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Lectures on Kerr/CFT

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Kerr/CFT and (far) Beyond: Lecture 4

(Towards) Microscopic Counting of General Black Holes

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Outline

- Reminder: extremal Kerr and the big picture.
- ***Microscopics of Kerr/CFT:*** proposal for precision counting.
- The elliptic genus index and spectral flow.
- Fractionation and length scales.

Extremal Black Holes: Overview

- ***Setting:***

  Black holes in $D = 4$ SUGRA, single center, asymptotically flat, $N = 8, 4, 2$ theory. Parameters (M, J, Q_I, P^I).

- ***Extremal limit:***

  $T_H = 0$.

  There is an AdS_2 factor in geometry.

- Distinguish ***3*** types of extremal Black Holes:

  i) BPS.

  ii) non-BPS extremal.

  iii) extremal Kerr.

Example 1

- ***Theory:*** M theory on $CY \times S^1$.

- ***Black hole:***
  $M5$ on $P \times S^1$ (P a divisor) with $P^3 \neq 0$ ("3 charges").
  Momentum $= n$ along S^1 (so 4 charges total).

- Black hole entropy: $S = 2\pi\sqrt{|n|P^3}$.

- ***Two extremal limits:***
  BPS: $n > 0$. The "fourth" charge break ***same*** SUSYs as P.
  non-BPS: $n < 0$. The "fourth" charge break ***opposite*** SUSYs so none are left.

Example 2: Kerr-Newman

- ***Theory:*** Einstein-Maxwell, "diagonal charges", $Q^4 \sim nP^3$. (Warning: the non-BPS branch is excluded by this truncation).

- ***Black hole: Kerr-Newman***

  Black hole entropy ($G_4 = 1$):
  $S = 2\pi \left[(M^2 - \frac{1}{2}Q^2) + \sqrt{M^2(M^2 - Q^2) - J^2} \right] = S_L + S_R$.

- ***BPS:*** $M^2 = Q^2$, $J = 0$, $S = 2\pi \cdot \frac{1}{2}Q^2$.

- ***Extremal Kerr:*** $Q^2 = 0$, $M^4 = J^2$, $S = 2\pi|J|$.

- ***BPS and Kerr both correspond to R in a specific state, L carries entropy***

The Big Picture

The ***general*** black hole — with parameters: (M, J, Q_I, P^I) — is described by a 2D CFT with L and R chiralities that interact weakly:

$$S = S_L + S_R \, , \quad \beta_H = \frac{1}{2}(\beta_L + \beta_R) \; .$$

R-movers have the ability to carry J.

BPS: $T_R \to 0$ (with $J = 0$). R-movers in ground state, J-carryers not excited. L-movers carry entropy.

Extremal Kerr: $T_R \to 0$ (with $J \neq 0$). R-movers in a definite state, with J-carryers excited. L-movers carry entropy.

Precision Counting: Extremal Kerr

Working assumption: the entropy of extremal Kerr comes from the "same" states (L-movers in our convention) as the BPS entropy.

The difference: for Kerr the R-movers are in a state that breaks SUSY spontaneously, instead of the SUSY preserving ground state.

Strategy for precision counting:

- Consider the CFT underlying the BPS counting.
- Keep the dynamical chirality (holomorphic=L-movers) intact.
- Modify the inert chirality (anti-holomorphic=R-movers) by spectral flow.

Setting

The $D1/D5$ on $K3 \times S^1$, described by the σ-model on

$$\mathcal{M}^k/\Sigma_k \ ,$$

with $\mathcal{M} = K3$, $k = n_1 n_5 + 1$. The central charge is $c = 6k$.

Excitations at level $h = p$ give asymptotic degeneracy

$$S = 2\pi\sqrt{kh} = 2\pi\sqrt{kp} \ .$$

The 4D version of the model involves adding a KK-monopole: basic reasoning remains, but some details change.

Counting in σ-model

Vertex operators:

$$\mathcal{V}(z,\bar z) = \mathcal{V}^L_{\rm int}(z) e^{iF_L\varphi_L(z)/\sqrt{2k}} \cdot \mathcal{V}^R_{\rm int}(\bar z) e^{iF_R\varphi_R(\bar z)/\sqrt{2k}} \ .$$

Bosonized R-currents: $J = \sqrt{2k}\partial\varphi_L$, $\bar J = \sqrt{2k}\bar\partial\varphi_R$.

Spacetime angular momentum: $F_{R,L} = 2j_{R,L}$.

