 2$ spatial dimensions, there follows the conductance cumulants

$$\langle\langle g^s \rangle\rangle \sim \begin{cases} (\bar{g})^{2-s}, & \text{if } s < s_0, \\ A_s \, (L/\ell)^{d-x^{(s)}}, & \text{if } s > s_0, \end{cases}$$

$$s_0 \approx \frac{2}{|\epsilon|}[1 + \mathcal{O}(\epsilon)],$$

$$d - x^{(s)} = |\epsilon| s^2 - (2 + |\epsilon|)s + (2 - |\epsilon|) + \mathcal{O}(\epsilon^2).$$

On the other hand, in the thick quantum wire limit of class D [Brouwer, Furusaki, Gruzberg, Mudry, (2000)]

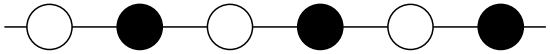
$$\langle\langle g^s \rangle\rangle \propto (L/\ell)^{-1/2}, \qquad s = 1, 2, \ldots,$$

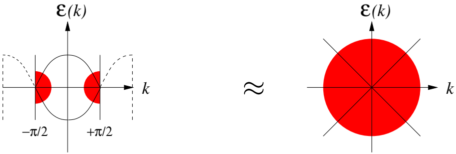
$$\langle\langle \ln g \rangle\rangle \propto -(L/\ell)^{1/2}, \qquad \operatorname{var} \ln g \sim L/\ell.$$

Discussion: Class DIII behaves similarly to class D. Lamacraft, Simons, and Zirnbauer (2004) using the one-dimensional NLσM (which is a principal chiral model for symmetry class DIII) and instanton methods reproduced the DMPK results from Brouwer, Furusaki, Gruzberg and Mudry (2000). So it is is not the NLσM which is called into question but the $2-|\epsilon|$ expansion.
Assume that the $2-|\epsilon|$ expansion is smoothly connected to the thick quantum wire limit. Then

- either the one-loop relevance of high-gradient operators is an artifact of the ϵ-expansion. These operators are in fact *irrelevant* once all orders in the ϵ-expansion are taken into account [Ludwig (1990); Brezin and Hikami (1997)],
- or high-gradient operators are truly relevant, however they are not independent, for they are non-linearly coupled through their full one-loop RG flow. Functional renormalization technique are required to study their flow [Mudry , Ryu, Furusaki (2003)].

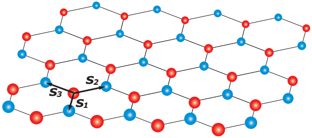
Dirac fermions in one-dimension is the rule:



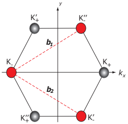


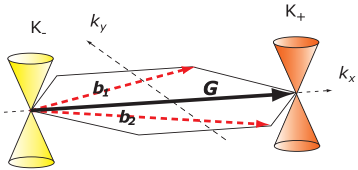
Although Fermi surfaces are the rule in two and more dimensions, there are counterexamples. Example 5: Graphene

Direct space



Reciprocal space





For spinless fermions at the Dirac point, i.e., half-filling,

$$\nu(\varepsilon) \propto |\varepsilon| + \mathcal{O}(\varepsilon^2), \qquad g = 2 \times \pi^{-1}.$$

We begin with the assumption that the static disorder enters only through real-valued and uncorrelated nearest-neighbor hoppings: the symmetry class is BDI. For weak disorder and at long wave length, the product of N single-particle Green functions averaged over disorder follows from

$$Z := \int \mathcal{D}[\psi^\dagger, \psi, \bar{\psi}^\dagger, \bar{\psi}] \exp(-S),$$

$$S := \sum_{\iota=1}^{k} \int \frac{d\bar{z}dz}{2\pi i} \left(\psi^{A\dagger}_{\iota} \bar{\partial} \psi_{A\iota} + \bar{\psi}^{A\dagger}_{\iota} \partial \bar{\psi}_{A\iota} \right) + \int \frac{d\bar{z}dz}{2\pi i} \left(\frac{g_{\mathrm{A}}}{2\pi} \mathcal{O}_{\mathrm{A}} + \frac{g_{\mathrm{M}}}{2\pi} \mathcal{O}_{\mathrm{M}} \right),$$

$$\mathcal{O}_{\mathrm{A}} := -J^{A}_{A} (-1)^{A} \bar{J}^{B}_{B} (-1)^{B} \equiv -\mathrm{str}\, J\, \mathrm{str}\, \bar{J}, \qquad J^{B}_{A} := \sum_{\iota=1}^{k} : \psi_{A\iota} \psi^{B\dagger}_{\iota} :,$$

$$\mathcal{O}_{\mathrm{M}} := -J^{B}_{A} \bar{J}^{A}_{B} (-1)^{A} \equiv -\mathrm{str}\, \left(J \bar{J} \right), \qquad \bar{J}^{B}_{A} := \sum_{\iota=1}^{k} : \bar{\psi}_{A\iota} \bar{\psi}^{B\dagger}_{\iota}$$

where $A, B = 1, \cdots, 2N$ with $k = 1$, in which case $g_{\mathrm{A}} \geq 0$ and $g_{\mathrm{M}} \geq 0$ can be thought of as the covariances of the disorder in class BDI. We can however treat the case of generic integer k and generic $g_{\mathrm{M}} \in \mathbb{R}$. Finally, $\mathrm{grade}(A)$ is 0 (bosons) for $A = 1, \cdots, N$ and 1 (fermions) for $A = N + 1, \cdots, 2N$.

