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Macroscopic and microscopic aspects of quantum black holes

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Quantum Black Holes

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Abstract: In recent years there has been enormous progress in understanding the entropy and other thermodynamic properties of black holes within string theory going well beyond the thermodynamic limit. It has become possible to begin exploring finite size effects in perturbation theory in inverse size and even nonperturbatively, with highly nontrivial agreements between thermodynamics and statistical mechanics. These lectures will review some of these developments emphasizing both the semiclassical and quantum aspects with the topics listed in the outline.

Keywords: black holes, superstrings.

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A good introductory textbook on general relativity from a modern perspective see [1]. For a more detailed treatment [2] which has become a standard reference among relativists, and [3] remains a classic for various aspects of general relativity. For quantum field theory in curved spacetime see [4]. For relevant aspects of string theory see [5, 6, 7, 8]. For additional details of some of the material covered here relating to $\mathcal{N} = 4$ dyons see [9].

1. Introduction

One of the important successes of string theory is that one can obtain a statistical understanding of the thermodynamic entropy[10, 11] of certain supersymmetric black holes in terms microscopic counting[12]. The entropy of black holes supplies us with very useful quantitative information about the fundamental degrees of freedom of quantum gravity.

Much of the earlier work was in the thermodynamic limit of large charges. In recent years there has been enormous progress in understanding the entropy and other thermodynamic properties of black holes within string theory going well beyond the thermodynamic limit. It has now become possible to begin exploring finite size effects in perturbation theory in inverse size and even nonperturbatively, with highly nontrivial agreements between thermodynamics and statistical mechanics. These lectures will describe some of this progress in our understanding of the quantum structure of black holes.

2. Black Holes

A black hole is at once the most simple and the most complex object.

It is the most simple in that it is completely specified by its mass, spin, and charge. This remarkable fact is a consequence of a the so called 'No Hair Theorem'. For an astrophysical object like the earth, the gravitational field around it depends not only on its mass but also on how the mass is distributed and on the details of the oblate-ness of the earth and on the shapes of the valleys and mountains. Not so for a black hole. Once a star collapses to form a black hole, the gravitational field around it forgets all details about the star that disappears behind the even horizon except for its mass, spin, and charge. In this respect, a black hole is very much like a structure-less elementary particle such as an electron.

And yet it is the most complex in that it possesses a huge entropy. In fact the entropy of a solar mass black hole is enormously bigger than the thermal entropy of the star that might have collapsed to form it. Entropy gives an account of the number of microscopic states of a system. Hence, the entropy of a black hole signifies an incredibly complex microstructure. In this respect, a black hole is very unlike an elementary particle.

Understanding the simplicity of a black hole falls in the realm of classical gravity. By the early seventies, full fifty years after Schwarzschild, a reasonably complete understanding of gravitational collapse and of the properties of an event horizon was achieved within classical general relativity. The final formulation began with the singularity theorems of Penrose, area theorems of Hawking and culminated in the laws of black hole mechanics.

Understanding the complex microstructure of a black hole implied by its entropy falls in the realm of quantum gravity and is the topic of present lectures. Recent developments have made it clear that a black hole is 'simple' not because it is like an elementary particle, but rather because it is like a statistical ensemble. An ensemble is also specified by a few a conserved quantum numbers such as energy, spin, and charge. The simplicity of a black hole is no different than the simplicity that characterizes a thermal ensemble.

To understand the relevant parameters and the geometry of black holes, let us first consider the Einstein-Maxwell theory described by the action

$$\frac{1}{16\pi G}\int R\sqrt{g}d^4x - \frac{1}{16\pi}\int F^2\sqrt{g}d^4x, \tag{2.1}$$

where G is Newton's constant, $F_{\mu\nu}$ is the electro-magnetic field strength, R is the Ricci scalar of the metric $g_{\mu\nu}$. In our conventions, the indices μ, ν take values $0, 1, 2, 3$ and the metric has signature $(-, +, +, +)$.

2.1 Schwarzschild Metric

Consider the Schwarzschild metric which is a spherically symmetric, static solution of the vacuum Einstein equations $R_{\mu\nu} - \frac{1}{2}g_{\mu\nu} = 0$ that follow from (2.1) when no electromagnetic fields are excited. This metric is expected to describe the spacetime outside a gravitationally collapsed non-spinning star with zero charge. The solution for the line element is given by

$$ds^2 \equiv g_{\mu\nu}dx^\mu dx^\nu = -(1-\frac{2GM}{r})dt^2 + (1-\frac{2GM}{r})^{-1}dr^2 + r^2 d\Omega^2,$$

where t is the time, r is the radial coordinate, and Ω is the solid angle on a 2-sphere. This metric appears to be singular at $r = 2GM$ because some of its components vanish or diverge, $g_{00} \to \infty$ and $g_{rr} \to \infty$. As is well known, this is not a real singularity. This is because the gravitational tidal forces are finite or in other words, components of Riemann tensor are finite in orthonormal coordinates. To better understand the nature of this apparent singularity, let us examine the geometry more closely near $r = 2GM$. The surface $r = 2GM$ is called the 'event horizon' of the Schwarzschild solution. Much of the interesting physics having to do with the quantum properties of black holes comes from the region near the event horizon.

To focus on the near horizon geometry in the region $(r - 2GM) \ll 2GM$, let us define $(r - 2GM) = \xi$, so that when $r \to 2GM$ we have $\xi \to 0$. The metric then takes the form

$$ds^2 = -\frac{\xi}{2GM}dt^2 + \frac{2GM}{\xi}(d\xi)^2 + (2GM)^2 d\Omega^2, \tag{2.2}$$

up to corrections that are of order $(\frac{1}{2GM})$. Introducing a new coordinate ρ,

$$\rho^2 = (8GM)\xi \quad \text{so that} \quad d\xi^2 \frac{2GM}{\xi} = d\rho^2,$$

the metric takes the form

$$ds^2 = -\frac{\rho^2}{16G^2M^2}dt^2 + d\rho^2 + (2GM)^2 d\Omega^2. \tag{2.3}$$

From the form of the metric it is clear that ρ measures the geodesic radial distance. Note that the geometry factorizes. One factor is a 2-sphere of radius $2GM$ and the other is the (ρ, t) space

$$ds_2^2 = -\frac{\rho^2}{16G^2M^2}dt^2 + d\rho^2. \tag{2.4}$$

We now show that this $1+1$ dimensional spacetime is just a flat Minkowski space written in funny coordinates called the Rindler coordinates.

2.2 Historical Aside

Apart from its physical significance, the entropy of a black hole makes for a fascinating study in the history of science. It is one of the very rare examples where a scientific idea has gestated and evolved over several decades into an important conceptual and quantitative tool almost entirely on the strength of theoretical considerations. That we can proceed so far with any confidence at all with very little guidance from experiment is indicative of the robustness of the basic tenets of physics. It is therefore worthwhile to place black holes and their entropy in a broader context before coming to the more recent results pertaining to the quantum aspects of black holes within string theory.

A black hole is now so much a part of our vocabulary that it can be difficult to appreciate the initial intellectual opposition to the idea of 'gravitational collapse' of a star and of a 'black hole' of nothingness in spacetime by several leading physicists, including Einstein himself.

To quote the relativist Werner Israel ,

"*There is a curious parallel between the histories of black holes and continental drift. Evidence for both was already non-ignorable by 1916, but both ideas were stopped in their tracks for half a century by a resistance bordering on the irrational.*"

On January 16, 1916, barely two months after Einstein had published the final form of his field equations for gravitation [13], he presented a paper to the Prussian Academy on behalf of Karl Schwarzschild [14], who was then fighting a war on the Russian front. Schwarzschild had found a spherically symmetric, static and exact solution of the full nonlinear equations of Einstein without any matter present.

The Schwarzschild solution was immediately accepted as the correct description within general relativity of the gravitational field outside a spherical mass. It would be the correct approximate description of the field around a star such as our sun. But something much more bizzare was implied by the solution. For an object of mass M, the solution appeared to become singular at a radius $R = 2GM/c^2$. For our sun, for example, this radius, now known as the Schwarzschild radius, would be about three kilometers. Now, as long the physical radius of the sun is bigger than three kilometers, the 'Schwarzschild's singularity' is of no concern because inside the sun the Schwarzschild solution is not applicable as there is matter present. But what if the entire mass of the sun was concentrated in a sphere of radius smaller than three kilometers? One would then have to face up to this singularity.

Einstein's reaction to the 'Schwarzschild singularity' was to seek arguments that would make such a singularity inadmissible. Clearly, he believed, a physical theory could not tolerate such singularities. This drove his to write as late as 1939, in a published paper,

"*The essential result of this investigation is a clear understanding as to why the 'Schwarzschild singularities' do not exist in physical reality.*"

This conclusion was however based on an incorrect argument. Einstein was not alone in this rejection of the unpalatable idea of a total gravitational collapse of a physical system. In the same year, in an astronomy conference in Paris, Eddington, one of the leading astronomers of the time, rubbished the work of Chandrasekhar who had concluded from his study of white dwarfs, a work that was to earn him the Nobel prize later, that a large enough star could collapse.

It is interesting that Einstein's paper on the inadmissibility of the Schwarzschild singularity appeared only two months before Oppenheimer and Snyder published their definitive work on stellar collapse with an abstract that read,

"*When all thermonuclear sources of energy are exhausted, a sufficiently heavy star will collapse.*"

Once a sufficiently big star ran out of its nuclear fuel, then there was nothing to stop the inexorable inward pull of gravity. The possibility of stellar collapse meant that a star could be compressed in a region smaller than its Schwarzschild radius and the 'Schwarzschild singularity' could no longer be wished away as Einstein had desired. Indeed it was essential to understand what it means to understand the final state of the star.

It is thus useful to keep in mind what seems now like a mere change of coordinates was at one point a matter of raging intellectual debate.

2.3 Rindler Coordinates

To understand Rindler coordinates and their relation to the near horizon geometry of the black hole, let us start with $1+1$ Minkowski space with the usual flat Minkowski metric,

$$ds^2 = -dT^2 + dX^2. \tag{2.5}$$

In light-cone coordinates,

$$U = (T + X) \quad V = (T - X), \tag{2.6}$$

the line element takes the form

$$ds^2 = -dU\ dV. \tag{2.7}$$

Now we make a coordinate change

$$U = \frac{1}{\kappa}e^{\kappa u}, \quad V = -\frac{1}{\kappa}e^{-\kappa v}, \tag{2.8}$$

to introduce the Rindler coordinates (u, v). In these coordinates the line element takes the form

$$ds^2 = -dU\ dV = -e^{\kappa(u-v)}du\ dv. \tag{2.9}$$

Using further coordinate changes

$$u = (t + x), \quad v = (t - x), \quad \rho = \frac{1}{\kappa}e^{\kappa x}, \tag{2.10}$$

we can write the line element as

$$ds^2 = e^{2\kappa x}(-dt^2 + dx^2) = -\rho^2\kappa^2 dt^2 + d\rho^2. \tag{2.11}$$

Comparing (2.4) with this Rindler metric, we see that the (ρ, t) factor of the Schwarzschild solution near $r \sim 2GM$ looks precisely like Rindler spacetime with metric

$$ds^2 = -\rho^2\kappa^2\ dt^2 + d\rho^2 \tag{2.12}$$

with the identification

$$\kappa = \frac{1}{4GM}.$$

This parameter κ is called the surface gravity of the black hole. For the Schwarzschild solution, one can think of it heuristically as the Newtonian acceleration GM/r_H^2 at the horizon radius $r_H = 2GM$. Both these parameters–the surface gravity κ and the horizon radius r_H play an important role in the thermodynamics of black hole.

This analysis demonstrates that the Schwarzschild spacetime near $r = 2GM$ is not singular at all. After all it looks exactly like flat Minkowski space times a sphere of radius $2GM$. So the curvatures are inverse powers of the radius of curvature $2GM$ and hence are small for large $2GM$.

2.4 Kruskal Extension

One important fact to note about the Rindler metric is that the coordinates u, v do not cover all of Minkowski space because even when the vary over the full range

$$-\infty \leq u \leq \infty, \quad -\infty \leq v \leq \infty$$

the Minkowski coordinate vary only over the quadrant

$$0 \leq U \leq \infty, \quad -\infty < V \leq 0. \tag{2.13}$$

If we had written the flat metric in these 'bad', 'Rindler-like' coordinates, we would find a fake singularity at $\rho = 0$ where the metric appears to become singular. But we can discover the 'good', Minkowski-like coordinates U and V and extend them to run from $-\infty$ to ∞ to see the entire spacetime.

Since the Schwarzschild solution in the usual (r, t) Schwarzschild coordinates near $r = 2GM$ looks exactly like Minkowski space in Rindler coordinates, it suggests that we must extend it in properly chosen 'good' coordinates. As we have seen, the 'good' coordinates near $r = 2GM$ are related to the Schwarzschild coordinates in exactly the same way as the Minkowski coordinates are related the Rindler coordinates.

In fact one can choose 'good' coordinates over the entire Schwarzschild spacetime. These 'good' coordinates are called the Kruskal coordinates. To obtain the Kruskal coordinates, first introduce the 'tortoise coordinate'

$$r^* = r + 2GM \log\left(\frac{r - 2GM}{2GM}\right). \tag{2.14}$$

In the (r^*, t) coordinates, the metric is conformally flat, *i.e.*, flat up to rescaling

$$ds^2 = (1 - \frac{2GM}{r})(-dt^2 + dr^{*2}). \tag{2.15}$$

Near the horizon the coordinate r^* is similar to the coordinate x in (2.11) and hence $u = t + r^*$ and $v = t - r^*$ are like the Rindler (u, v) coordinates. This suggests that we define U, V coordinates as in (2.8) with $\kappa = 1/4GM$. In these coordinates the metric takes the form

$$ds^2 = -e^{-(u-v)\kappa} dU\ dV = -\frac{2GM}{r} e^{-r/2GM} dU\ dV \tag{2.16}$$

We now see that the Schwarzschild coordinates cover only a part of spacetime because they cover only a part of the range of the Kruskal coordinates. To see the entire spacetime, we must extend the Kruskal coordinates to run from $-\infty$ to ∞. This extension of the Schwarzschild solution is known as the Kruskal extension.

Note that now the metric is perfectly regular at $r = 2GM$ which is the surface $UV = 0$ and there is no singularity there. There is, however, a real singularity at $r = 0$ which cannot be removed by a coordinate change because physical tidal forces become infinite. Spacetime stops at $r = 0$ and at present we do not know how to describe physics near this region.

2.5 Event Horizon

We have seen that $r = 2GM$ is not a real singularity but a mere coordinate singularity which can be removed by a proper choice of coordinates. Thus, locally there is nothing special about the surface $r = 2GM$. However, globally, in terms of the causal structure of spacetime, it is a special surface and is called the 'event horizon'. An event horizon is a boundary of region in spacetime from behind which no causal signals can reach the observers sitting far away at infinity.

To see the causal structure of the event horizon, note that in the metric (2.11) near the horizon, the constant radius surfaces are determined by

$$\rho^2 = \frac{1}{\kappa^2} e^{2\kappa x} = \frac{1}{\kappa^2} e^{\kappa u} e^{-\kappa v} = -UV = \text{constant} \tag{2.17}$$

These surfaces are thus hyperbolas. The Schwarzschild metric is such that at $r \gg 2GM$ and observer who wants to remain at a fixed radial distance $r =$ constant is almost like an inertial, freely falling observers in flat space. Her trajectory is time-like and is a straight line going upwards on a spacetime diagram. Near $r = 2GM$, on the other hand, the constant r lines are hyperbolas which are the trajectories of observers in uniform acceleration.

To understand the trajectories of observers at radius $r > 2GM$, note that to stay at a fixed radial distance r from a black hole, the observer must boost the rockets to overcome gravity. Far away, the required acceleration is negligible and the observers are almost freely falling. But near $r = 2GM$ the acceleration is substantial and the observers are not freely falling. In fact at $r = 2GM$, these trajectories are light like.

This means that a fiducial observer who wishes to stay at $r = 2GM$ has to move at the speed of light with respect to the freely falling observer. This can be achieved only with infinitely large acceleration. This unphysical acceleration is the origin of the coordinate singularity of the Schwarzschild coordinate system.

In summary, the surface defined by $r =$ contant is timelike for $r > 2GM$, spacelike for $r < 2GM$, and light-like or null at $r = 2GM$.

In Kruskal coordinates, at $r = 2GM$, we have $UV = 0$ which can be satisfied in two ways. Either $V = 0$, which defines the 'future event horizon', or $U = 0$, which defines the 'past event horizon'. The future event horizon is a one-way surface that signals can be sent into but cannot come out of. The region bounded by the event horizon is then a black hole. It is literally a hole in spacetime which is black because no light can come out of it. Heuristically, a black hole is black because even light cannot escape its strong gravitation pull. Our analysis of the metric makes this notion more precise. Once an observer falls inside the black hole she can never come out because to do so she will have to travel faster than the speed of light.

As we have noted already $r = 0$ is a real singularity that is inside the event horizon. Since it is a spacelike surface, once a observer falls insider the event horizon, she is sure to meet the singularity at $r = 0$ sometime in future no matter how much she boosts the rockets.

The summarize, an event horizon is a *stationary, null* surface. For instance, in our example of the Schwarzschild black hole, it is stationary because it is defined as a hypersurface $r = 2GM$ which does not change with time. More precisely, the time-like Killing vector $\frac{\partial}{\partial t}$ leaves it invariant. It is at the same time null because g^{rr} vanishes at $r = 2GM$. This surface that is simultaneously stationary and null, causally separates the inside and the outside of a black hole.

2.6 Black Hole Parameters

From our discussion of the Schwarzschild black hole we are ready to abstract some important general concepts that are useful in describing the physics of more general black holes.

To begin with, a *black hole* is an asymptotically flat spacetime that contains a region which is not in the backward lightcone of future timelike infinity. The boundary of such a region is a stationary null surface call the *event horizon*. The fixed t slice of the event horizon is a two sphere.

There are a number of important parameters of the black hole. We have introduced these in the context of Schwarzschild black holes. For a general black holes their actual values are different but for all black holes, these parameters govern the thermodynamics of black holes.

1. The radius of the event horizon r_H is the radius of the two sphere. For a Schwarzschild black hole, we have $r_H = 2GM$.

2. The area of the event horizon A_H is given by $4\pi r_H^2$. For a Schwarzschild black hole, we have $A_H = 16\pi G^2 M^2$.

3. The surface gravity is the parameter κ that we encountered earlier. As we have seen, for a Schwarzschild black hole, $\kappa = 1/4GM$.

3. Black Hole Entropy

3.1 Laws of Black Hole Mechanics

One of the remarkable properties of black holes is that one can derive a set of laws of black hole mechanics which bear a very close resemblance to the laws of thermodynamics. This is quite surprising because *a priori* there is no reason to expect that the spacetime geometry of black holes has anything to do with thermal physics.

(0) Zeroth Law: In thermal physics, the zeroth law states that the temperature T of body at thermal equilibrium is constant throughout the body. Otherwise heat will flow from hot spots to the cold spots. Correspondingly for stationary black holes one can show that surface gravity κ is constant on the event horizon. This is obvious for spherically symmetric horizons but is true also more generally for non-spherical horizons of spinning black holes.

(1) First Law: Energy is conserved, $dE = TdS + \mu dQ + \Omega dJ$, where E is the energy, Q is the charge with chemical potential μ and J is the spin with chemical potential Ω. Correspondingly for black holes, one has $dM = \frac{\kappa}{8\pi G} dA + \mu dQ + \Omega dJ$. For a Schwarzschild black hole we have $\mu = \Omega = 0$ because there is no charge or spin.

(2) Second Law: In a physical process the total entropy S never decreases, $\Delta S \geq 0$. Correspondingly for black holes one can prove the area theorem that the net area in any process never decreases, $\Delta A \geq 0$. For example, two Schwarzschild black holes with masses M_1 and M_2 can coalesce to form a bigger black hole of mass M. This is consistent with the area theorem since the area is proportional to the square of the mass and $(M_1 + M_2)^2 \geq M_1^2 + M_2^2$. The opposite process where a bigger black hole fragments is however disallowed by this law.

Thus the laws of black hole mechanics, crystallized by Bardeen, Carter, Hawking, and other bears a striking resemblance with the three laws of thermodynamics for a body in thermal equilibrium. We summarize these results below in Table(1).