Conformal weights:

$$\begin{aligned} h_R &= h^{\rm int}_R + \frac{1}{4k}F_R^2 \ , \\ h_L &= h^{\rm int}_L + \frac{1}{4k}F_L^2 \ . \end{aligned}$$

Momentum of the state:

$$p = h_L - h_R \ .$$

Extremal limit:

$$h_R^{\rm int} = 0 \quad \Rightarrow \; h_R = \frac{1}{4k} F_R^2 \,, \quad ({\rm extremal}) \,.$$

Origin of entropy: freedom in $\mathcal{V}^L_{\rm int}(z)$ with weight

$$h_L^{\rm int} = h_L - \frac{1}{4k} F_L^2 = p + h_R - \frac{1}{4k} F_L^2 = p + \frac{1}{4k} F_R^2 - \frac{1}{4k} F_L^2 \,.$$

The leading ***black hole entropy*** from Cardy's formula:

$$S = 2\pi \sqrt{\frac{ch}{6}} = 2\pi \sqrt{kp + j_R^2 - j_L^2} \,.$$

The BMPV black hole (rotating 5D BH with SUSY): special case $j_R = 0$.

Extremal 5D Kerr: $j_L = p = 0$ with entropy

$$S = 2\pi |j_R| \,.$$

Comments

- The ***4D version of computation***: add KK-monopole $\Rightarrow$ SUSY broken in L-sector $\Rightarrow$ there is no j_L. But j_R identified with $4D$ angular momentum J.

- ***Uncharged case***: $n_1 = n_5 = 0 \Rightarrow k = 1$ a very special case (the entire model is just elliptic genus of $K3$) so computation may not be justified.

- ***Answer analysis*** (inconclusive): for $p = 0$ the level k cancels so entropy would work out for Kerr no matter what central charge is claimed.

The Elliptic Genus

The ***partition function*** in the RR sector (with $(-)^F$ inserted):

$$Z(\tau, z, \bar{\tau}, \bar{z}) = \mathrm{Tr}_{\mathrm{RR}}[(-)^F y^{F_L} q^{L_0 - \frac{c_L}{24}} \bar{y}^{F_R} \bar{q}^{\bar{L}_0 - \frac{c_R}{24}}] \,,$$

where

$$q = e^{2\pi i \tau} \,, \quad y = e^{2\pi i z} \,, \quad \overline{y} = e^{-2\pi i \overline{z}} \,.$$

The $\mathcal{N} = 2$ SCA contains two supercurrents. Their zero-modes commute with the energy $[L_0, G_0^{\pm}] = 0$, and act as raising/lowering operators on the R-current $[F_L, G_0^{\pm}] = \pm G_0^{\pm}$.

Suppose some state $|\psi\rangle$ contributes to the partition function. Then $G_0^+|\psi\rangle$ generally contributes a state ***with the same energy*** and ***opposite statistics***. (and so does $G_0^-|\psi\rangle$).

Consequence: all contributions to the elliptic genus generally ***cancel in pairs***, if we take $z = 0$ ($y = 1$).

The exceptions: there are surviving states exactly if $G_0^+|\psi\rangle = G_0^-|\psi\rangle = 0$.

These are precisely the states with $L_0 - \frac{c}{24} = 0$, since the $\mathcal{N} = 2$ SCA gives

$$\{G_0^+, G_0^-\} = 2(L_0 - \frac{c}{24}) \, .$$

Conclusion: the elliptic genus index counts the states that are ***R ground states*** ($L_0 - \frac{c}{24} = 0$) in the holomorphic sector but ***arbitrary in the anti-holomorphic sector***.

Spectral Flow

The $\mathcal{N} = 2$ SCA permits a family of automorphisms. For any real η, the substitutions

$$\begin{aligned} L_0 &\rightarrow L_0 + \eta F_0 + k\eta^2 \ , \\ F_0 &\rightarrow F_0 + 2k\eta \ . \end{aligned}$$

leave the algebra invariant.

This process is called ***spectral flow***.

Example: spectral flow in the anti-holomorphic sector transforms the partition function as

$$Z(\tau, z, \overline{\tau}, \overline{z} + \overline{\eta\tau}) = e^{-2\pi i k(\overline{\eta}^2\overline{\tau} + 2\overline{\eta z})} Z(\tau, z, \overline{\tau}, \overline{z}) \ .$$

An Index for Extreme Kerr

The microstates proposed for extreme Kerr are in a definite state in the anti-holomorphic sector:

- The state in the anti-holomorphic sector carries angular momentum but is otherwise trivial.
- This anti-holomorphic state can be reached as spectral flow by $\eta = j_R/k$. This increases the angular momentum from 0 to j_R.

Accordingly, the proposed partition function for Kerr (taking $\overline{z} = 0$ after spectral flow) is

$$Z_{\mathrm{j_R}}(\tau, z, \overline{z}, \overline{\tau}) = \bar{q}^{-\frac{1}{k} j_R^2} Z_{\mathrm{j_R}=0}(\tau, z, 0, 0) \; .$$

This just formalizes the essentially trivial nature of the proposed counting.