The theory, a two-dimensional $\widehat{\mathrm{gl}}(M|M)_k$ Thirring model, has whenever $g_{\rm A} = g_{\rm M} = 0$ the global GL($M|M$) graded symmetry with currents that realize the $\widehat{\mathrm{gl}}(M|M)_k$ current algebra.

Can high-gradient operators become relevant in the family of two-dimensional $\widehat{\mathrm{gl}}(M|M)_k$ Thirring models with M and k positive integers due to the current-current interactions?

The strategy that we followed has three steps.

1. The first step consists of identifying all the independent "classical" high-gradient operators of order s.
2. The second step consists of normal-ordering all independent "classical" high-gradient operators of order s. This step depends crucially on the level k of the non-Abelian Thirring model. The inverse level $1/k$ plays the role of a "quantum parameter" that vanishes in the limit $k \to \infty$. The level $k = 1$ is thus the most "quantum".
3. The computation of the linearized RG flows for the high-gradient operators is the final step.

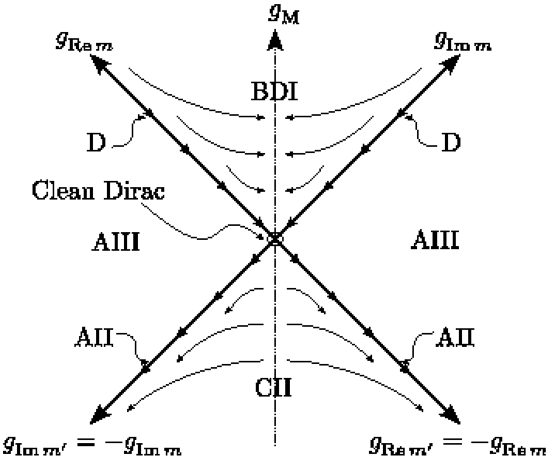
We could not solve the first step in its full generality. We were nevertheless able to construct two sets of high-gradient operators in the extreme "classical" limit $\widehat{\mathrm{gl}}(M|M)_k$ with $M, k \to \infty$ and the extreme "quantum" limit $\widehat{\mathrm{gl}}(M|M)_k$ with M a positive integer and $k = 1$, respectively, and carry out the second and third steps consistently.

In the extreme “classical” case, anomalous one-loop scaling dimensions for high-gradient operators of order *s* are distributed in a symmetric fashion about zero with the minimum and the maximum both depending quadratically on the order *s*, very much like for the family of NLσMs on the target spaces $\mathrm{U}(M+N)/\mathrm{U}(M)\times\mathrm{U}(N)$ with M and N positive integers. Hence, high-gradient operators must become (one-loop) relevant for both signs of the current-current interaction with increasing order *s* very much in the same way as their cousins do in both the compact family $\mathrm{U}(M+N)/\mathrm{U}(M)\times\mathrm{U}(N)$ and the non-compact family $\mathrm{U}(M,N)/\mathrm{U}(M)\times\mathrm{U}(N)$ with $M,N>1$.

In the extreme quantum case $k=1$, the spectrum of anomalous one-loop scaling dimensions of order *s* is always one-sided, i.e., positive for one sign of the current-current interaction. For $\widehat{\mathrm{gl}}(2N|2N)_{k=1}$ with N a positive integer the sign of the current-current interaction for which high-gradient operators are always irrelevant corresponds to the interpretation of the $\widehat{\mathrm{gl}}(2N|2N)_{k=1}$ Thirring model as a problem of Anderson localization in the symmetry class BDI. We have thus shown that the high-gradient operators in these random tight-binding models are irrelevant at one-loop order:

Phase diagram projected onto the critical sector

$$\mathrm{PSL}(2N|2N) \sim \mathrm{GL}(2N|2N)/\mathrm{U}(1) \times \mathrm{U}(1)$$



- Whether the apparent breakdown of one-parameter scaling due to the relevance of high-gradient operators at one loop is an artifact of the $2+\epsilon$ expansion or has a deeper meaning remains an outstanding problem for the description of Anderson localization using the NLσM approach.

- We have shown that graphene with real-valued nearest-neighbor random hopping only is, at the band center, an example of a critical theory for Anderson localization in two-dimensions with no relevant one-loop high-gradient operators.