Table 1: Laws of Black Hole Mechanics

Laws of Thermodynamics	**Laws of Black Hole Mechanics**
Temperature is constant throughout a body at equilibrium. T= constant.	Surface gravity is constant on the event horizon. κ =constant.
Energy is conserved. $dE = TdS + \mu dQ + \Omega dJ$.	Energy is conserved. $dM = \frac{\kappa}{8\pi} dA + \mu dQ + \Omega dJ$.
Entropy never decrease. $\Delta S \geq 0$.	Area never decreases. $\Delta A \geq 0$.

Here A is the area of the horizon, M is the mass of the black hole, and κ is the surface gravity which can be thought of roughly as the acceleration at the horizon[1].

3.2 Hawking temperature

This formal analogy is actually much more than an analogy. Bekenstein and Hawking discovered that there is a deep connection between black hole geometry, thermodynamics and quantum mechanics.

Bekenstein asked a simple-minded but incisive question. If nothing can come out of a black hole, then a black hole will violate the second law of thermodynamics. If we throw a bucket of hot water into a black hole then the net entropy of the world outside would seem to decrease. Do we have to give up the second law of thermodynamics in the presence of black holes?

Note that the energy of the bucket is also lost to the outside world but that does not violate the first law of thermodynamics because the black hole carries mass or equivalently energy. So when the bucket falls in, the mass of the black hole goes up accordingly to conserve energy. This suggests that one can save the second law of thermodynamics if somehow the black hole also has entropy. Following this reasoning and noting the formal analogy between the area of the black hole and entropy discussed in the previous section, Bekenstein proposed that a black hole must have entropy proportional to its area.

This way of saving the second law is however in contradiction with the classical properties of a black hole because if a black hole has energy E and entropy S, then it must also have temperature T given by

$$\frac{1}{T} = \frac{\partial S}{\partial E}.$$

[1] We have stated these laws for black holes without spin and charge but more general form is known.

For example, for a Schwarzschild black hole, the area and the entropy scales as $S \sim M^2$. Therefore, one would expect inverse temperature that scales as M

$$\frac{1}{T} = \frac{\partial S}{\partial M} \sim \frac{\partial M^2}{\partial M} \sim M. \tag{3.1}$$

Now, if the black hole has temperature then like any hot body, it must radiate. For a classical black hole, by its very nature, this is impossible. Hawking showed that after including quantum effects, however, it is possible for a black hole to radiate. In a quantum theory, particle-antiparticle are constantly being created and annihilated even in vacuum. Near the horizon, an antiparticle can fall in once in a while and the particle can escapes to infinity. In fact, Hawking's calculation showed that the spectrum emitted by the black hole is precisely thermal with temperature $T = \frac{\hbar\kappa}{2\pi} = \frac{\hbar}{8\pi GM}$. With this precise relation between the temperature and surface gravity the laws of black hole mechanics discussed in the earlier section become identical to the laws of thermodynamics. Using the formula for the Hawking temperature and the first law of thermodynamics

$$dM = TdS = \frac{\kappa\hbar}{8\pi G\hbar} dA,$$

one can then deduce the precise relation between entropy and the area of the black hole:

$$S = \frac{Ac^3}{4G\hbar}.$$

3.3 Euclidean Derivation of Hawking Temperature

Before discussing the entropy of a black hole, let us derive the Hawking temperature in a somewhat heuristic way using a Euclidean continuation of the near horizon geometry. In quantum mechanics, for a system with Hamiltonian H, the thermal partition function is

$$Z = \mathrm{Tr} e^{-\beta \hat{H}}, \tag{3.2}$$

where β is the inverse temperature. This is related to the time evolution operator $e^{-itH/\hbar}$ by a Euclidean analytic continuation $t = -i\tau$ if we identify $\tau = \beta\hbar$. Let us consider a single scalar degree of freedom Φ, then one can write the trace as

$$\mathrm{Tr} e^{-\tau \hat{H}/\hbar} = \int d\phi < \phi | e^{-\tau_E \hat{H}/\hbar} | \phi >$$

and use the usual path integral representation for the propagator to find

$$\mathrm{Tr} e^{-\tau \hat{H}/\hbar} = \int d\phi \int D\Phi e^{-S_E[\Phi]}.$$

Here $S_E[\Phi]$ is the Euclidean action over periodic field configurations that satisfy the boundary condition

$$\Phi(\beta\hbar) = \Phi(0) = \phi.$$

This gives the relation between the periodicity in Euclidean time and the inverse temperature,

$$\beta\hbar = \tau \quad \text{or} \quad T = \frac{\hbar}{\tau}. \tag{3.3}$$

Let us now look at the Euclidean Schwarzschild metric by substituting $t = -it_E$. Near the horizon the line element (2.11) looks like

$$ds^2 = \rho^2\kappa^2 dt_E^2 + d\rho^2.$$

If we now write $\kappa t_E = \theta$, then this metric is just the flat two-dimensional Euclidean metric written in polar coordinates provided the angular variable θ has the correct periodicity $0 < \theta < 2\pi$. If the periodicity is different, then the geometry would have a conical singularity at $\rho = 0$. This implies that Euclidean time t_E has periodicity $\tau = \frac{2\pi}{\kappa}$. Note that far away from the black hole at asymptotic infinity the Euclidean metric is flat and goes as $ds^2 = d\tau_E^2 + dr^2$. With periodically identified Euclidean time, $t_E \sim t_E + \tau$, it looks like a cylinder. Near the horizon at $\rho = 0$ it is nonsingular and looks like flat space in polar coordinates for this correct periodicity. The full Euclidean geometry thus looks like a cigar. The tip of the cigar is at $\rho = 0$ and the geometry is asymptotically cylindrical far away from the tip.

Using the relation between Euclidean periodicity and temperature, we then conclude that Hawking temperature of the black hole is

$$T = \frac{\hbar\kappa}{2\pi}. \tag{3.4}$$

3.4 Bekenstein-Hawking Entropy

Even though we have "derived" the temperature and the entropy in the context of Schwarzschild black hole, this beautiful relation between area and entropy is true quite generally essentially because the near horizon geometry is always Rindler-like. For *all* black holes with charge, spin and in number of dimensions, the Hawking temperature and the entropy are given in terms of the surface gravity and horizon area by the formulae

$$T_H = \frac{\hbar\kappa}{2\pi}, \quad S = \frac{A}{4G\hbar}.$$

This is a remarkable relation between the thermodynamic properties of a black hole on one hand and its geometric properties on the other.

The fundamental significance of entropy stems from the fact that even though it is a quantity defined in terms of gross thermodynamic properties, it contains nontrivial information about the *microscopic* structure of the theory through Boltzmann relation

$$S = k \log(d),$$

where d is the the degeneracy or the total number of microstates of the system of for a given energy, and k is Boltzmann constant. Entropy is not a kinematic quantity like energy or momentum but rather contains information about the total number microscopic degrees of freedom of the system. Because of the Boltzmann relation, one can learn a great deal about the microscopic properties of a system from its thermodynamics properties.

The Bekenstein-Hawking entropy behaves in every other respect like the ordinary thermodynamic entropy. It is therefore natural to ask what microstates might account for it. Since the entropy formula is given by this beautiful, general form

$$S = \frac{Ac^3}{4G\hbar},$$

that involves all three fundamental dimensionful constants of nature, it is a valuable piece of information about the degrees of freedom of a quantum theory of gravity.

4. Wald Entropy

In our discussion of Bekenstein-Hawking entropy of a black hole, the Hawking temperature could be deduced from surface gravity or alternatively the periodicity of the Euclidean time in the black hole solution. These are geometric asymptotic properties of the black hole solution. However, to find the entropy we needed to use the first law of black hole mechanics which was derived in the context of Einstein-Hilbert action

$$\frac{1}{16\pi} \int R\sqrt{g} d^4x.$$

Generically in string theory, we expect corrections (both in α' and g_s) to the effective action that has higher derivative terms involving Riemann tensor and other fields.

$$I = \frac{1}{16\pi} \int (R + R^2 + R^4 F^4 + \cdots).$$

How do the laws of black hole thermodynamics get modified?

4.1 Bekenstein-Hawking-Wald Entropy

Wald derived the first law of thermodynamics in the presence of higher derivative terms in the action. This generalization implies an elegant formal expression for the entropy S given a general action I including higher derivatives

$$S = 2\pi \int_{\rho^2} \frac{\delta I}{\delta R_{\mu\nu\alpha\beta}} \epsilon^{\mu\alpha} \epsilon^{\nu\beta} \sqrt{h} d^2\Omega,$$

where $\epsilon^{\mu\nu}$ is the binormal to the horizon, h the induced metric on the horizon, and the variation of the action with respect to $R_{\mu\nu\alpha\beta}$ is to be carried out regarding the Riemann tensor as formally independent of the metric $g_{\mu\nu}$.

As an example, let us consider the Schwarzschild solution of the Einstein Hilbert action. In this case, the event horizon is S^2 which has two normal directions along r and t. We can construct an antisymmetric 2-tensor $\epsilon_{\mu\nu}$ along these directions so that $\epsilon_{rt} = \epsilon_{tr} = -1$.

$$\mathcal{L} = \frac{1}{16\pi} R_{\mu\nu\alpha\beta} g^{\nu\alpha} g^{\mu\beta}, \qquad \frac{\partial \mathcal{L}}{\partial R_{\mu\nu\alpha\beta}} = \frac{1}{16\pi} \frac{1}{2} (g^{\mu\alpha} g^{\nu\beta} - g^{\nu\alpha} g^{\mu\beta})$$

Then the Wald entropy is given by

$$\begin{aligned} S &= \frac{1}{8} \int \frac{1}{2} (g^{\mu\alpha} g^{\nu\beta} - g^{\nu\alpha} g^{\mu\beta})(\epsilon_{\mu\nu}\epsilon_{\alpha\beta}) \sqrt{h} d^2\Omega \\ &= \frac{1}{8} \int g^{tt} g^{rr} \cdot 2 = \frac{1}{4} \int_{S^2} \sqrt{h} d^2\Omega = \frac{A_H}{4}, \end{aligned}$$

giving us the Bekenstein-Hawking formula as expected.

4.2 Wald entropy for extremal black holes

For non-spinning extremal black holes, the geometry is spherically symmetric. Moreover, the near horizon geometry becomes $AdS_2 \times S^2$ just as in the case of Reissner-Nordström black hole.

$$ds^2 = -(1 - r_+/r)(1 - r_-/r) dt^2 + \frac{dr^2}{(1 - r_+/r)(1 - r_-/r)} + r^2 (d\theta^2 + \sin^2\theta d\phi^2) \,. \quad (4.1)$$

Here (t, r, θ, ϕ) are the coordinates of space-time and r_+ and r_- are two parameters labelling the positions of the outer and inner horizon of the black hole respectively $(r_+ > r_-)$. The extremal limit corresponds to $r_- \to r_+$. We take this limit keeping the coordinates θ, ϕ, and

$$\sigma := \frac{(2r - r_+ - r_-)}{(r_+ - r_-)}, \quad \tau := \frac{(r_+ - r_-)t}{2r_+^2}, \quad (4.2)$$

fixed. In this limit the metric and the other fields take the form:

$$ds^2 = r_+^2\left(-(\sigma^2-1)d\tau^2 + \frac{d\sigma^2}{\sigma^2-1}\right) + r_+^2\left(d\theta^2 + \sin^2(\theta)d\phi^2\right) . \tag{4.3}$$

This is the metric of $AdS_2 \times S^2$, with AdS_2 parametrized by (σ, τ) and S^2 parametrized by (θ, ϕ). Although in the original coordinate system the horizons coincide in the extremal limit, in the (σ, τ) coordinate system the two horizons are at $\sigma = \pm 1$. The AdS_2 space has $SO(2,1) \equiv SL(2,\mathbb{R})$ symmetry– the time translation symmetry is enhanced to the larger $SO(2,1)$ symmetry. All known extremal black holes have this property. Henceforth, we will take this as a definition of the near horizon geometry of an extremal black hole. In four dimensions, we also have the S^2 factor with $SO(3)$ isometries. Our objective will be to exploit the $SO(2,1) \times SO(3)$ isometries of this spacetime to considerably simply the formula for Wald entropy.

Consider an arbitrary theory of gravity in four spacetime dimensions with metric $g_{\mu\nu}$ coupled to a set of $U(1)$ gauge fields $A_\mu^{(i)}$ ($i = 1, \ldots, r$ for a rank r gauge group) and neutral scalar fields ϕ_s ($s = 1, \ldots N$) . Let x^μ ($\mu = 0, \ldots, 3$ be local coordinates on spacetime and $\mathcal{L}$ be an arbitrary general coordinate invariant local lagrangian. The action is then

$$I = \int d^4x\sqrt{-det(g)}\,\mathcal{L}\,. \tag{4.4}$$

For an extermal black hole solution of this action, the most general form of the near horizon geometry and of all other fields consistent with $SO(2,1) \times SO(3)$ isometry is given by

$$ds^2 = v_1\left(-(\sigma^2-1)d\tau^2 + \frac{d\sigma^2}{\sigma^2-1}\right) + v_2(d\theta^2 + \sin^2(\theta)d\phi^2)\,, \tag{4.5}$$

$$F_{\sigma\tau}^{(i)} = e_i\ , \quad F_{\theta\phi}^{(i)} = \frac{p_i}{4\pi}\sin(\theta)\ , \quad \phi_s = u_s\ . \tag{4.6}$$

We can think of e_i and p_i ($i = 1, \ldots, r$) as the electric and magnetic fields respectively near the black hole horizon. The constants v_a ($a = 1, 2$) and u_s ($s = 1, \ldots, N$) are to be determined by solving the equations of motion. Let us define

$$f(u, v, e, p) := \int d\theta d\phi\sqrt{-\det(g)}\mathcal{L}|_{horizon}\,. \tag{4.7}$$

Using the fact that $\sqrt{-\det(g)} = \sin(\theta)$ on the horizon, we conclude

$$f(u, v, e, p) := 4\pi v_1 v_2 \mathcal{L}|_{horizon} \tag{4.8}$$

Finally we define the entropy function

$$\mathcal{E}(q, u, v, e, p) = 2\pi(e_i q_i - f(u, v, e, p))\,, \tag{4.9}$$

where we have introduced the quantities

$$q_i := \frac{\partial f}{\partial e_i} \tag{4.10}$$

which by definition can be identified with the electric charges carried by the black hole. This function called the 'entropy function' is directly related to the Wald entropy as we summarize below.

1. For a black hole with fixed electric charges $\{q_i\}$ and magnetic charges $\{p_i\}$, all near horizon parameters v, u, e are determined by extremizing $\mathcal{E}$ with respect to the near horizon parameters:

$$\frac{\partial \mathcal{E}}{\partial e_i} = 0 \quad i = 1, \ldots r\,; \tag{4.11}$$
$$\frac{\partial \mathcal{E}}{\partial v_a} = 0, \quad a = 1, 2; \tag{4.12}$$
$$\frac{\partial \mathcal{E}}{\partial u_s} = 0, \quad s = 1, \ldots N\,. \tag{4.13}$$

   Equation (4.11) is simply the definition of electric charge whereas the other two equations (4.12) and (4.13) are the equations of motion for the near horizon fields. This follows from the fact that the dependence of $\mathcal{E}$ on all the near horizon parameters other than e_i comes only through $f(u, v, e, p)$ which from (4.8) is proportional to the action near the horizon. Thus extremization of the near horizon action is the same as the extremization of $\mathcal{E}$. This determines the variables (u, v, e) in terms of (q, p) and as a result the value of the entropy function at the extremum $\mathcal{E}^*$ is a function only of the charges

$$\mathcal{E}^*(q, p) := \mathcal{E}(q, u^*(q, p), v^*(q, p), e^*(q, p), p)\,. \tag{4.14}$$

2. Once we have determined the near horizon geometry, we can find the entropy using Wald's formula specialized to the case of extermal black holes:

$$S_{wald} = -8\pi \int d\theta d\phi \frac{\partial S}{\partial R_{rtrt}} \sqrt{-g_{rr} g_{tt}}\,. \tag{4.15}$$

   With some algebra it is easy to see that the entropy is given by the value of the entropy function at the extremum:

$$S_{wald}(q, p) = \mathcal{E}^*(q, p)\,. \tag{4.16}$$

5. Exercises-I

Exercise 1.1: *Reissner-Nordström (RN) black hole*

The most general static, spherically symmetric, charged solution of the Einstein-Maxwell theory (2.1) gives the Reissner-Nordström (RN) black hole. In what follows we choose units so that $G = \hbar = 1$. The line element is given by

$$ds^2 = -\left(1 - \frac{2M}{r} + \frac{Q^2}{r^2}\right) dt^2 + \left(1 - \frac{2M}{r} + \frac{Q^2}{r^2}\right)^{-1} dr^2 + r^2 d\Omega^2, \tag{5.1}$$

and the electromagnetic field strength by

$$F_{tr} = Q/r^2.$$

The parameter Q is the charge of the black hole and M is the mass. For $Q = 0$ this reduces to the Schwarzschild black hole.

1. *Identify the horizon for this metric and examine the near horizon geometry to show that it has two-dimensional Rindler spacetime as a factor.*
2. *Using the relation to the Rindler geometry determine the surface gravity κ as for the Schwarzschild black hole and thereby determine the temperature and entropy of the black hole. Show that in the extremal limit $M \to Q$ the temperature vanishes but the entropy has a nonzero limit.*
3. *Show that for the extremal Reissner-Nordström black hole the near horizon geometry is of the form $AdS_2 \times S^2$.*

Exercise 1.2 *Uniformly accelerated observer and Rindler coordinates*

Consider an astronaut in a spaceship moving with constant acceleration a in Minkowski spactime with Minkowski coordinates $(T, \vec{X})$. This means she feels a constant normal reacting from the floor of the spaceship in her rest frame:

$$\frac{d^2\vec{X}}{dt^2} = \vec{a}\,, \quad \frac{dT}{d\tau} = 1 \tag{5.2}$$

where τ is proper time and $\vec{a}$ is the acceleration 3-vector.

1. *Write the equation of motion in a covariant form and show that her 4-velocity $u^\mu := \frac{dX^\mu}{d\tau}$ is timelike whereas her 4-acceleration a^μ is spacelike.*
2. *Show that if she is moving along the x direction, then her trajectory is of the form*

$$T = \frac{1}{a}\sinh(a\tau)\,, \quad X = \frac{1}{a}\cosh(a\tau) \tag{5.3}$$

   which is a hyperboloid. Find the acceleration 4-vector.

3. *Show that it is natural for her to use her proper time as the time coordinate and introduce a coordinate frame of a family of observers with*

$$T = \zeta \, \sinh(a\eta)\,, \quad X = \zeta \, \cosh(a\eta)\,. \tag{5.4}$$

By examining the metric, show that $v = \eta - \zeta$ and $u = \eta + \zeta$ are precisely the Rindler coordinates introduced earlier with the acceleration parameter a identified with the surface gravity κ.

Exercise 1.3 *Perturbative half-BPS states*

Consider a heterotic string wrapping w times around a circle carrying momentum n along the circle. Recall that the heterotic strings consists of a right-moving superstring and a left-moving bosonic string. In the NSR formalism in the light-cone gauge, the worldsheet fields are:

- Right moving superstring $X^i(\sigma^-)$ $\tilde{\psi}^i(\sigma^-)$ $\qquad i = 1 \cdots 8$
- Left-moving bosonic string $X^i(\sigma^+), X^I(\sigma^+)$ $\qquad I = 1 \cdots 16,$

where X^i are the bosonic transverse spatial coordinates, $\tilde{\psi}^i$ are the worldsheet fermions, and X^I are the coordinates of an internal $E_8 \times E_8$ torus. A BPS state is obtained by keeping the right-movers in the ground state (that is, setting the right-moving oscillator number $\tilde{N} = \frac{1}{2}$ in the NS sector and $\tilde{N} = 0$ in the R sector).

1. *Using Virasoro constraints show that the mass of these states satisfies a BPS bound.*
2. *Show that the degeneracy $d(n, w)$ of such perturbative BPS-states with winding w and momentum n depend only on the T-duality invariant $Q^2/2 = nw := N$.* and hence we can talk about $d(N)$.
3. *Calculate the canonical partition function $Z(\beta) :=$ Tr $\left(e^{-\beta L_0}\right) := \sum e^{-\beta(N-1)} d(N)$.*

Exercise 1.4 *Cardy formula*

The degeneracy $d(N)$ can be obtained from the canonical partition function by the inverse Laplace transform

$$d(N) = \frac{1}{2\pi i} \int d\beta e^{\beta N} Z(\beta). \tag{5.5}$$

We would like to find an asymptotic expansion of $d(N)$ for large N. This is given by the 'Cardy formula' which utilizes the modular properties of the partition function.

1. *Show that $Z(\beta)$ is related to the modular form $\Delta(\tau)$ of weight 12 by $Z(\beta) = 1/\Delta(\tau)$, with $\beta := -2\pi i \tau$.*
2. *Using the modular properties of $Z(\beta)$, show that for large N the degeneracy scales as $d(N) \sim \exp{(4\pi\sqrt{N})}$.*

6. Exercises II

Exercise 2.1: *Elements of String Compactifications*

The heterotic string theory in ten dimensions has 16 supersymmetries. The bosonic massless fields consist of the metric g_{MN}, a 2-form field $B^{(2)}$, 16 abelian 1-form gauge fields $A^{(r)}$ $r = 1, \ldots 16$, and a real scalar field ϕ called the dilaton. The Type-IIB string theory in ten dimensions has 32 supersymmetries. The bosonic massless fields consist of the metric g_{MN}; two 2-form fields $C^{(2)}, B^{(2)}$; a self-dual 4-form field $C^{(4)}$; and a complex scalar field λ called the dilaton-axion field.

One of the remarkable strong-weak coupling dualities is the 'string-string' duality between heterotic string compactified on $T^4 \times T^2$ and Type-IIB string compactified on $K3 \times T^2$. One piece of evidence for this duality is obtained by comparing the massless spectrum for these compactifications and certain half-BPS states in the spectrum.

1. *Show that the heterotic string compactified on $T^4 \times S^1 \times \tilde{S}^1$ leads a four dimensional theory with $\mathcal{N} = 4$ supersymmetry with 22 vector multiplets.*
2. *Show that the Type-IIB string compactified on $K3 \times S^1 \times \tilde{S}^1$ leads a four dimensional theory with $\mathcal{N} = 4$ supersymmetry with 22 vector multiplets.*
3. *Show that the Kaluza-Klein monopole in Type-IIB string associated with the circle $\tilde{S}^1$ has the right structure of massless fluctuations to be identified with the half-BPS perturbative heterotic string in the dual description.*

Exercise 2.2: *Wald entropy for extremal black holes*

The entropy function formalism developed in §4 allows one to compute the entropy of various extermal black holes very efficiently by simply solving certain algebraic equations (instead of partial differential equations). It also allows one to incorporate effects of higher derivative corrections to the two-derivative action with relative ease.

1. *Using the Einstein-Hilbert action (2.1) show that the Wald entropy of the Schwarzschild black hole equals its Bekenstein-Hawking Entropy.*
2. *Using the two-derivative effective action (9.15) of string theory compute the Bekenstein-Hawking Entropy of extremal quarter-BPS black holes in string theory with charge vectors Q and P.*

7. Tutorial IA: Extremal Black Holes

7.1 Reissner-Nordström Metric

From the metric (5.1) we see that the event horizon for this solution is located at where $g^{rr} = 0$, or

$$1 - \frac{2M}{r} + \frac{Q^2}{r^2} = 0.$$

Since this is a quadratic equation in r,

$$r^2 - 2QMr + Q^2 = 0,$$

it has two solutions.

$$r_{\pm} = M \pm \sqrt{M^2 - Q^2}.$$

Thus, r_+ defines the outer horizon of the black hole and r_- defines the inner horizon of the black hole. The area of the black hole is $4\pi r_+^2$.

Following the steps similar to what we did for the Schwarzschild black hole, we can analyze the near horizon geometry to find the surface gravity and hence the temperature:

$$T = \frac{\kappa\hbar}{2\pi} = \frac{\sqrt{M^2 - Q^2}}{2\pi(2M(M + \sqrt{M^2 - Q^2}) - Q^2)} \tag{7.1}$$

$$S = \pi r_+^2 = \pi(M + \sqrt{M^2 - Q^2})^2. \tag{7.2}$$

These formulae reduce to those for the Schwarzschild black hole in the limit $Q = 0$.

7.2 Extremal Black Holes

For a physically sensible definition of temperature and entropy in (7.1) the mass must satisfy the bound $M^2 \geq Q^2$. Something special happens when this bound is saturated and $M = |Q|$. In this case $r_+ = r_- = |Q|$ and the two horizons coincide. We choose Q to be positive. The solution (5.1) then takes the form,

$$ds^2 = -(1 - Q/r)^2 dt^2 + \frac{dr^2}{(1 - Q/r)^2} + r^2 d\Omega^2, \tag{7.3}$$

with a horizon at $r = Q$. In this extremal limit (7.1), we see that the temperature of the black hole goes to zero and it stops radiating but nevertheless its entropy has a finite limit given by $S \to \pi Q^2$. When the temperature goes to zero, thermodynamics does not really make sense but we can use this limiting entropy as the definition of the zero temperature entropy.

For extremal black holes it more convenient to use isotropic coordinates in which the line element takes the form

$$ds^2 = H^{-2}(\vec{x})dt^2 + H^2(\vec{x})d\vec{x}^2$$

where $d\vec{x}^2$ is the flat Euclidean line element $\delta_{ij}dx^i dx^j$ and $H(\vec{x})$ is a harmonic function of the flat Laplacian

$$\delta^{ij}\frac{\partial}{\partial x^i}\frac{\partial}{\partial x^j}.$$

The extremal Reissner-Nordström solution is obtained by choosing

$$H(\vec{x}) = \left(1 + \frac{Q}{r}\right),$$

and the field strength is given by $F_{0i} = \partial_i H(\vec{x})$.

One can in fact write a multi-centered Reissner-Nordström solution by choosing a more general harmonic function

$$H = 1 + \sum_{i=1}^{N} \frac{Q_i}{|\vec{x} - \vec{x}i|}. \tag{7.4}$$

The total mass M equals the total charge Q and is given additively

$$Q = \sum Q_i. \tag{7.5}$$

The solution is static because the electrostatic repulsion between different centers balances gravitational attraction among them.

Note that the coordinate ρ in the isotropic coordinates should not be confused with the coordinate r in the spherical coordinates. In the isotropic coordinates the line-element is

$$ds^2 = -\left(1 + \frac{Q}{\rho}\right)^2 dt^2 + (1 + \frac{Q}{\rho})^{-2}(d\rho^2 + \rho^2 d\Omega^2),$$

and the horizon occurs at $\rho = 0$. Contrast this with the metric in the spherical coordinates (7.3) that has the horizon at $r = M$. The near horizon geometry is quite different from that of the Schwarzschild black hole. The line element is

$$\begin{aligned} ds^2 &= -\frac{\rho^2}{Q^2}dt^2 + \frac{Q^2}{\rho^2}(d\rho^2 + \rho^2 d\Omega^2) \\ &= (-\frac{\rho^2}{Q^2}dt^2 + \frac{Q^2}{\rho^2}dr^2) + (Q^2 d\Omega^2). \end{aligned}$$

The geometry thus factorizes as for the Schwarzschild solution. One factor the 2-sphere S^2 of radius Q but the other (r,t) factor is now not Rindler any more but is a two-dimensional Anti-de Sitter or AdS_2. The geodesic radial distance in AdS_2 is $\log r$. As a result the geometry looks like an infinite throat near $r=0$ and the radius of the mouth of the throat has radius Q.

Extremal RN black holes are interesting because they are stable against Hawking radiation and nevertheless have a large entropy. We now try to see if the entropy can be explained by counting of microstates. In doing so, supersymmetry proves to be a very useful tool.

7.3 Supersymmetry, BPS representation, and Extremality

Some of the special properties of external black holes can be understood better by embedding them in supergravity. We will be interested in these lectures in string compactifications with $\mathcal{N}=4$ supersymmetry in four spacetime dimensions. The $\mathcal{N}=4$ supersymmetry algebra contains in addition to the usual Poincaré generators, sixteen real supercharges Q^i_α where $\alpha=1,2$ is the usual Weyl spinor index of 4d Lorentz symmetry. and the internal index $i=1,\ldots,4$ in the fundamental $\mathbf{4}$ representation of an $SU(4)$, the R-symmetry of the superalgebra. The relevant anticommutators for our purpose are

$$\begin{aligned} \{Q^a_\alpha, \bar{Q}_{\dot\beta b}\} &= 2P_\mu \sigma^\mu_{\alpha\dot\beta}\delta^i_j \\ \{Q^a_\alpha, Q^b_\beta\} = \epsilon_{\alpha\beta}Z^{ab} \qquad & \{\bar{Q}_{\dot\alpha a}, \bar{Q}_{\dot\beta b}\} = \bar{Z}_{ab}\epsilon_{\dot\alpha\dot\beta} \end{aligned} \tag{7.6}$$

where σ^μ are (2×2) matrices with $\sigma_0 = -\mathbf{1}$ and $\sigma^i fori = 1,2,3$ are the usual Pauli matrices. Here P_μ is the momentum operator and Q are the supersymmetry generators and the complex number Z^{ab} is the central charge matrix.

Let us first look at the representations of this algebra when the central charge is zero. In this case the massive and massless representation are qualitatively different.

1. Massive Representation, $M>0, P^\mu = (M,0,0,0)$
   In this case, (7.6) becomes $\{Q^a_\alpha, \bar{Q}_{\dot\beta b}\} = 2M\delta_{\alpha\dot\beta}\delta^a_b$ and all other anti-commutators vanish. Up to overall scaling, these are the commutation relations for eight complex fermionic oscillators. Each oscillator has a two-state representation, filled or empty, and hence the total dimension of the representation is $2^8 = 256$ which is CPT self-conjugate.

2. Massless Representation $M=0, P^\mu = (E,0,0,E)$
   In this case (7.6) becomes $\{Q^1_\alpha, \bar{Q}_{\dot\beta 1}\} = 2E\delta_{\alpha\dot\beta}$ and all other anti-commutators vanish. Up to overall scaling, these are now the anti-commutation relations of

two fermionic oscillators and hence the total dimension of the representation is $2^4 = 16$ which is also CPT-self-conjugate.

The important point is that for a massive representation, with $M = \epsilon > 0$, no matter how small ϵ, the supermultiplet is long and precisely at $M = 0$ it is short. Thus the size of the supermultiplet has to change discontinuously if the state has to acquire mass. Furthermore, the size of the supermultiplet is determined by the number of supersymmetries that are *broken* because those have non-vanishing anti-commutations and turn into fermionic oscillators.

Note that there is a bound on the mass $M \geq 0$ which simply follows from the fact the using (7.6) one can show that the mass operator on the right hand side of the equation equals a positive operator, the absolute value square of the supercharge on the left hand side. The massless representation saturates this bound and is 'small' whereas the massive representation is long.

There is an analog of this phenomenon also for nonzero Z_{ab}. As explained in the appendix, the central charge matrix Z_{ab} can be brought to the standard form by an $U(4)$ rotation

$$\tilde{Z} = UZU^T, \;\; U \in U(4)\;, \qquad \tilde{Z}_{ab} = \left(\begin{array}{c|c} Z_1\varepsilon & 0 \\ \hline 0 & Z_2\varepsilon \end{array}\right)\;, \quad \varepsilon = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\,. \tag{7.7}$$

so we have two 'central charges' Z_1 and Z_2. Without loss of generality we can assume $|Z_1| \geq |Z_2|$. Using the supersymmetry algebra one can prove the BPS bound $M - |Z_1| \geq 0$ by showing that this operator is equal to a positive operator (see appendix for details). States that saturate this bound are the BPS states. There are three types of representations:

- If $M = |Z_1| = |Z_2|$, then eight of of the sixteen supersymmetries are preserved. Such states are called half-BPS. The broken supersymmetries result in four complex fermionic zero modes whose quantization furnishes a 2^4-dimensional short multiplet

- If $M = |Z_1| > |Z_2|$, then and four out of the sixteen supersymmetries are preserved. Such states are called quarter-BPS. The broken supersymmetries result in six complex fermionic zero modes whose quantization furnishes a 2^6-dimensional intermediate multiplet.

- If $M > |Z_1| > |Z_2|$, then no supersymmetries are preserved. Such states are called non-BPS.The sixteen broken supersymmetries result in eight complex fermionic zero modes whose quantization furnishes a 2^8-dimensional long multiplet.

The significance of BPS states in string theory and in gauge theory stems from the classic argument of Witten and Olive which shows that under suitable conditions, the spectrum of BPS states is stable under smooth changes of moduli and coupling constants. The crux of the argument is that with sufficient supersymmetry, for example $\mathcal{N} = 4$, the coupling constant does not get renormalized. The central charges Z_1 and Z_2 of the supersymmetry algebra depend on the quantized charges and the coupling constant which therefore also does not get renormalized. This shows that for BPS states, the mass also cannot get renormalized because if the quantum corrections increase the mass, the states will have to belong a long representation . Then, the number of states will have to jump discontinuously from, say from 16 to 256 which cannot happen under smooth variations of couplings unless there is some kind of a 'Higgs Mechanism' or there is some kind of a phase transition[2]

As a result, one can compute the spectrum at weak coupling in the region of moduli space where perturbative or semiclassical counting methods are available. One can then analytically continue this spectrum to strong coupling. This allows us to obtain invaluable non-perturbative information about the theory from essentially perturbative commutations.

8. Tutorial IB: BPS states in string theory

8.1 BPS dyons in theories with $\mathcal{N} = 4$ supersymmetry

The massless spectrum of the toroidally compactified heterotic string on T^6 contains 28 different "photons" or $U(1)$ gauge fields – one from each of the 22 vector multiplets and 6 from the supergravity multiplet. As a result, the electric charge of a state is specified by a 28-dimensional charge vector Q and the magnetic charge is specified by a 28-dimensional charge vector P. Thus, a dyonic state is specified by the charge vector

$$\Gamma = \begin{pmatrix} Q \\ P \end{pmatrix} \tag{8.1}$$

where Q and P are the electric and magnetic charge vectors respectively. Both Q and P are elements of a self-dual integral lattice $\Pi^{22,6}$ and can be represented as 28-dimensional

[2] Such 'phase transitions' do occur and the degeneracies can jump upon crossing certain walls in the moduli space. This phenomenon called 'wall-crossing' occurs not because of Higgs mechanism but because at the walls, single particle states have the same mass as certain multi-particle states and can thus mix with the multi-particle continuum states. The wall-crossing phenomenon complicates the analytic continuation of the degeneracy from weak coupling from strong coupling since one may encounter various walls along the way. However, in many cases, the jumps across these walls can be taken into account systematically.

column vectors in $\mathbb{R}^{22,6}$ with integer entries, which transform in the fundamental representation of $O(22, 6; \mathbb{Z})$. We will be interested in BPS states.

- For half-BPS state the charge vectors Q and P must be parallel. These states are dual to perturbative BPS states.
- For a quarter-BPS states the charge vectors Q and P are not parallel. There is no duality frame in which these states are perturbative.

There are three invariants of $O(22, 6; \mathbb{Z})$ quadratic in charges given by P^2, Q^2, $Q \cdot P$. These three T-duality invariants will be useful in later discussions.

8.2 Spectrum of half-BPS states

An instructive example of BPS of states is provided by an infinite tower of BPS states that exists in perturbative string theory.

Consider a perturbative heterotic string state wrapping around S^1 with winding number w and quantized momentum n. Let the radius of the circle be R and $\alpha' = 1$, then one can define left-moving and right-moving momenta as usual,

$$p_{L,R} = \sqrt{\frac{1}{2}} \left(\frac{n}{R} \pm wR \right). \tag{8.2}$$

The Virasoro constraints are then given by

$$\tilde{L}_0 - \frac{M^2}{4} + \frac{p_R^2}{2} = 0 \tag{8.3}$$

$$L_0 - \frac{M^2}{4} + \frac{p_L^2}{2} = 0, \tag{8.4}$$

where N and $\tilde{N}$ are the left-moving and right-moving oscillation numbers respectively.

The left-moving oscillator number is then

$$L_0 = \sum_{n=1}^{\infty} (\sum_{i=1}^{8} n a_{-n}^i a_n^i + \sum_{I=1}^{16} n \beta_{-n}^I \beta_{-n}^I) - 1 := N - 1, \tag{8.5}$$

where a^i are the left-moving Fourier modes of the fields X^i, and β^I are the Fourier modes of the fields X^I. From the Virasoro constraint (8.3) we see that a BPS state with $\tilde{N} = 0$ saturates the BPS bound

$$M = \sqrt{2} p_R, \tag{8.6}$$

and thus $\sqrt{2}p_R$ can be identified with the central charge of the supersymmetry algebra. The right-moving ground state after the usual GSO projection is indeed 16-dimensional as expected for a BPS-state in a theory with $\mathcal{N} = 4$ supersymmetry. To see this, note that the right-moving fermions satisfy anti-periodic boundary condition in the NS sector and have half-integral moding, and satisfy periodic boundary conditions in the R sector and have integral moding. The oscillator number operator is then given by

$$\tilde{L}_0 = \sum_{n=1}^{\infty} \sum_{i=1}^{8} (n\tilde{a}^i_{-n}\tilde{a}^i_n + r\tilde{\psi}^i_{-r}\tilde{\psi}^i_r - \frac{1}{2}) := \tilde{N} - \frac{1}{2}. \tag{8.7}$$

with $r \equiv -(n - \frac{1}{2})$ in the NS sector and by

$$\tilde{L}_0 = \sum_{n=1}^{\infty} \sum_{i=1}^{8} (n\tilde{a}^i_{-n}\tilde{a}^i_n + r\tilde{\psi}^i_{-r}\tilde{\psi}^i_r) \tag{8.8}$$

with $r \equiv (n - 1)$ in the R sector.

In the NS-sector then one then has $\tilde{N} = \frac{1}{2}$ and the states are given by

$$\tilde{\psi}^i_{-\frac{1}{2}}|0>, \tag{8.9}$$

that transform as the vector representation $\mathbf{8_v}$ of $SO(8)$. In the R sector the ground state is furnished by the representation of fermionic zero mode algebra $\{\psi^i_0, \psi^j_0\} = \delta^{ij}$ which after GSO projection transforms as $\mathbf{8_s}$ of $SO(8)$. Altogether the right-moving ground state is thus 16-dimensional $\mathbf{8_v} \oplus \mathbf{8_s}$.

We thus have a perturbative BPS state which looks pointlike in four dimensions with two integral charges n and w that couple to two gauge fields $g_{5\mu}$ and $B_{5\mu}$ respectively. It saturates a BPS bound $M = \sqrt{2}p_R$ and belongs to a 16-dimensional short representation. This point-like state is our 'would-be' black hole. Because it has a large mass, as we increase the string coupling it would begin to gravitate and eventually collapse to form a black hole.

Microscopically, there is a huge multiplicity of such states which arises from the fact that even though the right-movers are in the ground state, the string can carry arbitrary left-moving oscillations subject to the Virasoro constraint. Using $M = \sqrt{2}p_R$ in the Virasoro constraint for the left-movers gives us

$$N - 1 = \frac{1}{2}(p_R^2 - p_L^2) := Q^2/2 = nw. \tag{8.10}$$

We would like to know the degeneracy of states for a given value of charges n and w which is given by exciting arbitrary left-moving oscillations whose total worldsheet energy adds up to N. Let us take $w = 1$ for simplicity and denote the degeneracy by $d(n)$ which we want to compute. As usual, it is more convenient to evaluate the canonical partition function

$$Z(\beta) = \mathrm{Tr}\left(e^{-\beta L_0}\right) \tag{8.11}$$

$$\equiv \sum_{-1}^{\infty} d(n) q^n \quad q := e^{-\beta}\,. \tag{8.12}$$

This is the canonical partition function of 24 left-moving massless bosons in $1+1$ dimensions at temperature $1/\beta$. The micro-canonical degeneracy $d(n)$ is given then given as usual by the inverse Laplace transform

$$d(n) = \frac{1}{2\pi i}\int d\beta e^{\beta n} Z(\beta). \tag{8.13}$$

Using the expression (8.5) for the oscillator number s and the fact that

$$\mathrm{Tr}(q^{-s\alpha_{-n}\alpha_n}) = 1 + q^s + q^{2s} + q^{3s} + \cdots = \frac{1}{(1-q^s)}\,, \tag{8.14}$$

the partition function can be readily evaluated to obtain

$$Z(\beta) = \frac{1}{q}\prod_{s=1}^{\infty}\frac{1}{(1-q^s)^{24}} \quad . \tag{8.15}$$

It is convenient to introduce a variable τ by $\beta := -2\pi i\tau$, so that$q := e^{2\pi i\tau}$. The function

$$\Delta(\tau) = q\prod_{s=1}^{\infty}(1-q^s)^{24}, \tag{8.16}$$

is the famous discriminant function. Under modular transformations

$$\tau \to \frac{a\tau+b}{c\tau+d} \quad a,b,c,d \in \mathbb{Z}\,, \quad with \quad ad-bc=1 \tag{8.17}$$

it transforms as a modular form of weight 12:

$$\Delta(\frac{a\tau+b}{c\tau+d}) = (c\tau+d)^{12}\Delta(\tau)\,, \tag{8.18}$$

This remarkable property allows us to relate high temperature ($\beta \to 0$) to low tempreature ($\beta \to \infty$) and derive a simple explicit expression for the asymptotic degeneracies $d(n)$ for n very large.

8.3 Cardy Formula

We would like to evaluate this integral (8.13) for large N which corresponds to large worldsheet energy. We would therefore expect that the integral will receive most of its contributions from high temperature or small β region of the integrand. To compute the large N asymptotics, we then need to know the small β asymptotics of the partition function. Now, $\beta \to 0$ corresponds to $q \to 1$ and in this limit the asymptotics of $Z(\beta)$ are very difficult to read off from (8.15) because its a product of many quantities that are becoming very large. It is more convenient to use the fact that $Z(\beta)$ is the inverse of $\Delta(\tau)$ which is a modular form of weight 12 we can conclude

$$Z(\beta) = (\beta/2\pi)^{12} Z(\frac{4\pi^2}{\beta}). \tag{8.19}$$

This allows us to relate the $q \to 1$ or high temperature asymptotics to $q \to 0$ or low temperature asymptotics as follows. Now, $Z(\tilde{\beta}) = Z\left(\frac{4\pi^2}{\beta}\right)$ asymptotics are easy to read off because as $\beta \to 0$ we have $\tilde{\beta} \to \infty$ or $e^{-\tilde{\beta}} = \tilde{q} \to 0$. As $\tilde{q} \to 0$

$$Z(\tilde{\beta}) = \frac{1}{\tilde{q}} \prod_{n=1}^{\infty} \frac{1}{(1-\tilde{q}^n)^{24}} \sim \frac{1}{\tilde{q}}. \tag{8.20}$$

This allows us to write

$$d(N) \sim \frac{1}{2\pi i} \int \left(\frac{\beta}{2\pi}\right)^{12} e^{\beta N + \frac{4\pi^2}{\beta}} d\beta. \tag{8.21}$$

This integral can be evaluated easily using saddle point approximation. The function in the exponent is $f(\beta) \equiv \beta N + \frac{4\pi^2}{\beta}$ which has a maximum at

$$f'(\beta) = 0 \quad \text{or} \quad N - \frac{4\pi^2}{\beta_c} = 0 \quad \text{or} \quad \beta_c = \frac{2\pi}{\sqrt{N}}. \tag{8.22}$$

The value of the integrand at the saddle point gives us the leading asymptotic expression for the number of states

$$d(n) \sim \exp{(4\pi\sqrt{n})}. \tag{8.23}$$

This implies that the black holes corresponding to these states should have nonzero entropy that goes in general as

$$S \sim 4\pi\sqrt{nw}. \tag{8.24}$$

We would now like to identify the black hole solution corresponding to this state and test if this microscopic entropy agrees with the macroscopic entropy of the black hole.

The formula that we derived for the degeneracy $d(N)$ is valid more generally in any $1+1$ CFT. In a general the partition function is a modular form of weight $-k$

$$Z(\beta) \sim Z\left(\frac{4\pi^2}{\beta}\right)\beta^k.$$

which allows us to high temperature asymptotics to low temperature asymptotics for $Z(\tilde{\beta})$ because

$$\tilde{\beta} \equiv \frac{4\pi^2}{\beta} \to \infty \qquad \text{as} \quad \beta \to 0. \tag{8.25}$$

At low temperature only ground state contributes

$$\begin{aligned} Z(\tilde{\beta}) &= \operatorname{Tr}\exp(-\tilde{\beta}(L_0 - c/24)) \\ &\sim \exp(-E_0\tilde{\beta}) \sim \exp(\frac{\tilde{\beta}c}{24}), \end{aligned}$$

where c is the central charge of the theory. Using the saddle point evaluation as above we then find.

$$d(N) \sim \exp\left(2\pi\sqrt{\frac{cN}{6}}\right). \tag{8.26}$$

In our case, because we had 24 left-moving bosons, $c = 24$, and then (8.26) reduces to (8.23).

9. Tutorial IIA: Elements of String Theory

9.1 $\mathcal{N} = 4$ string compactifications and BPS spectrum

Superstring theories are naturally formulated in ten-dimensional Lorentzian spacetime $\mathcal{M}_{10}$. A 'compactification' to four-dimensions is obtained by taking $\mathcal{M}_{10}$ to be a product manifold $\mathbb{R}^{1,3} \times X_6$ where X_6 is a compact Calabi-Yau threefold and $\mathbb{R}^{1,3}$ is the noncompact Minkowski spacetime. We will focus in these lectures on a compactification of Type-II superstring theory when X_6 is itself the product $X_6 = K3 \times T^2$. A highly nontrivial and surprising result from the 90s is the statement that this compactification is quantum equivalent or 'dual' to a compactification of heterotic string theory on $T^4 \times T^2$ where T^4 is a four-dimensional torus [15, 16]. One can thus describe the theory either in the Type-II frame or the heterotic frame.

The four-dimensional theory in $\mathbb{R}^{1,3}$ resulting from this compactification has $\mathcal{N} = 4$ supersymmetry[3]. The massless fields in the theory consist of 22 vector multiplets in addition to the supergravity multiplet. The massless moduli fields consist of the S-modulus λ taking values in the coset

$$SL(2,\mathbb{Z})\backslash SL(2;\mathbb{R})/O(2;\mathbb{R}), \tag{9.1}$$

and the T-moduli μ taking values in the coset

$$O(22,6;\mathbb{Z})\backslash O(22,6;\mathbb{R})/O(22;\mathbb{R}) \times O(6;\mathbb{R}). \tag{9.2}$$

The group of discrete identifications $SL(2,\mathbb{Z})$ is called S-duality group. In the heterotic frame, it is the electro-magnetic duality group [17, 18] whereas in the type-II frame, it is simply the group of area- preserving global diffeomorphisms of the T^2 factor. The group of discrete identifications $O(22,6;\mathbb{Z})$ is called the T-duality group. Part of the T-duality group $O(19,3;\mathbb{Z})$ can be recognized as the group of geometric identifications on the moduli space of K3; the other elements are stringy in origin and have to do with mirror symmetry.

At each point in the moduli space of the internal manifold $K3 \times T^2$, one has a distinct four- dimensional theory. One would like to know the spectrum of particle states in this theory. Particle states are unitary irreducible representations, or supermultiplets, of the $\mathcal{N} = 4$ superalgebra. The supermultiplets are of three types which

[3] This supersymmetry is a super Lie algebra containing $ISO(1,3) \times SU(4)$ as the bosonic subalgebra where $ISO(1,3)$ is the Poincaré symmetry of the $\mathbb{R}^{1,3}$ spacetime and $SU(4)$ is an internal symmetry called R-symmetry in physics literature. The odd generators of the superalgebra are called supercharges. With $\mathcal{N} = 4$ supersymmetry, there are eight complex supercharges which transform as a spinor of $ISO(1,3)$ and a fundamental of $SU(4)$.

have different dimensions in the rest frame. A long multiplet is 256- dimensional, an intermediate multiplet is 64-dimensional, and a short multiplet is 16- dimensional. A short multiplet preserves half of the eight supersymmetries (*i.e.* it is annihilated by four supercharges) and is called a half-BPS state; an intermediate multiplet preserves one quarter of the supersymmetry (*i.e.* it is annihilated by two supercharges), and is called a quarter-BPS state; and a long multiplet does not preserve any supersymmetry and is called a non-BPS state. One consequence of the BPS property is that the spectrum of these states is 'topological' in that it does not change as the moduli are varied, except for jumps at certain walls in the moduli space [19].

An important property of the BPS states that follows from the superalgebra is that their mass is determined by the charges and the moduli [19]. Thus, to specify a BPS state at a given point in the moduli space, it suffices to specify its charges. The charge vector in this theory transforms in the vector representation of the T-duality group $O(22, 6; \mathbb{Z})$ and in the fundamental representation of the S-duality group $SL(2, \mathbb{Z})$. It is thus given by a vector $\Gamma^{i\alpha}$ with integer entries

$$\Gamma^{i\alpha} = \begin{pmatrix} Q^i \\ P^i \end{pmatrix} \quad where \qquad i = 1, 2, \ldots 28; \quad \alpha = 1, 2 \tag{9.3}$$

transforming in the $(2, 28)$ representation of $SL(2, \mathbb{Z}) \times O(22, 6; \mathbb{Z})$. The vectors Q and P can be regarded as the quantized electric and magnetic charge vectors of the state respectively. They both belong to an even, integral, self-dual lattice $\Pi^{22,6}$. We will assume in what follows that $\Gamma = (Q, P)$ in (9.3) is primitive in that it cannot be written as an integer multiple of (Q_0, P_0) for Q_0 and P_0 belonging to $\Pi^{22,6}$. A state is called purely electric if only Q is non-zero, purely magnetic if only P is non- zero, and dyonic if both P and Q are non-zero.

To define S-duality transformations, it is convenient to represent the S-modulus as a complex field S taking values in the upper half plane. An S-duality transformation

$$\gamma \equiv \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2; \mathbb{Z}) \tag{9.4}$$

acts simultaneously on the chargesand the S-modulus by

$$\begin{pmatrix} Q \\ P \end{pmatrix} \to \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} Q \\ P \end{pmatrix}; \qquad S \to \frac{aS + b}{cS + d} \tag{9.5}$$

To define T-duality transformations, it is convenient to represent the T-moduli by a 28×28 of matrix μ_I^A satisfying

$$\mu^t \, L \, \mu = L \tag{9.6}$$

with the identification that $\mu \sim k\mu$ for every $k \in O(22;\mathbb{R}) \times O(6;\mathbb{R})$. Here L is the (28×28) matrix

$$L_{IJ} = \begin{pmatrix} -\mathbf{C}_{16} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{I}_6 \\ \mathbf{0} & \mathbf{I}_6 & \mathbf{0} \end{pmatrix}, \tag{9.7}$$

with $\mathbf{I}_s$ the $s \times s$ identity matrix and $\mathbf{C}_{16}$ is the Cartan matrix of $E_8 \times E_8$. The T-moduli are then represented by the matrix

$$\mathcal{M} = \mu^t \mu \tag{9.8}$$

which satisifies

$$\mathcal{M}^t = \mathcal{M}, \qquad \mathcal{M}^t L \mathcal{M} = L \tag{9.9}$$

In this basis, a T-duality transformation can then be represented by a (28×28) matrix R with integer entries satisfying

$$R^t L R = L, \tag{9.10}$$

which acts simultaneously on the charges and the T-moduli by

$$Q \to RQ; \quad P \to RP; \quad \mu \to \mu R^{-1} \tag{9.11}$$

Given the matrix μ_I^A, one obtains an embedding $\Lambda^{22,6} \subset \mathbb{R}^{22,6}$ of $\Pi^{22,6}$ which allows us to define the moduli-dependent charge vectors Q and P by

$$Q^A = \mu_I^A Q_I \quad P^A = \mu_I^A P_I \, . \tag{9.12}$$

Note that while Q^I are integers Q^A are not. In what follows we will not always write the indices explicitly assuming that it will be clear from the context. In any case, the final answers will only depend on the T-duality invariants which are all integers. The matrix L has a 22-dimensional eigensubspace with eigenvalue -1 and a 6- dimensional eigensubspace with eigenvalue $+1$. Given Q and P, one can define the 'right-moving' charges[4] Q_R and P_R as the projections of Q and P respectively onto the subspace with eigenvalue $+1$. and the 'left-moving' charges as the projections

$$Q_{R,L} = \frac{(1 \pm L)}{2} Q \, ; \quad P_{R,L} = \frac{(1 \pm L)}{2} P \tag{9.13}$$

The right-moving charges since for the heterotic string, Q_R are related to the right-moving momenta. The central charges Z_1 and Z_2 defined in §A.2 are given in terms of these right-moving charges by

[4]The right- moving charges couple to the graviphoton vector fields associated with the right-moving chiral currents in the conformal field theory of the dual heterotic string.

If the vectors Q and P are nonparallel, then the state is quarter-BPS. On the other hand, if $Q = pQ_0$ and $P = qQ_0$ for some $Q_0 \in \Pi^{22,6}$ with p and q relatively prime integers, then the state is half-BPS.

An important piece of nonperturbative information about the dynamics of the theory is the exact spectrum of all possible dyonic BPS- states at all points in the moduli space. More specifically, one would like to compute the number $d(\Gamma)|_{\lambda,\mu}$ of dyons of a given charge Γ at a specific point (λ, μ) in the moduli space. Computation of these numbers is of course a very complicated dynamical problem. In fact, for a string compactification on a general Calabi-Yau threefold, the answer is not known. One main reason for focusing on this particular compactification on $K3 \times T^2$ is that in this case the dynamical problem has been essentially solved and the exact spectrum of dyons is now known. Furthermore, the results are easy to summarize and the numbers $d(\Gamma)|_{\lambda,\mu}$ are given in terms of Fourier coefficients of various modular forms.

In view of the duality symmetries, it is useful to classify the inequivalent duality orbits labeled by various duality invariants. This leads to an interesting problem in number theory of classification of inequivalent duality orbits of various duality groups such as $SL(2,\mathbb{Z}) \times O(22,6;\mathbb{Z})$ in our case and more exotic groups like $E_{7,7}(\mathbb{Z})$ for other choices of compactification manifold X_6. It is important to remember though that a duality transformation acts simultaneously on charges and the moduli. Thus, it maps a state with charge Γ at a point in the moduli space (λ, μ) to a state with charge Γ' but at some other point in the moduli space (λ', μ'). In this respect, the half-BPS and quarter-BPS dyons behave differently.

- For half-BPS states, the spectrum does not depend on the moduli. Hence $d(\Gamma)|_{\lambda',\mu'} = d(\Gamma)|_{\lambda,\mu}$. Furthermore, by an S-duality transformation one can choose a frame where the charges are purely electric with $P = 0$ and $Q \neq 0$. Single-particle states have Q primitive and the number of states depends only on the T-duality invariant integer $n \equiv Q^2/2$. We can thus denote the degeneracy of half-BPS states $d(\Gamma)|_{S',\mu'}$ simply by $d(n)$.

- For quarter-BPS states, the spectrum does depend on the moduli, and $d(\Gamma)|_{\lambda',\mu'} \neq d(\Gamma)|_{\lambda,\mu}$. However, the partition function turns out to be independent of moduli and hence it is enough to classify the inequivalent duality orbits to label the partition functions. For the specific duality group $SL(2,\mathbb{Z}) \times O(22,6;\mathbb{Z})$ the partition functions are essentially labeled by a single discrete invariant [20, 21, 22].

$$I = \gcd(Q \wedge P)\,, \tag{9.14}$$

The degeneracies themselves are Fourier coefficients of the partition function. For a given value of I, they depend only on[5] the moduli and the three T-duality invariants $(m, n, \ell) \equiv (P^2/2, Q^2/2, Q \cdot P)$. Integrality of (m, n, ℓ) follows from the fact that both Q and P belong to $\Pi^{22,6}$. We can thus denote the degeneracy of these quarter-BPS states $d(\Gamma)|_{\lambda,\mu}$ simply by $d(m, n, l)|_{\lambda,\mu}$. For simplicity, we consider only $I = 1$ in these lectures.

9.2 String-String duality

It will be useful to recall a few details of the string-string duality between heterotic compactified on $T^4 \times S^1 \times \tilde{S}^1$ and Type-IIB compactified on $K3 \times S^1 \times \tilde{S}^1$. Two pieces of evidence for this duality will be relevant to our discussion.

• *Low energy effective action*

Both these compactifications result in $\mathcal{N} = 4$ supergravity in four dimensions. With this supersymmetry, the two-derivative effective action for the massless fields receives no quantum corrections. Hence, if the two theories are to be dual to each other, they must have identical 2-derivative action.

This is indeed true. Even though the field content and the action are very different for the two theories in ten spacetime dimensions, upon respective compactifications, one obtains $\mathcal{N} = 4$ supergravity with 22 vector multiplets coupled to the supergravity multiplet. This has been discussed briefly in one of the tutorials. For a given number of vector multiplets, the two-derivative action is then completely fixed by supersymmetry and hence is the same for the two theories. This was one of the properties that led to the conjecture of a strong-weak coupling duality between the two theories.

For our purposes, we will be interested in the 2-derivative action for the bosonic fields. This is a generalization of the Einstein-Hilbert-Maxwell action (2.1) which couples the metric, the moduli fields and 28 abelian gauge fields:

$$\begin{aligned} I = \frac{1}{32\pi} \int d^4x \sqrt{-\det G}\, S\, [R_G &+ \frac{1}{S^2} G^{\mu\nu}(\partial_\mu S \partial_\nu S - \frac{1}{2}\partial_\mu a \partial_\nu a) + \frac{1}{8} G^{\mu\nu} Tr(\partial_\mu M L \partial_\nu M L) \\ &- G^{\mu\mu'} G^{\nu\nu'} F^{(i)}_{\mu\nu} (LML)_{ij} F^{(j)}_{\mu'\nu'} - \frac{a}{S} G^{\mu\mu'} G^{\nu\nu'} F^{(i)}_{\mu\nu} L_{ij} \tilde{F}^{(j)}_{\mu'\nu'}] \qquad i, j = 1, \ldots, 28. \end{aligned} \tag{9.15}$$

The expectation value of the dilaton field S is related to the four-dimensional string coupling g_4

$$S \sim \frac{1}{g_4^2}\,, \tag{9.16}$$

[5] There is an additional dependence on arithmetic T-duality invariants but the degeneracies for states with nontrivial values of these T-duality invariants can be obtained from the degeneracies discussed here by demanding S-duality invariance [22].

and a is the axion field. The metric $G_{\mu\nu}$ is the metric in the string frame and is related to the metric $g_{\mu\nu}$ in Einstein frame by the Weyl rescaling

$$g_{\mu\nu} = SG_{\mu\nu} \tag{9.17}$$

• *BPS spectrum*

Another requirement of duality is that the spectrum of BPS states should match for the two dual theories. Perturbative states in one description will generically get mapped to some non-perturbative states in the dual description. As a result, this leads to highly nontrivial predictions about the nonpertubative spectrum in the dual description given the perturbative spectrum in one description.

As an example, consider the perturbative BPS-states in the heterotic string discussed in the tutorial. A heterotic string wrapping w times on S^1 and carrying momentum n gets mapped in Type-IIA to the NS5-brane wrapping w times on $K3 \times S^1$ and carrying momentum n. One can go from Type-IIA to Type-IIB by a T-duality along the $\tilde{S}^1$ circle. Under this T-duality, the NS5-brane gets mapped to a KK-monopole with monopole charge w associated with the circle $\tilde{S}^1$ and carrying momentum n. This thus leads to a prediction that the spectrum of KK-monopole carrying momentum in Type-IIB should be the same as the spectrum of perturbative heterotic string discussed earlier. We will verify this highly nontrivial prediction in the next subsection for the case of $w = 1$.

9.3 Kaluza-Klein monopole and the heterotic string

The metric of the Kaluza-Klein monopole is given by the so called Taub-NUT metric

$$ds^2_{TN} = \left(1 + \frac{R_0}{r}\right)\left(dr^2 + r^2(d\theta^2 + \sin^2\theta d\phi^2)\right) + R_0^2\left(1 + \frac{R_0}{r}\right)^{-1}(2\,d\psi + \cos\theta d\phi)^2 \tag{9.18}$$

with the identifications:

$$(\theta, \phi, \psi) \equiv (2\pi - \theta, \phi + \pi, \psi + \frac{\pi}{2}) \equiv (\theta, \phi + 2\pi, \psi + \pi) \equiv (\theta, \phi, \psi + 2\pi)\,. \tag{9.19}$$

Here R_0 is a constant determining the size of the Taub-NUT space $\mathcal{M}_{TN}$. This metric satisfies the Einstein equations in four-dimensional Euclidean space. The metric (9.18) admits a normalizable self-dual harmonic form ω, given by

$$\omega^{KK} = \frac{r}{r + R_0}d\sigma_3 + \frac{R_0}{(r + R_0)^2}dr \wedge \sigma_3\,, \qquad \sigma_3 \equiv \left(d\psi + \frac{1}{2}\cos\theta d\phi\right)\,. \tag{9.20}$$

We are interested in the Type-IIB string theory compactified on $K_3 \times \tilde{S}^1 \times S^1$ in the presence of a Kaluza-Klein monopole, with $\tilde{S}^1$ identified with the asymptotic circle of the Taub-NUT space labeled by the coordinate ψ in (9.18). Thus, we want analyze the massless fluctuations of Type-IIB string on $K_3 \times S^1 \times \mathcal{M}_{TN}$ space. Let y and $\tilde{y}$ be the coordinates of S^1 and $\tilde{S}^1$ respectively with $y \sim y + 2\pi R$ and $\tilde{y} \sim \tilde{y} + 2\pi\tilde{R}$. When the radius R of the S^1 is large compared to the size of the $K3$ and the radius $\tilde{R}$ of the $\tilde{S}^1$ circle, we obtain an 'effective string' wrapping the S^1 with massless spectrum that agrees with the massless spectrum of a fundmental heterotic string wrapping S^1. These massless modes can be deduced as follows:

- The center-of-mass of the KK-monopole can be located anywhere in $\mathbb{R}^3$ and its position is specified by a vector $\vec{a}$. Thus, we have

$$r := |\vec{x} - \vec{a}|\,, \quad \cos\theta := \frac{x^3 - a^3}{r}\,, \quad \tan\phi := \frac{x^1 - a^1}{x^2 - a^2}\,. \tag{9.21}$$

  if (x^1, x^2, x^3) are the coordinates of $\mathbb{R}^3$. We can allow these coordinates to fluctuate in the t and y directions and hence we will obtain three non-chiral massless $a^i(t, y)$ scalar fields along the effective string associated with oscillations of the three coordinates of the center-of-mass of the KK monopole.
- There are two additional non-chiral scalar fields $b(t, y)$ and $c(t, y)$ obtained by reducing the two 2-form fields $B^{(2)}$ and C^2 of Type-IIB along the harmonic 2-form (9.20):

$$B^{(2)} = b(t, y) \cdot \omega^{KK} \quad C^{(2)} = c(t, y) \cdot \omega^{KK} \tag{9.22}$$

- There are 3 right-moving $a^r_R(t + y)\,, r = 1, 2, 3$ and 19 left-moving scalars $a^s_L(t - y)\,, s = 1, \ldots, 19$ obtained by reducing the self-dual 4-form field $C^{(4)}$ of type IIB theory. This works as follows. The field $C^{(4)}$ can be reduced taking it as a tensor product of the harmonic 2-form (9.20) and a harmonic 2-form $\omega^{K_3}_\alpha$ for $\alpha = 1, \ldots, 22$ on K_3. This gives rise to rise to a chiral scalar field on the world-volume. The chirality of the scalar field is correlated with whether the corresponding harmonic 2-form $\omega^{K_3}_\alpha$ is self-dual or anti-self-dual. Since $K3$ has three self-dual $\omega^{K_3+}_r$ and nineteen anti-selfdual harmonic 2-forms $\omega^{K_3-}_s$, we get 3 right-moving and 19 left-moving scalars:

$$C^{(4)} = \sum_{r=1}^{3} a^s_R(t + y) \cdot \omega^{K_3-}_s \wedge \omega^{KK} + \sum_{s=1}^{19} a^s_L(t - y) \cdot \omega^{K_3-}_s \wedge \omega^{KK}\,. \tag{9.23}$$

The KK-monopole background breaks 8 of the 16 supersymmetries of Type-II on $K3 \times S^1$. Consequently, there are eight right-moving fermionic fields

$$S^a(t + y) \quad a = 1, \ldots, 8$$

which arise as the goldstinos of these eight broken supersymmetries. This is precisely the field content of the $1+1$ dimensional worldsheet theory of the heterotic string wrapping S^1 as we discussed in the tutorial (8.2).

10. Tutorial IIB: Some calculations using the entropy function

10.1 Entropy of Reisnner-Nordström black holes

Consider the Einstein-Maxell theory given by the action (2.1) and a solution given by

$$\begin{aligned} ds^2 &= v_1\left(-(\sigma^2-1)d\tau^2+\frac{d\sigma^2}{\sigma^2-1}\right)\;+v_2\left(d\theta^2+sin^2(\theta)d\phi^2\right) \\ F_{\sigma\tau} &= e\,, \quad F_{\theta\phi}=\frac{p}{4\pi}\sin\left(\theta\right) \end{aligned} \tag{10.1}$$

Substituting into the action we obtain the entropy function

$$\begin{aligned} \mathcal{E}(q,v,e,q,p) &\equiv 2\pi\left(e_iq_i-f(v,e,p)\right) \\ &= 2\pi\left[eq-\;4\pi v_1\,v_2\left\{\frac{1}{16\pi}\left(-\frac{2}{v_1}+\frac{2}{v_2}\right)+\frac{1}{2v_1^2}e^2-\frac{1}{32\pi^2v_2^2}p^2\right\}\right] \end{aligned} \tag{10.2}$$

The extermization equations

$$\frac{\partial\mathcal{E}}{\partial e}=0\,,\quad \frac{\partial\mathcal{E}}{\partial v_1}=0\,,\quad \frac{\partial\mathcal{E}}{\partial v_2}=0 \tag{10.3}$$

can be easily solved to obtain

$$v_1=v_2=\frac{q^2+p^2}{4\pi}\,,\quad e=\frac{q}{4\pi} \tag{10.4}$$

and

$$S_{wald}(q,p)=\mathcal{E}^*(q,p)=\frac{q^2+p^2}{4}\,. \tag{10.5}$$

10.2 Entropy of dyonic black holes

In this case, the fields near the horizon take the form

$$\begin{aligned} ds^2 &= \frac{v_1}{16}\left(-(\sigma^2-1)d\tau^2+\frac{d\sigma^2}{\sigma^2-1}\right)\;+\frac{v_2}{16}\left(d\theta^2+sin^2\theta d\phi^2\right) \\ F^{(i)}_{\sigma\tau} &= \frac{1}{4}e_i\,,\quad F^{(i)}_{\theta\phi}=\frac{1}{16\pi}p_i\,,\quad M_{ij}=u_{ij},\quad S=u_s,\quad a=u_a\,. \end{aligned} \tag{10.6}$$

Substituting into the action we get

$$f(u_S,u_a,u_M,\vec{v},\vec{e},\vec{p}) \equiv \int d\theta d\phi \sqrt{-\det G}\,\mathcal{L}$$
$$= \frac{1}{8}\,v_1\,v_2\,u_S\left[-\frac{2}{v_1}+\frac{2}{v_2}+\frac{2}{v_1^2}e_i(Lu_ML)_{ij}e_j - \frac{1}{8\pi^2 v_2^2}p_i(Lu_ML)_{ij}p_j + \frac{u_a}{\pi u_S v_1 v_2}e_iL_{ij}p_j\right]. \tag{10.7}$$

Hence the entropy function becomes

$$\begin{aligned}\mathcal{E}(q,u_S,u_a,u_M,v,e,p) &:= 2\pi\left(e_iq_i - f(u_S,u_a,u_M,v,e,p)\right)\\ &= 2\pi\Bigg[e_iq_i - \frac{1}{8}\,v_1\,v_2\,u_S\Bigg\{-\frac{2}{v_1}+\frac{2}{v_2}+\frac{2}{v_1^2}e_i(Lu_ML)_{ij}e_j\\ &\qquad -\frac{1}{8\pi^2v_2^2}p_i(Lu_ML)_{ij}p_j+\frac{u_a}{\pi u_Sv_1v_2}e_iL_{ij}p_j\Bigg\}\Bigg].\end{aligned} \tag{10.8}$$

Eliminating e_i from (10.2) using the equation $\partial\mathcal{E}/\partial e_i = 0$ we get:

$$\begin{aligned}&\mathcal{E}(q,u_S,u_a,u_M,v,e(u,v,q,p),p)\\ &= 2\pi\left[\frac{u_S}{4}(v_2-v_1)+\frac{v_1}{v_2u_S}\,q^Tu_Mq+\frac{v_1}{64\pi^2v_2u_S}(u_S^2+u_a^2)p^TLu_MLp-\frac{v_1}{4\pi v_2u_S}\,u_a\,q^Tu_MLp\right].\end{aligned} \tag{10.9}$$

We can simplify the formulæ by defining new charge vectors:

$$Q_i = 2q_i, \qquad P_i = \frac{1}{4\pi}\,L_{ij}p_j\,. \tag{10.10}$$

In terms of $\vec{Q}$ and $\vec{P}$ the entropy function $\mathcal{E}$ is given by:

$$\mathcal{E} = \frac{\pi}{2}\left[u_S(v_2-v_1)+\frac{v_1}{v_2u_S}\left(Q^Tu_MQ+(u_S^2+u_a^2)\,P^Tu_MP-2\,u_a\,Q^Tu_MP\right)\right]. \tag{10.11}$$

Substituting (10.19) into (10.11) and using (10.15), 10.16, we get:

$$\mathcal{E} = \frac{\pi}{2}\left[u_S(v_2-v_1)+\frac{v_1}{v_2}\left\{\frac{Q^2}{u_S}+\frac{P^2}{u_S}(u_S^2+u_a^2)-2\,\frac{u_a}{u_S}\,Q\cdot P\right\}\right]. \tag{10.12}$$

Note that we have expressed the right hand side of this equation in an T-duality invariant form. Written in this manner, eq.10.12 is valid for general $\vec{P}$, $\vec{Q}$ satisfying

$$P^2>0, \quad Q^2>0, \quad (Q\cdot P)^2<Q^2P^2\,. \tag{10.13}$$

We now need to find the extremum of $\mathcal{E}$ with respect to u_S, u_a, u_{Mij}, v_1 and v_2. In general this leads to a complicated set of equations. We can simplify the analysis by using the $O(22,6;\mathbb{R})$ symmetries (9.11) of the two-derivative action (9.15) which induces the following transformations on the various parameters:

$$\begin{aligned} e_i \to \Omega_{ij} e_j, \qquad p_i \to \Omega_{ij} p_j, \qquad u_M \to \Omega u_M \Omega^T\,, \\ q_i \to (\Omega^T)^{-1}_{ij} q_j\,, \qquad Q_i \to (\Omega^T)^{-1}_{ij} Q_j, \qquad P_i \to (\Omega^T)^{-1}_{ij} P_j\,. \end{aligned} \tag{10.14}$$

The entropy function (10.11) is invariant under these transformations. Since at its extremum with respect to u_{Mij} the entropy function depends only on $\vec{P}$, $\vec{Q}$, v_1, v_2, u_S and u_a it must be a function of the $O(22,6)$ invariant combinations:

$$Q^2 = Q_i L_{ij} Q_j, \quad P^2 = P_i L_{ij} P_j, \quad Q\cdot P = Q_i L_{ij} P_j\,, \tag{10.15}$$

besides v_1, v_2, u_S and u_a. Let us for definiteness take $Q^2>0$, $P^2>0$, and $(Q\cdot P)^2 < Q^2P^2$. In that case with the help of an $SO(22,6)$ transformation we can make

$$(I_r - L)_{ij} Q_j = 0, \quad (I_r - L)_{ij} P_j = 0\,, \tag{10.16}$$

where I_r denotes the $r\times r$ identity matrix. This is most easily seen by diagonalizing L to the form

$$\begin{pmatrix} -I_{22} & 0 \\ 0 & I_6 \end{pmatrix}. \tag{10.17}$$

In this case Q and P satisfying (10.16) will have

$$Q_i = 0, \quad P_i = 0, \quad \text{for } 1\le i\le 22\,. \tag{10.18}$$

Let us now see that for P and Q satisfying this condition, every term in (10.11) is extremized with respect to u_M for

$$u_M = I_r\,. \tag{10.19}$$

Clearly a variation δu_{Mij} with either i or j in the range $[7,r]$ will give vanishing contribution to each term in $\delta\mathcal{E}$ computed from (10.11). On the other hand due to the constraint (9.9) on M, any variation δM_{ij} (and hence δu_{Mij}) with $1\le i,j\le 6$ must vanish, since in this subspace satisfying (9.9) requires M to be both symmetric and orthogonal. Thus each term in $\delta\mathcal{E}$ vanishes under all allowed variations of u_M.

We should emphasize that (10.19) is not the only possible value of u_M that extremizes $\mathcal{E}$. Any u_M related to (10.19) by an $O(22,6)$ transformation that preserves the vectors $\vec{Q}$ and $\vec{P}$ will extremize $\mathcal{E}$. Thus there is a family of extrema representing flat directions of $\mathcal{E}$. However as we have argued in §4, the value of the entropy is independent of the choice of u_M.

It remains to extremize $\mathcal{E}$ with respect to v_1, v_2, u_S and u_a. Extremization with respect to v_1 and v_2 give:

$$v_1 = v_2 = u_S^{-2} \left(Q^2 + P^2(u_S^2 + u_a^2) - 2u_a \, Q \cdot P \right) . \tag{10.20}$$

Substituting this into (10.12) gives:

$$\mathcal{E} = \frac{\pi}{2} \frac{1}{u_S} \left\{ Q^2 - 2 \, u_a \, Q \cdot P + P^2(u_S^2 + u_a^2) \right\} . \tag{10.21}$$

It is convenient to write it in a manifestly $SL(,\mathbb{Z})$ invariant way as

$$\mathcal{E} = \frac{\pi}{2} \frac{1}{\lambda_2} \, |Q + \lambda P|^2 \, . \tag{10.22}$$

if we write $\lambda = u_a + i u_S$.

Finally, extremizing with respect to u_a, u_S we get

$$u_S = \frac{\sqrt{Q^2 P^2 - (Q \cdot P)^2}}{P^2} \, , \qquad u_a = \frac{Q \cdot P}{P^2}, \qquad v_1 = v_2 = 2P^2 \, . \tag{10.23}$$

The black hole entropy, given by the value of $\mathcal{E}$ for this configuration, is

$$S_{BH} = \pi \sqrt{Q^2 P^2 - (Q \cdot P)^2} \, . \tag{10.24}$$

11. Spectrum of quarter-BPS dyons

In this section we will derive the spectrum of quarter-BPS dyons in the simplest string compactification with $\mathcal{N} = 4$ in four spacetime dimensions. Surprisingly, the partition function for counting these dyons turns out to involve interesting mathematical objects called Siegel modular forms which are a natural generalizations for the group $Sp(2, \mathbb{Z})$ of usual modular forms of the group $Sp(1, \mathbb{Z}) \sim SL(2, \mathbb{Z})$.

11.1 Siegel modular forms

Let $Sp(2, \mathbb{Z})$ be the group of (4×4) matrices g with integer entries satisfying $gJg^t = J$ where

$$J \equiv \begin{pmatrix} 0 & -I_2 \\ I_2 & 0 \end{pmatrix} \tag{11.1}$$

is the symplectic form. We can write the element g in block form as

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix}, \tag{11.2}$$

where A, B, C, D are all (2×2) matrices with integer entries. Then the condition $gJg^t = J$ implies

$$AB^t = BA^t, \qquad CD^t = DC^t, \qquad AD^t - BC^t = \mathbf{1}\,, \tag{11.3}$$

Let $\mathbb{H}_2$ be the (genus two) Siegel upper half plane, defined as the set of (2×2) symmetric matrix Ω with complex entries

$$\Omega = \begin{pmatrix} \tau & z \\ z & \sigma \end{pmatrix} \tag{11.4}$$

satisfying

$$\mathrm{Im}(\tau) > 0, \quad \mathrm{Im}(\sigma) > 0, \quad \det(\mathrm{Im}(\Omega)) > 0\,. \tag{11.5}$$

An element $g \in Sp(2, \mathbb{Z})$ of the form (11.2) has a natural action on $\mathbb{H}_2$ under which it is stable:

$$\Omega \to (A\Omega + B)(C\Omega + D)^{-1}. \tag{11.6}$$

The matrix Ω can be thought of as the period matrix of a genus two Riemann surface[6] on which there is a natural symplectic action of $Sp(2, \mathbb{Z})$.

A Siegel form $F(\Omega)$ of weight k is a holomorphic function $\mathbb{H}_2 \to \mathbb{C}$ satisfying

$$F[(A\Omega + B)(C\Omega + D)^{-1}] = \{\det(C\Omega + D)\}^k F(\Omega). \tag{11.7}$$

[6]See [23, 24, 25] for a discussion of the connection with genus-two Riemann surfaces.

A Siegel modular form can be written in terms of its Fourier series

$$F(\Omega) = \sum a(n,r,m)\, q^n y^r p^m\,. \tag{11.8}$$

The Siegel modular form which makes its appearance in the present physics problem of counting $\mathcal{N} = 4$ dyons is the Igusa form Φ_{10} which is the unique (cusp) form[7] of weight 10. This Siegel modular form is a very interesting mathematical object and has a number of useful properties directly relevant for the present physical application. In particular, it can be constructed very explicitly in two different ways in terms of familiar modular forms and theta functions by using two different 'lifts'[8]

- *Additive lift*
  Consier the function $\psi(\tau, z)$

  $$\psi(\tau, z) = \eta^{18}(\tau)\, \vartheta_1^2(\tau, z)\,. \tag{11.9}$$

  which is a weak Jacobi form of weight 1 and index 10 (see §B.2 for definitions). It admits a Fourier expansion

  $$\psi(\tau, z) = \sum_{n,r} c_{10}(n,r) q^n y^r \qquad q := e^{2\pi i\tau}\; y := e^{2\pi i z}\,. \tag{11.10}$$

  From the properties of weak Jacobi forms, it follows that the Fourier coefficients $c_{10}(n,r)$ depend only on the combination $4n - r^2$ and hence we can write $c_{10}(n,r) = C_{10}(4n - r^2)$ for some function C_{10}. The additive lift then gives the Fourier expansion of the Igusa cusp form in terms of the Fourier coefficients of $\psi(\tau, z)$ as

  $$\Phi_{10}(\Omega) = \sum_{n,m,l} a(m,n,l) p^m q^n y^l\,, \quad p := e^{2\pi i\sigma}\,, \tag{11.11}$$

  where $a(m,n,l)$ are defined by

  $$a(n,r,m) = \sum_{\substack{d|(n,r,m)\\ d\geq 1}} d^{k-1} C_{10}(\frac{4mn - r^2}{d^2}) \tag{11.12}$$

  This lift is 'additive' in that it gives a sum representation of the Igusa form.

[7] It is called a 'cusp' form because it vanishes at 'cusps' which correspond to $z = 0$ and its images.

[8] These constructions are called lifts because they allow us to construct the Igusa cusp form which is a function of three variables using the Fourier expansions of a weak Jacobi forms which are functions of only two variables.

- *Multiplicative lift*

  Consider the function $\chi(\tau, z)$

$$\chi(\tau, z) = 8\left(\frac{\vartheta_2(\tau, z)^2}{\vartheta_2(\tau)^2} + \frac{\vartheta_3(\tau, z)^2}{\vartheta_3(\tau)^2} + \frac{\vartheta_4(\tau, z)^2}{\vartheta_4(\tau)^2}\right) , \tag{11.13}$$

  which is weak Jacobi form of weight 0 and index 1 with a Fourier expansion

$$\chi(\tau, z) = \sum_{n,r} c_0(n, l) q^n y^l \qquad q := e^{2\pi i\tau} \ , \ y := e^{2\pi iz} \, . \tag{11.14}$$

  This function arises in physics applications as the elliptic genus of the $K3$ surface (see appendix (C) for details). Once again, $c_0(n, l)$ depend only on the combination $d := 4n - l^2$ and hence we can write

$$c_0(n, l) = C_0(4n - l^2) \tag{11.15}$$

  which defines the function $C_0(d)$. The multiplicative lift gives a product representation of the Igusa cusp form in terms of $C_0(d)$:

$$\Phi_{10}(\Omega) = pqy \prod_{(s,t,r)>0} (1 - p^s q^t y^r)^{C_0(4st - r^2)}, \tag{11.16}$$

  in terms of C_0 given by (C.10, 11.14). Here the notation $(s, t, r) > 0$ means that either $s > 0, t, r \in \mathbb{Z}$, or $s = 0, t > 0, r \in \mathbb{Z}$, or $s = t = 0, r < 0$.

  This lift is 'multiplicative' in that it gives a product representation of the Igusa form.

11.2 Summary of Results

Siegel forms occur naturally in the context of counting of quarter-BPS dyons. The partition function for these dyons depends on three (complexified) chemical potentials (σ, τ, z), conjugate to the three T-duality invariant integers (m, n, ℓ) respectively and is given by

$$\mathrm{Z}(\Omega) = \frac{1}{\Phi_{10}(\Omega)} \, . \tag{11.17}$$

Note that this is very analogous to the case of half-BPS states discussed in the tutorials where the partition function was

$$\mathrm{Z}(\tau) = \frac{1}{\Delta(\tau)} \, . \tag{11.18}$$

was the inverse of a modular form $\Delta(\tau)$ of weight 12 of the group $Sp(1, \mathbb{Z})$.

Given the partition function (11.18), one can extract the black hole degeneracies from the Fourier coefficients. However, there is one complication that also turns out to have interesting physical implications. The Igusa cusp form has double zeros at $z = 0$ and its images. The partition function is therefore a *meromorphic* Siegel form (11.7) of weight -10 with double poles at these divisors. As a result, different Fourier contours would give different answers for the degeneracies and there appears to be an ambiguity in the choice of the Fourier contour.

This ambiguity turns out to have a very nice physical interpretation. The spectrum of quarter-BPS dyons actually has a moduli dependence. For a given charge vector Γ, there are single-centered black hole solutions that exist everywhere in the moduli space. However, in addition, there can be two-centered solutions such that one center carries charge Γ_1 and the other Γ_2 with $\Gamma = \Gamma_1 + \Gamma_2$. A simple example is when one charge center has charge $(Q, 0)$ and the other has charge $(0, P)$. The distance between these two centers is fixed in terms of the charges and the moduli fields.

As one changes the moduli, the distance between the two centers can go to infinity and the two-centered solution can decay at certain walls *i. e.* surfaces of co-dimension one. Thus, on one side of the wall, we have only a single-centered black hole whereas on the other side one has the single-centered black hole as well as the two-centered black hole. Hence the degeneracy on one side of the wall is different from the degeneracy on the other side of the all. Upon crossing the wall, the degeneracy jumps. This phenomenon is known as the 'wall- crossing phenomenon'. The moduli space is thus divided up into chambers separated by walls. The degeneracy is different from chamber to chamber.

This dependence of the degeneracy on the chamber in the moduli space is nicely captured by the dependence of the Fourier coefficients on the choice of the contour. As we will explain below, the choice of the contour depends on the moduli in a precise way. As the moduli are varied, the contour is deformed. The dependence of the contour on the moduli is such that as the moduli hit a wall in the moduli space, the contour hits a pole of the partition function. The poles are thus nicely correlated with the walls. Crossing the wall in the moduli space corresponds to crossing a pole in the contour space. The jump in the degeneracy upon crossing the wall is given by the residue at the pole that is crossed by the contour.

To see this more precisely, note that the three quadratic T- duality invariants of a given dyonic state can be organized as a 2×2 symmetric matrix

$$\Lambda = \begin{pmatrix} Q \cdot Q & Q \cdot P \\ Q \cdot P & P \cdot P \end{pmatrix} = \begin{pmatrix} 2n & \ell \\ \ell & 2m \end{pmatrix}, \tag{11.19}$$

where the dot products are defined using the $O(22, 6; \mathbb{Z})$ invariant metric L. The matrix

Ω in (11.18) and (11.4) can be viewed as the matrix of complex chemical potentials conjugate to the charge matrix Λ. The charge matrix Λ is manifestly T-duality invariant. Under an S-duality transformation (9.4), it transforms as

$$\Lambda \to \gamma \Lambda \gamma^t \tag{11.20}$$

There is a natural embedding of this physical S-duality group $SL(2,\mathbb{Z})$ into $Sp(2,\mathbb{Z})$:

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix} = \begin{pmatrix} (\gamma^t)^{-1} & \mathbf{0} \\ \mathbf{0} & \gamma \end{pmatrix} = \begin{pmatrix} d & -c & 0 & 0 \\ -b & a & 0 & 0 \\ 0 & 0 & a & b \\ 0 & 0 & c & d \end{pmatrix} \in Sp(2,\mathbb{Z}). \tag{11.21}$$

The embedding is chosen so that $\Omega \to (\gamma^T)^{-1}\Omega\gamma^{-1}$ and $\mathrm{Tr}(\Omega \cdot \Lambda)$ in the Fourier integral is invariant. This choice of the embedding ensures that the physical degeneracies extracted from the Fourier integral are S-duality invariant if we appropriately transform the moduli at the same time as we explain below.

To specify the contours, it is useful to define the following moduli-dependent quantities. One can define the matrix of right-moving T-duality invariants

$$\Lambda_R = \begin{pmatrix} Q_R \cdot Q_R & Q_R \cdot P_R \\ Q_R \cdot P_R & P_R \cdot P_R \end{pmatrix} . \tag{11.22}$$

which depends both on the integral charge vectors N, M as well as the T-moduli μ One can then define two matrices naturally associated to the S-moduli $\lambda = \lambda_1 + i\lambda_2$ and the T- moduli μ respectively by

$$\mathcal{S} = \frac{1}{\lambda_2}\begin{pmatrix} |\lambda|^2 & \lambda_1 \\ \lambda_1 & 1 \end{pmatrix} \quad , \quad \mathcal{T} = \frac{\Lambda_R}{|\det(\Lambda_R)|^{\frac{1}{2}}} . \tag{11.23}$$

Both matrices are normalized to have unit determinant. In terms of them, we can construct the moduli-dependent 'central charge matrix'

$$\mathcal{Z} = |\det(\Lambda_R)|^{\frac{1}{4}} \big(\mathcal{S} + \mathcal{T}\big), \tag{11.24}$$

whose determinant equals the BPS mass

$$M_{Q,P} = |\det \mathcal{Z}|. \tag{11.25}$$

We define

$$\tilde{\Omega} \equiv \begin{pmatrix} \sigma & -z \\ -z & \tau \end{pmatrix} \tag{11.26}$$

related to Ω by an $SL(2,\mathbb{Z})$ transformation

$$\tilde{\Omega} = \hat{S}\Omega\hat{S}^{-1} \quad where \quad \hat{S} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \tag{11.27}$$

so that, under a general S-duality transformation γ, we have the transformation $\tilde{\Omega} \to \gamma\tilde{\Omega}\gamma^T$ as $\Omega \to (\gamma^T)^{-1}\Omega\gamma^{-1}$.

With these definitions, $\Lambda, \Lambda_R, \mathcal{Z}$ and $\tilde{\Omega}$ all transform as $X \to \gamma X \gamma^T$ under an S-duality transformation (9.4) and are invariant under T-duality transformations. The moduli-dependent Fourier contour can then be specified in a duality-invariant fashion by [26]

$$\mathcal{C} = \{\mathrm{Im}\tilde{\Omega} = \varepsilon^{-1}\mathcal{Z}; \quad 0 \le \mathrm{Re}(\tau), \mathrm{Re}(\sigma), \mathrm{Re}(z) < 1\}, \tag{11.28}$$

where $\varepsilon \to 0^+$. For a given set of charges, the contour depends on the moduli λ, μ through the definition of the central charge vector (11.24). The degeneracies $d(m,n,l)|_{\lambda,\mu}$ of states with the T-duality invariants (m,n,l), at a given point (λ,μ) in the moduli space are then given by[9]

$$d(m,n,l)|_{\lambda,\mu} = \int_{\mathcal{C}} e^{-i\pi \mathrm{Tr}(\Omega\cdot\Lambda)}\ \mathrm{Z}(\Omega)\, d^3\Omega\,. \tag{11.29}$$

This contour prescription thus specifies how to extract the degeneracies from the partition function for a given set of charges and in any given region of the moduli space. In particular, it also completely summarizes all wall-crossings as one moves around in the moduli space for a fixed set of charges. Even though the indexed partition function has the same functional form throughout the moduli space, the spectrum is moduli dependent because of the moduli dependence of the contours of Fourier integration and the pole structure of the partition function. Since the degeneracies depend on the moduli *only* through the dependence of the contour $\mathcal{C}$, moving around in the moduli space corresponds to deforming the Fourier contour.

With this understanding of the wall crossing and the contour prescription, we have completely specified how to extract dyon degeneracies from the Fourier coefficients of the partition function. The partition function in turn is constructed explicitly in terms of Fourier coefficients of known objects such as ψ or χ. We will not here analyze wall-crossing in any further detail which can be found in [20, 27, 26].

To summarize, given the partition function the degeneracies are extracted as above. It remains to derive the partition function. The the logic of the derivation is as follows:

[9]The physical degeneracies have an additional multiplicative factor of $(-1)^{\ell+1}$ which we omit here for simplicity of notation in later chapaters.

1. We derive the degeneracy for a special charge configuration in one corner of the moduli space.
2. Using constraints from wall-crossing, we extend this answer for the same set of charges to all over the moduli space.
3. Using duality symmetries, we extend this answer to all possible values of charges.

With this general strategy in mind, we turn to the derivation of the dyon partition function for a special representative set of charges in a certain weakly coupled region of the moduli space.

12. Derivation of the microscopic partition function

The product representation of the Igusa form is particularly useful for the physics application because it is closely related to the generating function for the elliptic genera of symmetric products of $K3$ introduced earlier. This is a consequence of the fact that the multiplicative lift of the Igusa form is obtained starting with the elliptic genus of a single copy $K3$ as the input. The generating function for the elliptic genera of symmetric products of K3 is defined by

$$\widehat{Z}(\sigma, \tau, z) := \sum_{m=-1}^{\infty} \chi_{m+1}(\tau, z) p^m \tag{12.1}$$

where $\chi_m(\tau, z)$ is the elliptic genus of $\mathrm{Sym}^m(K3)$ with $\chi_0(\tau, z) \equiv 1$ and $\chi_1(\tau, z) \equiv \chi(\tau, z)$. A standard orbifold computation [28] gives

$$\widehat{Z}(\sigma, \tau, z) = \frac{1}{p} \prod_{s>0, t\geq 0, l} \frac{r}{(1 - p^s q^t y^r)^{C_0(4st - r^2)}} \tag{12.2}$$

in terms of the Fourier coefficients C_0 of the elliptic genus of a single copy of $K3$. As we will explain in the next section, this partition function captures the degeneracies of five-dimensional Strominger-Vafa black holes with charges D1-D5-P.

Comparing the product representation for the Igusa form (11.16) with (12.2), we get the relation:

$$\mathrm{Z}(\Omega) = \frac{1}{\Phi_{10}(\sigma, \tau, z)} = \frac{\widehat{Z}(\sigma, \tau, z)}{\psi(\tau, z)} . \tag{12.3}$$

This relation of the Igusa form to the elliptic genera of symmetric products of $K3$ and the degeneracies of five-dimensional D1-D5-P black holes has a deeper physical significance and allows for a microscopic derivation of the counting formula as we explain below.

12.1 A representative charge configuration

Consider four-dimensional BPS-states in Type IIB on $K3 \times S^1 \times \tilde{S}^1$ with the following charge configuration:

- 1 KK-monopole associated with the circle $\tilde{S}^1$.
- 1 D5-branes wrapping $K3 \times S^1$
- m D1-branes wrapping S^1
- n units of momentum along the circle S^1
- l units of momentum along the circle $\tilde{S}^1$

We would like to compute $d(m, n, l)$ which is the number of quantum states with these quantum numbers counting bosons with +1 and fermions with -1. Let F be the spacetime fermion number then we could try to compute

$$\operatorname*{Tr}_{m,n,l} \left[(-1)^F\right]. \tag{12.4}$$

However, this vanishes. If a state breaks $2n$ supersymmetries, then it has $2n$ real fermion zero modes which are the Goldstinoes of the broken symmetry. Quantization of each pair leads to Bose-Fermi degeneracy so the trace above vanishes. This can be remedied by inserting $(2h)^n$ where h is the 'helicity', that is, the third component of angular momentum in the rest frame. For states paired by a complex fermion the effect of this insertion is to 'soak up' the fermion zero mode since this mode has spin half. Thus, we compute

$$d(m, n, l) = \operatorname*{Tr}_{m,n,l} \left[(-1)^F (2h)^6\right] \tag{12.5}$$

since for a quarter-BPS state, out of the 16 supersymmetries 12 are broken. In practice, this means we just ignore the 12 fermionic zero modes from broken supersymmetry and evaluate simply $\mathrm{Tr}(-1)^F$ over the remaining modes. The index thus defined receives contribution only from the BPS states.

It turns out that we can relate these unknown degeneracies $d(m, n, l)$ of 4d-states to known degeneracies of the D1-D5-P configuration in five dimensions which are much easier to compute. This is known as the 4d-5d lift [29]. The main idea is to use the fact that the geometry of the Kaluza-Klein monopole (9.18) in the charge configuration above asymptotes to $\mathbb{R}^3 \times \tilde{S}^1$ at asymptotic infinity $r \to \infty$ but reduces to flat Euclidean space $\mathbb{R}^4$ near the core of the monopole at $r \to 0$. Thus at asymptotic infinity we have a KK-monopole in four-dimensional flat Minkowski spacetime which near the core looks like a five-dimensional flat Minkowski spacetime. Our charge configuration then reduces essentially to the *five-dimensional* Strominger-Vafa black hole [12] with angular momentum [30] discussed in the previous subsection.

Our strategy will be to compute the grand canonical partition function introducing chemical potentials (σ, τ, z) conjugate to the charges (m, n, l) and the 'fugacities'

$$p := e^{2\pi i\sigma}\,, \quad q := e^{2\pi i\tau}\,, \quad y := e^{2\pi i z}\,. \tag{12.6}$$

The partition function is then

$$Z(\sigma, \tau, z) = \sum_{m,n,l} p^m q^n y^l (-1)^l \, d(m, n, l)\,. \tag{12.7}$$

The factor of $(-1)^l$ is introduced for convenience which can be absorbed by $z \to z+1/2$.

Since $d(m, n, l)$ is a topological quantity protected from quantum corrections, the dyon partition function it does not depend on the coupling or the moduli such as the radius $\tilde{R}$. We can focus on the region near the core by taking the radius of the circle $\tilde{S}^1$ goes to infinity so that in this limit we have a weakly coupled problem. In this limit, the charge l corresponding to the momentum around this circle gets identified with the angular momentum l in five dimensions. The total partition function at weak coupling at large radius $\tilde{R}$ is thus a product of three factors

$$Z(\Omega) = Z_{D1}(p, q, y)\, Z_{KK}(q)\, Z_{CM}(q, y)\,. \tag{12.8}$$

The three factors arise as follows.

1. The factor $Z_{D1}(\sigma, \tau, z)$ counts the bound states of the D1-brane bound to a single D5-brane, carrying arbitrary momentum and angular momentum.
2. The factor $Z_{KK}(\tau)$ counts the bound states of momentum n with the Kaluza-Klein monopole. The KK-monopole cannot carry any momentum along the $\tilde{S}^1$ directions nor does it carry any D1-brane charge. Hence the partition function depends only τ.
3. The factor $Z_{CM}(\tau, z)$ counts the bound states of the center of mass motion of the Strominger-Vafa black hole in the Kaluza-Klein geometry [23, 31]. It carries no D1-brane charge and hence depends only τ and z.

At weak coupling, these three systems reduce to decoupled bosonic and fermionic oscillators and our computation is reduced to something very similar to the warm-up exercise. Each oscillator carries certain quantum numbers (s, t, r) which can contribute to the total charge (m, n, l) of our interest. Each bosonic oscillator contributes

$$\sum_{k=0}^{\infty} e^{2\pi i k(s\sigma, t\tau, rz} = \left(1 - p^s q^t y^r\right)^{-1}\,. \tag{12.9}$$

Each fermionic oscillator contributes

$$\sum_{k=0}^{1} e^{2\pi i k(s\sigma,t\tau,rz}(-1)^k = \big(1 - p^s q^t y^r\big) \tag{12.10}$$

where the $(-1)^k$ is present because of $(-1)^F$. The partition function will be thus of the general form

$$Z(\Omega) \sim \prod_{s,t,r} \frac{r}{(1 - p^s q^t y^r)^{f(s,t,r)}}\,, \tag{12.11}$$

where $f(s,t,r)$ is the difference between the number of bosonic oscillators and the number of fermionic oscillators for given charges (s,t,r) . All physics is now contained in these numbers. In the remaining subsections we discuss systematically various contribution to the partition function to determine $f(s,t,r)$ for our system.

12.2 Motion of the D1-brane relative to the D5-brane

As a warm up, let us first consider D1-brane (or fundamental Type-II string) in flat space wrapped around a circle S^1 or radius R with coordinate $y \sim y + 2\pi R$. The fluctuations of the D1-brane consists of 8 transverse bosons $\phi^i(t,y)$ as well as 8 left-chiral fermions $S^a(t+y)$ and 8 right-chiral fermions $\tilde{S}^a(t-y)$ where t is the time coordinate, $i = 1,\ldots,8$, and $a = 1,\ldots,8$. The fluctuations are of the form

$$\phi^i(t,y) = \phi^i_0 + p^i_0 t + \sum_{n>0} \phi^i_n e^{-\frac{n}{R}(t-y)} + \sum_{n>0} \tilde{\phi}^i_n e^{-\frac{n}{R}(t+y)} + c.c. \tag{12.12}$$

For the fermions we have similarly

$$S^a(t-y) = \sum_{n>0} S^a_n e^{-\frac{n}{R}(t-y)} + c.c. \tag{12.13}$$

$$\tilde{S}^a(t+y) = \sum_{n>0} \tilde{S}^a_n e^{-\frac{n}{R}(t+y)} + c.c. \tag{12.14}$$

We can quantize this system as usual. Then ϕ^i_n and $\tilde{\phi}^i_n$ are bosonic oscillators with frequencies n/R and occupation numbers N^i_n and $\tilde{N}^i_n$ respectively. Similarly, S^a_n and $\tilde{S}^a_n$ are fermionic oscillators with frequencies n/R and occupation numbers M^i_n and $\tilde{M}^i_n$ respectively. The total left-moving momentum along S^1 is

$$P = \frac{1}{R}\sum_{i=1}^{8}\sum_{n=1}^{\infty} n(N^i_n - \tilde{N}^i_n) + \frac{1}{R}\sum_{a=1}^{8}\sum_{n=1}^{\infty} n(M^a_n - \tilde{M}^a_n) \tag{12.15}$$

and the total energy is

$$E = \frac{1}{R}\sum_{i=1}^{8}\sum_{n=1}^{\infty} n(N^i_n + \tilde{N}^i_n) + \frac{1}{R}\sum_{a=1}^{8}\sum_{n=1}^{\infty} n(M^a_n + \tilde{M}^a_n) \tag{12.16}$$

To obtain a BPS state we want to minimize the energy given fixed momentum P. This implies

$$\tilde{N}_n^i = 0\,, \quad \tilde{M}_n^i = 0 \quad E = P\,. \tag{12.17}$$

We would like to know how many BPS states there are for a given charge P. This is a combinatorial problem of finding $d(P)$ which is the number of ways to choose a set of integers $\{N_n^i, M_n^a\}$ satisfying the constraint

$$\frac{1}{R}\left(\sum_{n=1}^{\infty}\left(\sum_{i=1}^{8} nN_n^i + \sum_{a=1}^{8}\; n(M_n^a\right)\right) = P\,. \tag{12.18}$$

As usual it is easier to pass to the canonical ensemble. computing

$$Z(\tau) := \sum_{\{N_n^i, M_n^a\}} q^N \equiv \sum_P d(N) q^N\,, \quad q := e^{2\pi i\tau}\,, \tag{12.19}$$

ignoring the constraint. Here we have use for convenience $N = RP$ which is an integer or equivalently can absorb R into τ. One can then obtain $d(N)$ by inverse Laplace transform using

$$Z(\tau) := \sum_P d(N) q^N\,, \quad d(N) = \int_0^1 e^{-2\pi i N\tau} Z(\tau) d\tau\,. \tag{12.20}$$

The partition function is readily evaluated and is given by

$$Z(\tau) = \frac{\prod_{n=1}^{\infty}(1+q^n)^8}{\prod_{n=1}^{\infty}(1-q^n)^8} \tag{12.21}$$

From this one can find that

$$d(N) \sim e^{2\pi\sqrt{2N}}\,, \tag{12.22}$$

which follows also from the Cardy formula using the fact that for 8 free bosons and 8 free fermions the central charge is 12.

After this warm-up exercise, let us turn to the problem of motion of m D1-branes bound to a single D5-brane. Now, *a priori* the D1-brane can again oscillate in all 8 transverse directions. However, if we switch on a 2-form field along 2-cycles of $K3$, then open strings connecting D1-branes and D5-branes become tachyonic. Condensation into ground state binds the D1-branes to the D5-branes and as a result they can oscillate only along the directions along the $K3$.

We are interested in a configuration with m units of D1-brane charge n units of momentum, and l units of angular momentum. If m is divisible by s then we have to consider both the configuration with m D1-branes winding number 1 as well as the configuration with m/s D1-branes with winding number s. Similarly, the momentum and angular momentum can be shared among these m or m/s D1-branes. As usual, it is more convenient to relax all constraints on the charges and compute instead the canonical partition function. So, we introduce chemical (complexified) chemical potentials σ, τ, z conjugate to the integers m, n, l and compute the unrestricted sum by summing over all possible charges (r, s, t). The degeneracies $d_{D1}(m, n, l)$ can then be extracted by an inverse Fourier transform.

Consider a D1-brane wound r times along the S^1, carrying momentum s along the S^1 with angular momentum $J_L = t/2$. Let

$$Z_{D1} = \frac{1}{p} \prod_{s>0, t\geq 0, l} \frac{r}{(1 - p^s q^t y^r)^{c(s,t,r)}} \,. \tag{12.23}$$

Now, a D1-brane wrapping s times around a circle R is like a D1-brane wrapping once on a circle of effective radius $R_e = 2\pi R s$. If we want it to carry physical momentum t, then since

$$\frac{t}{R} = \frac{ts}{nR} = \frac{ts}{R_e} \tag{12.24}$$

Because of conformal invariance, the partition function does not depend on the overall scale R. We thus conclude that the partition function for winding s and physical momentum t is the same as the partition function for winding 1 and physical momentum st. In other words,

$$c(s, t, r) = c_0(st, r) \,. \tag{12.25}$$

These coefficients are nothing but the $c_0(n, l)$ defined in (11.14) of the elliptic genus $\chi(\tau, z)$ of a single copy of K_3. Hence $c(s, t, r) = c_0(st, r) = C_0(4st - r^2)$ from (11.15). Indeed, our computation of Z_{D1} is one way to derive the generating function $\hat{Z}$ for the elliptic genera of symmetric products of $K3$. In summary,

$$Z_{D1}(\sigma, \tau, z) = \hat{Z}(\sigma, \tau, z) \,. \tag{12.26}$$

Comment: The problem of counting microstates of m D1-branes bound to a D5-brane is the counting problem that arises in computing the microstates of the well-known Strominger-Vafa black hole in five dimensions. The microscopic configuration there consists of Q_5 D5-branes wrapping $K3 \times S^1$, Q_1 D1-branes wrapping the S^1, with total momentum n along the circle. We have chosen $Q_5 = 1$ and $Q_1 = m$ but more generally, we can simply replace m by $Q_1 Q_5$. The bound states are described by an effective string wrapping the circle carrying left-moving momentum n. The central charge of the system can be computed at weak coupling and is given by $6m$. Applying Cardy's formula then gives

$$d_m(n) = \exp(2\pi \sqrt{mn}). \tag{12.27}$$

This implies a microscopic entropy $S = \log d = 2\pi\sqrt{Q_1 Q_5 n}$. The corresponding BPS black hole solutions with three charges in five dimensions can be found in supergravity and the resulting entropy matches precisely with the macroscopic entropy.

12.3 Dynamics of the KK-monopole

In the previous subsection we have worked out the low-energy massless fluctuations of the KK-monopole. If we excite only the left-movers then we have 24 bosons carrying momentum t. The KK-monopole cannot support any momentum along $theS^1$ circle. Summing over all momenta gives rise to the partition function

$$Z_{KK}(\tau) = \frac{1}{q}\prod_{t=1}^{\infty}\frac{1}{(1-q^t)^{24}} = \frac{1}{\eta^{24}(\tau)} \tag{12.28}$$

The factor of $1/q$ comes because the ground state carries some 'zero point' momentum -1. Altogether, we recognize this as precisely the partition function of the left-moving BPS oscillations of the heterotic string as expected from duality.

12.4 D1-D5 center-of-mass oscillations in the KK-monopole background

Now it remains for us to find the contribution to the partition function from the oscillations of the center of mass of the D1-D5 system moving in the background the KK-monopole. This is easy to evaluate using the fact that for large radius near the center of the KK-monopole, the Taub-NUT space is essentially flat Euclidean space $\mathcal{R}^4$. The partition function of four bosons and four fermions is simply

$$Z_{CM}(\tau, z) = \frac{\eta^6(\tau)}{\theta_1^2(\tau, z)}\,. \tag{12.29}$$

Putting this all together we find the desired result

$$Z(\Omega) = \frac{\hat{Z}(\sigma, \tau, z)}{\psi(\tau, z)} = \frac{1}{\Phi_{10}(\Omega)}\,. \tag{12.30}$$

13. Comparison of entropy and degeneracy

Now we turn to the comparison of the macroscopic entropy with microscopic degeneracy both to the leading order and to the first subleading order.

13.1 Subleading corrections to the Wald entropy

We have already computed the leading order entropy in the tutorial (10.2). We would now like to see how to take the effects of higher order corrections. Let us suppose the Lagrangian is of the form

$$\mathcal{L} = \mathcal{L}_0 + \epsilon\mathcal{L}_1\,, \tag{13.1}$$

where the term of order ϵ is a small correction from higher-derivative terms. The entropy function defined using this Lagrangian will also be of the form

$$\mathcal{E} = \mathcal{E}_0 + \epsilon\mathcal{E}_1\,. \tag{13.2}$$

The solutions of the extremization equations will also have an expansion

$$\begin{aligned} &e^*(q,p) = e^*_{(0)} + \epsilon e^*_{(1)} + \dots\,; \\ &u^*(q,p) = u^*_{(0)} + \epsilon u^*_{(1)} + \dots\,; \quad v^*(q,p) = v^*_{(0)} + \epsilon v^*_{(1)} + \dots \quad . \end{aligned} \tag{13.3}$$

To compute the entropy we have to compute the value of the entropy function $\mathcal{E}^*$ at the extermum

$$\mathcal{E}^*(q,p) = \mathcal{E}_0(q,u^*,v^*,e^*,p) + \epsilon\mathcal{E}_1(q,u^*,v^*,e^*,p)\,. \tag{13.4}$$

If we are interested in the first subleading correction to order ϵ we simply expand these functions to obtain

$$\mathcal{E}^*(q,p) = \mathcal{E}_0(q,u^*_0,v^*_0,e^*_0,p) + \epsilon\mathcal{E}_1(q,u^*_0,v^*_0,e^*_0,p) + O(\epsilon^2)\,. \tag{13.5}$$

The important point is that to $O(\epsilon)$ one could have had terms like

$$\frac{\partial\mathcal{E}_0}{\partial e}\,, \quad \frac{\partial\mathcal{E}_0}{\partial v}\,, \quad \frac{\partial\mathcal{E}_0}{\partial u}\,, \tag{13.6}$$

evaluated at the leading order extremum values u^*_0, v^*_0, e^*_0. However, these all vanish because to the leading order, the extremum values of near horizon fields are found precisely by setting all terms in (13.6) to zero. Hence, to find the first subleading correction, it is not necessary to solve the extermization equations all over again. It suffices to evaluate the correction to the entropy $\mathcal{E}_1$ at the extremum values found using the zeroth order entropy function $\mathcal{E}_0$. This greatly simplify practical computations.

To illustrate these ideas, we apply them to the heterotic action for the dyonic black holes of our interest. The heterotic supergravity action (9.15) is only the leading 2-derivative supergravity approximation to the full string effective action. The theory has a 4-derivative correction to the effective action given by the lagrangian

$$\Delta\mathcal{L} = \phi(\lambda, \bar{\lambda}) \left(R_{\mu\nu\alpha\beta} R^{\mu\nu\alpha\beta} - 4 R_{\mu\nu} R^{\mu\nu} \right) , \tag{13.7}$$

where $\phi(\lambda, \bar{\lambda})$ is a nontrivial function of axion-dilaton $\lambda := a + iS$:

$$\phi(\lambda, \bar{\lambda}) = -\frac{1}{64\pi^2} \left[12 \log(S) + 24 \log\left(\eta(a - iS)\right) + 24 \log\left(\eta(a + iS)\right) \right] \tag{13.8}$$

It is easy to check that this induces a correction to the entropy function of the form

$$\mathcal{E}_1 = 64\pi^2 \phi(\lambda, \bar{\lambda}) \, . \tag{13.9}$$

Consequently, the Wald entropy corrected to this order is then given by

$$S_{wald} = \pi\sqrt{Q^2P^2 - (Q\cdot P)^2} + 64\pi^2 \phi \left(a = \frac{Q\cdot P}{P^2} \, , \, S = \frac{\sqrt{Q^2P^2 - (Q\cdot P)^2}}{P^2} \right) + \ldots \tag{13.10}$$

13.2 Asymptotic expansion of the microscopic degeneracy

Given the exact formula for the degeneracies, one can try to extract the asymptotic degeneracies in the limit with m, n are both large and positive. Since the Fourier integral now involves three variables, the calculation is more involved than the Cardy formula that we encountered for modular forms of single variable. The answer however is simple. The statistical entropy $\log(d)$ is obtained by minimizing the following function with respect to λ

$$\mathcal{E}_B(\lambda) = \frac{\pi}{2\lambda_2} \left| Q + \lambda P \right|^2 - 64\pi^2 \phi(\lambda, \bar{\lambda}) + O(Q^{-2}) \, , \tag{13.11}$$

where ϕ is the same function introduced in (13.8). As a result the statistical entropy matches beautifully with the thermodynamic Wald entropy given by (13.10). We should emphasize that the origin of the function ϕ in the two computations is of totally different origin. In the computation of the Wald entropy $S_{wald}(Q, P)$ it arises from specific terms in the effective action of massless fields in string theory. In the computation of the statistical entropy $\log d(Q, P)$, on the other hand, it arises from the asymptotic expansion of the Fourier coefficients of the partition function for quarter-BPS dyons which for some reason is related the Igusa cusp form. This thus points to a highly nontrivial internal consistency in the structure of string theory and gives us some confidence that we may be on the right track in the search for a quantum theory of gravity.

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A. $\mathcal{N} = 4$ supersymmetry

We summarize here some facts about the representation of the $\mathcal{N} = 4$ superalgebra. For more details see REFWessBagger.

A.1 Massless supermultiplets

There are two massless representations that will be of interest to us.

1. Supergravity multiplet:
   It contains the metric $g_{\mu\nu}$, six vectors $A_\mu^{(ab)}$, and two gravitini $\psi^a_{\mu\alpha}$.

2. Vector Multiplet:
   It contains a vector A_μ, six scalar fields $X^{(ab)}$, and the gaugini χ^a_α,

The low energy massless spectrum of a supergravity theory consists of the supergravity multiplet and n_v vector multiplets. Supersymmetry then completely fixes the form of the two derivative action. The compactification of heterotic string theory on T^6 leads to a theory in four spacetime dimensions with $\mathcal{N} = 4$ supersymmetry and 28 abelian gauge fields which corresponds to $28 - 6 = 22$ vector multiplets.

A.2 General BPS representations

In the rest frame of the dyon, the $\mathcal{N} = 4$ supersymmetry algebra takes the form

$$\{Q^a_\alpha, Q^{\dagger b}_{\dot\beta}\} = M\delta_{\alpha\dot\beta}\,\delta^{ab}\ , \qquad \{Q^a_\alpha, Q^b_\beta\} = \epsilon_{\alpha\beta}Z^{ab}\ , \qquad \{Q^{\dagger a}_{\dot\alpha}, Q^{\dagger b}_{\dot\beta}\} = \epsilon_{\dot\alpha\dot\beta}\bar{Z}^{ab} \tag{A.1}$$

where $a, b = 1, \ldots 4$ are $SU(4)$ R-symmetry indices and α, β are Weyl spinor indices. In a given charge sector, the central charge matrix encodes information about the charges and the moduli. To write it explicitly, we first define a central charge vector in $\mathcal{C}^6$

$$Z^m(\Gamma) = \frac{1}{\sqrt{\tau_2}}(Q^m_R - \tau P^m_R)\ , \qquad m = 1, \ldots 6\ , \tag{A.2}$$

which transforms in the (complex) vector representation of $Spin(6)$. Using the equivalence $Spin(6) = SU(4)$, we can relate it to the antisymmetric representation of Z_{ab} by

$$Z_{ab}(\Gamma) = \frac{1}{\sqrt{\tau_2}}(Q_R - \tau P_R)^m\lambda^m_{ab}\ , \qquad m = 1, \ldots 6 \tag{A.3}$$

where λ^m_{ab} are the Clebsch-Gordon matrices. Since $Z(\Gamma)$ is antisymmetric, it can be brought to a block-diagonal form by a $U(4)$ rotation

$$\tilde{Z} = UZU^T,\ \ U \in U(4)\ , \qquad \tilde{Z}_{ab} = \left(\begin{array}{c|c} Z_1\varepsilon & 0 \\ \hline 0 & Z_2\varepsilon \end{array}\right)\ ,\quad \varepsilon = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \tag{A.4}$$

where Z_1 and Z_2 are non-negative real numbers. A $U(2)$ rotation in the 12 plane and another $U(2)$ rotation in the 34 plane will not change the block diagonal form. Since ε is the invariant tensor of $SU(2)$, the $U(2) \times U(2)$ transformation can only change independently the phases of Z_1 and Z_2. We will therefore treat more generally Z_1 and Z_2 as complex numbers.

We now split the $SU(4)$ index as $a = (r, i)$, where $r, i = 1, 2$ and i represents the block number. Defining the following fermionic oscillators

$$\mathcal{A}^i_\alpha = \frac{1}{\sqrt{2}}(\mathcal{Q}^{1i}_\alpha + \epsilon_{\alpha\beta}\mathcal{Q}^{\dagger\, 2i}_\beta), \quad \mathcal{B}^i_\alpha = \frac{1}{\sqrt{2}}(\mathcal{Q}^{1i}_\alpha - \epsilon_{\alpha\beta}\mathcal{Q}^{\dagger\, 2i}_\beta)\ , \qquad \mathcal{Q}^a = U^a_b\, Q^b \tag{A.5}$$

the supersymmetry algebra takes the form

$$\{\mathcal{A}^{i\dagger}_{\dot\alpha}, \mathcal{A}^j_\beta\} = (M + Z_i)\, \delta_{\dot\alpha\beta}\, \delta^{ij}\ , \qquad \{\mathcal{B}^{i\dagger}_{\dot\alpha}, \mathcal{B}^j_\beta\} = (M - Z_i)\, \delta_{\dot\alpha\beta}\, \delta^{ij} \tag{A.6}$$

with all other anti-commutators being zero.

Let us conclude by giving an explicit representation for λ^m_{ab}. An $SU(4)$ rotation which rotates the supercharges, $Q' = UQ$, acts on the Clebsch-Gordon matrices as

$$U\lambda^m U^T = R^m{}_n(U)\lambda^m \tag{A.7}$$

where $R^m{}_n$ is an $SO(6)$ rotation matrix. The Clebsch-Gordon matrices λ^m_{ab} are given by the components $(C\Gamma^m)_{ab}$ where Γ^m are the Dirac matrices of $Spin(5)$ in the Weyl basis satisfying the Clifford algebra $\{\Gamma^m, \Gamma^n\} = 2\delta^{mn}$, and C is the charge conjugation matrix. The Gamma matrices are given explicitly in terms of Pauli matrices by

$$\Gamma^1 = \sigma_1 \times \sigma_1 \times 1\ , \quad \Gamma^4 = \sigma_2 \times 1 \times \sigma_1 \tag{A.8}$$

$$\Gamma^2 = \sigma_1 \times \sigma_2 \times 1\ , \quad \Gamma^5 = \sigma_2 \times 1 \times \sigma_2 \tag{A.9}$$

$$\Gamma^3 = \sigma_1 \times \sigma_3 \times 1\ , \quad \Gamma^6 = \sigma_2 \times 1 \times \sigma_3, \tag{A.10}$$

where the The charge conjugation matrix is defined by $C\Gamma^m C^{-1} = -\Gamma^{m*}$

$$C = \sigma_1 \times \sigma_2 \times \sigma_2, \quad \Gamma = \sigma_3 \times 1 \times 1, \quad C\Gamma^m = \begin{pmatrix} \lambda^m_{ab} & 0 \\ 0 & \bar\lambda^m_{\dot a\dot b} \end{pmatrix} \tag{A.11}$$

where the un-dotted indices transform in the spinor representation of $Spin(6)$ or the $\mathbf{4}$ of $SU(4)$ whereas the the dotted indices transform in the conjugate spinor representation of $Spin(6)$ or the $\bar{\mathbf{4}}$ of $SU(4)$. The matrices λ^m_{ab} thus defined have the required antisymmetry and transform properties as in (A.7).

B. Modular Cornucopia

We assemble here together some properties of modular forms and Jacobi forms.

B.1 Modular forms

Let $\mathbb{H}$ be the upper half plane, *i.e.*, the set of complex numbers τ whose imaginary part satisfies $\text{Im}(\tau) > 0$. Let $SL(2, \mathbb{Z})$ be the group of matrices

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \tag{B.1}$$

with integer entries such that $ad - bc = 1$.

A *modular form* $f(\tau)$ of weight k on $SL(2, \mathbb{Z})$ is a holomorphic function on $\mathcal{H}$, that transforms as

$$f(\frac{a\tau + b}{c\tau + d}) = (c\tau + d)^k f(\tau) \quad \forall \quad \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2, \mathbb{Z}), \tag{B.2}$$

for an integer k (necessarily even if $f(0) \neq 0$). It follows from the definition that $f(\tau)$ is periodic under $\tau \to \tau + 1$ and can be written as a Fourier series

$$f(\tau) = \sum_{n=-\infty}^{\infty} a(n) q^n \,, \quad q := e^{2\pi i \tau} \,, \tag{B.3}$$

and is bounded as $\text{Im}(\tau) \to \infty$. If $a(0) = 0$, then the modular form vanishes at infinity and is called a *cusp form*. Conversely, one may weaken the growth condition at ∞ to $f(\tau) = \mathcal{O}(q^{-N})$ rather than $\mathcal{O}(1)$ for some $N \geq 0$; then the Fourier coefficients of f have the behavior $a(n) = 0$ for $n < -N$. Such a function is called a *weakly holomorphic modular form*.

The vector space over $\mathbb{C}$ of holomorphic modular forms of weight k is usually denoted by M_k. Similarly, the space of cusp forms of weight k and the space of weakly holomorphic modular forms of weight k are denoted by S_k and $M^!_k$ respectively. We thus have the inclusion

$$S_k \subset M_k \subset M^!_k \,. \tag{B.4}$$

The growth properties of Fourier coefficients of modular forms are known:

1. $f \in M^!_k \Rightarrow a_n = \mathcal{O}(e^{C\sqrt{n}})$ as $n \to \infty$ for some $C > 0$;

2. $f \in M_k \Rightarrow a_n = \mathcal{O}(n^{k-1})$ as $n \to \infty$;

3. $f \in S_k \Rightarrow a_n = \mathcal{O}(n^{k/2})$ as $n \to \infty$.

Some important modular forms on $SL(2,\mathbb{Z})$ are:

1. The *Eisenstein series* $E_k \in M_k$ $(k \geq 4)$. The first two of these are

$$E_4(\tau) = 1 + 240 \sum_{n=1}^{\infty} \frac{n^3 q^n}{1-q^n} = 1 + 240q + \dots\,, \tag{B.5}$$

$$E_6(\tau) = 1 - 504 \sum_{n=1}^{\infty} \frac{n^5 q^n}{1-q^n} = 1 - 504q + \dots\,. \tag{B.6}$$

2. The *discriminant function* Δ. It is given by the product expansion

$$\Delta(\tau) = q \prod_{n=1}^{\infty} (1-q^n)^{24} = q - 24q^2 + 252q^3 + ... \tag{B.7}$$

   or by the formula $\Delta = \left(E_4^3 - E_6^2\right)/1728$.

The two forms E_4 and E_6 generate the ring of modular forms, so that any modular form of weight k can be written (uniquely) as a sum of monomials $E_4^\alpha E_6^\beta$ with $4\alpha + 6\beta = k$. We also have $M_k = \mathbf{C} \cdot E_k \oplus S_k$ and $S_k = \Delta \cdot M_{k-12}$, so that any $f \in M_k$ also has a unique expansion as $\sum\limits_{0 \leq n \leq k/12} \alpha_n\, E_{k-12n}\, \Delta^n$ (with $E_0 = 1$). From either representation, we see that a modular form is uniquely determined by its weight and first few Fourier coefficients.

B.2 Jacobi forms

Consider a holomorphic function $\varphi(\tau, z)$ from $\mathbb{H} \times \mathbb{C}$ to $\mathbb{C}$ which is "modular in τ and elliptic in z" in the sense that it transforms under the modular group as

$$\varphi(\frac{a\tau+b}{c\tau+d}, \frac{z}{c\tau+d}) = (c\tau+d)^k\, e^{\frac{2\pi i m c z^2}{c\tau+d}}\, \varphi(\tau, z)\,, \quad \forall \quad \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2;\mathbb{Z}) \tag{B.8}$$

and under the translations of z by $\mathbb{Z}\tau + \mathbb{Z}$ as

$$\varphi(\tau, z + \lambda\tau + \mu) = e^{-2\pi i m(\lambda^2\tau + 2\lambda z)} \varphi(\tau, z), \quad \forall \quad \lambda, \mu \in \mathbb{Z}\,, \tag{B.9}$$

where k is an integer and m is a positive integer.

These equations include the periodicities $\varphi(\tau+1, z) = \varphi(\tau, z)$ and $\varphi(\tau, z+1) = \varphi(\tau, z)$, so φ has a Fourier expansion

$$\varphi(\tau, z) = \sum_{n,r} c(n,r)\, q^n\, y^r\ , \qquad\qquad (q := e^{2\pi i\tau},\ y := e^{2\pi i z})\ . \tag{B.10}$$

Equation (B.9) is then equivalent to the periodicity property

$$c(n,r) = C(4nm - r^2;\, r)\ , \qquad \text{where } C(d;r) \text{ depends only on } r \pmod{2m}\ . \tag{B.11}$$

The function $\varphi(\tau, z)$ is called a *holomorphic Jacobi form* (or simply a *Jacobi form*) of weight k and index m if the coefficients $C(d;r)$ vanish for $d < 0$, *i.e.* if

$$c(n,r) = 0 \qquad \text{unless} \qquad 4mn \geq r^2\ . \tag{B.12}$$

It is called a *Jacobi cusp form* if it satisfies the stronger condition that $C(d;r)$ vanishes unless d is strictly positive, *i.e.*

$$c(n,r) = 0 \qquad \text{unless} \qquad 4mn > r^2\ , \tag{B.13}$$

and conversely, it is called a *weak Jacobi form* if it satisfies the weaker condition

$$c(n,r) = 0 \qquad \text{unless} \qquad n \geq 0 \tag{B.14}$$

rather than (B.12).

B.3 Theta functions

In this section, we collect definitions and useful properties of theta function. The Jacobi theta function is defined by

$$\theta[{}^a_b](v|\tau) = \sum_{n\in\mathbb{Z}} q^{\frac{1}{2}(n-a)^2} e^{2\pi i(v-b)(n-a)}\ , \tag{B.15}$$

where a, b are real and $q = e^{2\pi i\tau}$. It satisfies the modular properties

$$\theta[{}^a_b](v|\tau+1) = e^{-i\pi a(a-1)}\theta[{}^{a}_{a+b-\frac{1}{2}}](v|\tau) \tag{B.16}$$

$$\theta[{}^a_b]\left(\frac{v}{\tau}| - \frac{1}{\tau}\right) = e^{2i\pi ab + i\pi\frac{v^2}{\tau}}\theta[{}^a_b](v|\tau) \tag{B.17}$$

The Jacobi-Erderlyi theta functions are the values at half periods,

$$\theta_1(z|\tau) = \theta[{}^{\frac{1}{2}}_{\frac{1}{2}}](z|\tau), \quad \theta_2(z|\tau) = \theta[{}^{\frac{1}{2}}_{0}](z|\tau), \quad \theta_3(z|\tau) = \theta[{}^0_0](z|\tau), \quad \theta_4(z|\tau) = \theta[{}^{0}_{\frac{1}{2}}](z|\tau) \tag{B.18}$$

In particular,

$$\theta_1(v/\tau, -1/\tau) = i\sqrt{-i\tau}\,e^{i\pi v^2/\tau}\theta_1(v,\tau) \tag{B.19}$$

The Dedekind η function is defined as

$$\eta(\tau) = q^{\frac{1}{24}}\prod_{n=1}^{\infty}(1-q^n)\,. \tag{B.20}$$

It satisfies the modular property

$$\eta\left(-\frac{1}{\tau}\right) = \sqrt{-i\tau}\eta(\tau) \tag{B.21}$$

It is related to the Jacobi-Erderlyi theta functions by the identities

$$\frac{\partial}{\partial v}\theta_1(v)|_{v=0} = 2\pi\ \eta^3(\tau) \tag{B.22}$$

$$\theta_2(0|\tau)\theta_3(0|\tau)\theta_4(0|\tau) = 2\eta^3 \tag{B.23}$$

The partition function of a single left-moving boson is given by

$$Z_{boson}(\tau) := \mathrm{Tr}(q^{L_0}) = \frac{1}{\eta(\tau)}\,. \tag{B.24}$$

C. A few facts about $K3$

C.1 K3 as an Orbifold

"Kummer's third surface" or K3 has played an important role in many developments concerning duality. Let us recall some of its properties. $K3$ is a four dimensional manifold which has $SU(2)$ holonomy. To understand what this means, consider a generic 4d real manifold. If you take a vector in the tangent space at point P, parallel transport it, and come back to point P, then, in general, it will be rotated by an $SO(4)$ matrix:

$$V_i(P) \to O_{ij}\ V_i(P) \qquad O_{ij} \in SO(4). \tag{C.1}$$

Such a manifold is then said to to have $SO(4)$ holonomy. In the case of K3, the holonomy is a subgroup of $SO(4)$, namely $SU(2)$. The smaller the holonomy group, the more "symmetric" the space. For example, for a torus, the holonomy group consists of just the identity because the space is flat and Riemann curvature is zero; so, upon parallel transport along a closed loop, a vector comes back to itself. For a K3, there *is* nonzero curvature but it is not completely arbitrary: the Riemann tensor is non-vanishing but the Ricci tensor R_{ij} vanishes. Therefore, K3 can alternatively be defined as the manifold of compactification that solves the vacuum Einstein equations.

Only other thing about K3 that we need to know is the topological information. A surface can have nontrivial cycles which cannot be shrunk to a point. For example, a torus has two nontrivial 1-cycles. The number of nontrivial k-cycles which cannot be smoothly deformed into each other is given by the k-th Betti number b_k of the surface. The number of non-trivial k-cycles is in one to one correspondence with the number of harmonic k-forms on the surface given by the k-th de-Rham cohomology [5, 6]. A harmonic k-form F_k satisfies the Laplace equation, or equivalently satisfies the equations

$$d^* F_k = 0, \qquad dF_k = 0 \tag{C.2}$$

A manifold always has a harmonic 0-form, *viz.*, a constant, and a harmonic 4-form, *viz.*, the volume from, assuming we can integrate on it. K3 has no harmonic 1-forms or 3-forms, but has 22 harmonic 2-forms. So, the Betti numbers for K3 are:

$$b_0 = 1, \qquad b_1 = 0, \qquad b_2 = 22, \qquad b_3 = 0, \qquad b_4 = 1. \tag{C.3}$$

Out of the 22 2-forms, 19 are anti-self-dual, and 3 are self-dual. In other words,

$$b_2^s = 3, \qquad b_2^a = 19. \tag{C.4}$$

This is all the information one needs to compute the massless spectrum of compactifications on K3.

K3 has a simple description as a $\mathbf{Z}_2$ orbifold of a 4-torus. Let (x_1, x_2, x_3, x_4) be the real coordinates of the torus $\mathbf{T}^4$. Let us further take the torus to be a product $\mathbf{T}^4 = \mathbf{T}^2 \times \mathbf{T}^2$. Let us introduce complex coordinates (z_1, z_2), $z_1 = x_1 + ix_2$ and $z_2 = x_3 + ix_4$. The 2-torus with coordinate z_1 is defined by the identifications $z_1 \sim z_1 + 1 \sim z_1 + i$, and similarly for the other torus. The tangent space group is $Spin(4) \equiv SU(2)_1 \times SU(2)_2$, and the vector representation is $\mathbf{4v} \equiv (\mathbf{2}, \mathbf{2})$. If we take a subgroup $SU(2)_1 \times U(1)$ of $Spin(4)$, then the vector decomposes as

$$\mathbf{4v} = \mathbf{2}_+ \oplus \bar{\mathbf{2}}_-. \tag{C.5}$$

The coordinates(z_1, z_2) transform as the doublet $\mathbf{2}_+$ and $(\bar{z}_1, \bar{z}_2)$ as the $\bar{\mathbf{2}}_-$. The $\mathbf{Z}_2 = \{1, I\}$ is generated by

$$I : (z_1, z_2) \to (-z_1, -z_2). \tag{C.6}$$

This $\mathbf{Z}_2$ is a subgroup and in fact the center of $SU(2)_1$. Consequently, as we shall see, the resulting manifold has $SU(2)$, indeed a $\mathbf{Z}_2$ holonomy. For a torus coordinatized by z_1, there are 4 fixed points of $z_1 \to -z_1$ Altogether, on $\mathbf{T}^4/\mathbf{Z}_2$, there are 16 fixed points.

Let us calculate the number of harmonic forms on this orbifold. To begin with, we have on the torus $\mathbf{T}^4$, the following harmonic forms:

$$
\begin{array}{ll}
1 & 1 \\
4 & dx^i \\
6 & dx^i \wedge dx^j \\
4 & dx^i \wedge dx^j \wedge dx^l \\
1 & dx^i \wedge dx^j \wedge dx^k \wedge dx^l .
\end{array}
\tag{C.7}
$$

The first column gives the number of forms indicated in the second column where the indices i, j, k, l take values $1, \cdots 4$. Under the reflection I, only the even forms $1, dx^i \wedge dx^j$, and $dx^i \wedge dx^j \wedge dx^k \wedge dx^l$ survive.

$$
\begin{array}{llcl}
\text{0-form} & 1 & & 1 \\
1 & 4 & & 0 \\
2 & 6 & \overset{\frac{1+I}{2}}{\longrightarrow} & 6 \\
3 & 4 & & 0 \\
4 & 1 & & 1
\end{array}
\quad ,
\tag{C.8}
$$

where the second column give the number of forms on the torus and the third column the number of forms that survive the projection. Let us look at the 2-forms from the torus that survive the $\mathbf{Z}_2$ projection. By taking the combinations

$$
dx^i \wedge dx^j \pm \frac{1}{2}\epsilon^{ijkl} dx^k \wedge dx^l
$$

we see that three of these 2-forms are self-dual and the remaining three are anti-self-dual.

At the fixed point of the orbifold symmetry there is a curvature singularity. The singularity can be repaired as follows. We cut out a ball of radius R around each point, which has a boundary $S^3/\mathbf{Z}_2$, replace it with a noncompact smooth manifold that is also Ricci flat and has a boundary $S^3/\mathbf{Z}_2$, and then take the limit $R \to 0$. The required noncompact Ricci-flat manifold with boundary $S^3/\mathbf{Z}_2$ is known to exist and is called the Eguchi-Hanson space. The Betti number of the Eguchi Hanson space are $b_0 = b_4 = 1$ ad $b_2^a = 1$. Therefore, each fixed point contributes an anti-self-dual 2-form which corresponds to a nontrivial 2-cycle in the Eguchi-Hanson space that would be stuck at the fixed point in the limit $R \to 0$.

Altogether, we get $b_0 = 1, b_2^s = 3$, $b_2^a = 3 + 16 = 19, b_4 = 1$, and $b_1 = b_3 = 0$ giving us the cohomology of K3. It obviously has $SU(2)$ holonomy. Away from the fixed point, a parallel transported vector goes back to itself, because all the curvature is concentrated at the fixed points. As we go around the fixed point a vector is returned to its reflected image (for instance, $(dz_1, dz_2) \to -(dz_1, dz_2)$), *i. e.*, transformed by an element of $SU(2)$.

In string theory there is no need to repair the singularity by hand. We shall see in §5.3 and §5.4 that the twisted states in the spectrum of Type-II string moving on an orbifold automatically take care of the repairing. The twisted states somehow know about the Eguchi-Hanson manifold that would be necessary to geometrically repair the singularity.

C.2 Elliptic genus of $K3$

Consider a two-dimensional superconformal field theories (SCFT) with $(2, 2)$ or more worldsheet supersymmetry[10]. We denote the superconformal field theory by $\sigma(\mathcal{M})$ when it corresponds to a sigma model with a target manifold $\mathcal{M}$. Let H be the Hamiltonian in the Ramond sector, and J be the left- moving $U(1)$ R-charge. The elliptic genus $\chi(\tau, z; \mathcal{M})$ is then defined [32, 33, 34] as a trace over the Hilbert space $\mathcal{H}_R$ in the Ramond sector

$$\chi(\tau, z; \mathcal{M}) = \operatorname*{Tr}_{\mathcal{H}_R} \left(q^H y^J (-1)^F\right) . \tag{C.9}$$

where F is the fermion number. An elliptic genus so defined satisfies the modular transformation property (B.8) as a consequence of modular invariance of the path integral. Similarly, it satisfies the elliptic transformation property (B.9) as a consequence of spectral flow. Furthermore, in a unitary SCFT, the positivity of the Hamiltonian implies that the elliptic genus is a weak Jacobi form.

A particularly useful example in the present context is $\sigma(K3)$, which is a $(4, 4)$ SCFT whose target space is a $K3$ surface. The elliptic genus is a topological invariant and is independent of the moduli of the $K3$. Hence, it can be computed at some convenient point in the $K3$ moduli space, for example, at the orbifold point where the $K3$ is the Kummer surface. At this point, the $\sigma(K3)$ SCFT can be regarded as a $\mathbb{Z}_2$ orbifold of the $\sigma(T^4)$ SCFT which is an SCFT with a torus T^4 as the target space. A simple computation using standard techniques of orbifold conformal field theory yields [35] the formula for the elliptic genus we introduced earlier in (C.10):

$$\chi(\tau, z) = 8 \left(\frac{\vartheta_2(\tau, z)^2}{\vartheta_2(\tau)^2} + \frac{\vartheta_3(\tau, z)^2}{\vartheta_3(\tau)^2} + \frac{\vartheta_4(\tau, z)^2}{\vartheta_4(\tau)^2} \right) . \tag{C.10}$$

The first term can be seen to arise from the untwisted projected partition function, the second from the twisted, unprojected partition function and the third from the twisted, projected partition function.

[10] An SCFT with (r, s) supersymmetries has r left- moving and s right-moving supersymmetries.

Note that for $z = 0$, the trace (C.9) reduces to the Witten index of the SCFT and correspondingly the elliptic genus reduces to the Euler character of the target space manifold. In our case, one can readily verify from (**??**) and (C.10) that $\chi(\tau, 0; K3)$ equals 24 which is the Euler character of $K3$.

C.3 Type IIB string on $K3$

Consider II-B compactified on K3. The resulting theory in the remaining 6-dimensional Minkowski space has $(0, 2)$ chiral supersymmetry. To discuss the spectrum let us recall that massless states are labeled by the representations of the little group in six dimensions which is $Spin(4) = SU(2) \times SU(2)$. With $(0, 2)$ supersymmetry, only two massless supermultiplets are possible. In terms of representations of the little group the supermultiplets are given by

1. The gravity multiplet:
   a graviton $(\mathbf{3}, \mathbf{3})$,
   five self-dual 2-forms $5(\mathbf{1}, \mathbf{3})$,
   gravitini $4(\mathbf{2}, \mathbf{3})$,
2. The tensor multiplet:
   an anti-self-dual 2-form $(\mathbf{3}, \mathbf{1})$,
   fermions $4(\mathbf{2}, \mathbf{1})$, five scalars $(\mathbf{1}, \mathbf{1})$.

The gravitini are right-handed whereas the fermions in the tensor multiplets are left-handed.

We can explicitly work out the spectrum of Type-IIB on a K3 that is a $\mathbf{Z}_2$ orbifold. Let us take $X^m, m = 6, 7, 8, 9$ to be the coordinates of the internal torus and $X^i, i = 2, 3, 4, 5$, to be the noncompact light-cone coordinates. It is convenient to decompose the little group in ten dimensions $SO(8)$ as

$$\begin{aligned} Spin(8) &\supset Spin(4)_I \times Spin(4)_E \\ &\equiv SU(2)_{1I} \times SU(2)_{2I} \times SU(2)_{1E} \times SU(2)_{2E}, \end{aligned} \tag{C.11}$$

where the subscript I is for internal, E is for external. With this embedding, the representations decompose as

$$\begin{aligned} \mathbf{8v} &= (\mathbf{4v}, \mathbf{1}) \oplus (\mathbf{1}, \mathbf{4v}) \equiv (\mathbf{2}, \mathbf{2}, \mathbf{1}, \mathbf{1}) \oplus (\mathbf{1}, \mathbf{1}, \mathbf{2}, \mathbf{2}), \\ \mathbf{8s} &= (\mathbf{2s}, \mathbf{2s}) \oplus (\mathbf{2c}, \mathbf{2c}) \equiv (\mathbf{2}, \mathbf{1}, \mathbf{2}, \mathbf{1}) \oplus (\mathbf{1}, \mathbf{2}, \mathbf{1}, \mathbf{2}), \\ \mathbf{8c} &= (\mathbf{2s}, \mathbf{2c}) \oplus (\mathbf{2c}, \mathbf{2s}) \equiv (\mathbf{2}, \mathbf{1}, \mathbf{1}, \mathbf{2}) \oplus (\mathbf{1}, \mathbf{2}, \mathbf{2}, \mathbf{1}). \end{aligned} \tag{C.12}$$

The orbifold group is a $\mathbf{Z}_2$ subgroup of $SU(2)_{LI}$ which acts as $-\mathbf{1}$ on the doublet representation $\mathbf{2}$.

• *Untwisted sector*
The states in the untwisted sector are obtained by keeping the $\mathbf{Z}_2$ invariant states of the original 10-dimensional states.

$$(\mathbf{8v} \oplus \mathbf{8c}) \otimes (\mathbf{8v} \oplus \mathbf{8c}). \tag{C.13}$$

For example, the bosons (labeled by $SU(2)_{1E} \times SU(2)_{2E}$ quantum numbers are

$$\begin{aligned} &[4(\mathbf{1},\mathbf{1}) \otimes 4(\mathbf{1},\mathbf{1})] \oplus [(\mathbf{2},\mathbf{2}) \otimes (\mathbf{2},\mathbf{2})] \\ &[2(\mathbf{1},\mathbf{2}) \otimes 2(\mathbf{1},\mathbf{2})] \oplus [2(\mathbf{2},\mathbf{1}) \otimes 2(\mathbf{2},\mathbf{1})] \end{aligned} \tag{C.14}$$

This gives rise to a graviton, 25 scalars, 5 self-dual and 5 anti-self-dual 2-forms. The fermions can be obtained similarly which give the superpartners required by supersymmetry. Together, we get the gravity multiplet and five tensor multiplets.
• *Twisted Sector*
There are 16 twisted sectors coming from the 16 fixed points. The bosonic fields and fermionic fields are twisted according to their transformation property under the $\mathbf{Z}_2$. We see from that four fermions that transform as $2(\mathbf{2},\mathbf{1})$ and four bosons that transform as $(\mathbf{2},\mathbf{2})$ are $\mathbf{Z}_2$ invariant and are not twisted where as the four other are twisted. The ground state energy is zero because there are equal number of bosons and fermions that are twisted. The untwisted fermions have zero modes. The zero mode algebra gives rise to a four dimensional representation $(\mathbf{2},\mathbf{1}) \oplus 2(\mathbf{1},\mathbf{1})$. Therefore the massless representation is

$$[(\mathbf{2},\mathbf{1}) \oplus 2(\mathbf{1},\mathbf{1})] \otimes [(\mathbf{2},\mathbf{1}) \oplus 2(\mathbf{1},\mathbf{1})] \tag{C.15}$$

which gives precisely the particle content of a tensor multiplet. Therefore, the twisted sector contributes 16 tensor multiplets.

The massless spectrum of Type-IIB on a K3 orbifold thus consists of a gravity multiplet and 21 tensor multiplet together from the untwisted and the untwisted sector. There are 105 scalars that parametrizes the moduli space $O(21,5;\mathbf{Z})\backslash O(21,5;\mathbf{R})/O(21;\mathbf{R})\times O(5;\mathbf{R})$.

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