Index vs. Partition Function

Unitarity and the $\mathcal{N}=2$ SCA implies a bound on the R charges for the R ground states:

$$-k \leq \overline{F}_0 \leq k \ .$$

A weakness in the original microscopic counting of black hole microstates (for nonrotating black holes): the microstates were generally rotating, with angular momentum in the range

$$|F_R/2| \leq k/2 \ .$$

The description of extreme Kerr has the analogous weakness: the microstates have angular momenta in the range

$$|F_R/2 - j_R| \leq k/2 \ ,$$

rather than the precise Kerr value.

This does not seem a critical issue for large angular momenta $j_R \gg k/2$.

The Structure of Kerr Counting

Answer analysis: the entropy follows from Cardy's fomula ($c = 6k$)

$$S = 2\pi\sqrt{kh_L} = 2\pi |j_R| \;\Rightarrow\; kh_L = j_R^2 \,.$$

There are (at least) two proposed assignments:

- Proposed microscopic counting: $k = 1$, $h_L = j_R^2$.
- Standard Kerr/CFT: $k = 2j_R$, $h_L = \frac{c}{24} = \frac{k}{4} = \frac{1}{2}j_R$.

So: is the central charge for 4D extreme Kerr $c = 12J$, or not?

Plan: first detour briefly into Kerr/CFT, then consider fractionation.

Two Central Charges in Kerr/CFT

Reminder: the entropy of 4D Kerr,

$$S_{\text{Kerr}} = S_L + S_R = 2\pi \left(G_4 M^2 + \sqrt{G_4^2 M^4 - J^2} \right) \ .$$

At extremality $J = G_4 M^2$, the ground state degeneracy $S_L = 2\pi |J|$.

Original Kerr/CFT: consider the Near Horizon Extreme Kerr (NHEK) geometry.

Focus on diffeomorphisms acting on the azimuthal coordinate ϕ (the $U(1)$ Killing vector is $i\partial_\phi$).

Result: these diffeomorphisms form a Virasoro algebra with $c = 12J$.

An alternative procedure: the horizon of the ***near*** NHEK geometry spins at nearly the speed of light.

Strategy: study the DLCQ limit towards the rotating frame.

Focus on diffeomorphisms that trivialize in the strict limit: these pertain to excitations in the R-sector, those that actually carry angular momentum.

The theory of excitations reduce to (a chiral sector of) the standard Brown-Hennaux computation in AdS_3, with central charge $c = 12J$.

Conclusion: $c = 12J$ has been computed from "mesoscopic" considerations, in two distinct ways.

Disclosure: some questions about these procedures remain.

Fractionation

Back from the detour, we return to the question: is the central charge for 4D extreme Kerr $c = 12J$, or not?

Remark: it could be that both are right (but not for the same question).

Reminder: the best understood system (the D1/D5 symmetric orbifold) exhibits ***fractionation***.

The central charge of the SUSY sigma model on $\mathcal{M}^k/\Sigma_k$ is $c = 6k$.

At ***high energy*** the spectrum is that of a free gas with $c = 6k$ d.o.f.'s $\Rightarrow$ the entropy is given by Cardy formula $S = 2\pi\sqrt{kh}$.

The important modes at ***low energy*** are the twisted sector states with conformal weight quantized in units of $1/k$.

Interpretation: the low energy modes "live" in the CFT with target space $\mathcal{M}$; but their quantized energy is measured in units of $1/k$.

So: this class of excitations ***by itself*** has the entropy of a gas with $c = 6$ and $h_{\rm eff} = kh$.

Cardy's formula computes the entropy of such a gas as $S = 2\pi\sqrt{1 \cdot h_{\rm eff}} = 2\pi\sqrt{k \cdot h}$, just like the high energy theory.

The upshot: Cardy's formula $S = 2\pi\sqrt{kh}$ is always justified for CFT energies $h \gg k$; but for these CFT's the sum over all sectors means it applies all the way down to $h \geq \frac{c}{24} = \frac{k}{4}$.

This phenomenon is known as ***fractionation***, and the twisted sector states dominating at low energy are the ***long strings***.

In summary: this type of theory has the property that, as we vary the energy scale, the "effective" central charge adjusts in such a manner that the entropy (as computed by Cardy's formula) remains the same.

The Kerr Black Hole Revisited

Reminder: the proposed microscopic counting describes the extremal Kerr BH as a CFT with $k = 1$ and $h_L = j_R^2$.

The standard Kerr/CFT describes the extremal Kerr BH as a CFT with $k = 2J$ and $h_L = \frac{1}{2}J$.

The lesson of fractionation: these results could well be consistent.

Consequence of interpretation: the former result (the proposed microscopic counting) is just the long string sector.

The full description would be something along the lines of a σ-model on $\mathcal{M}^{2J}/\Sigma^{2J}$.

This is the best proposal that I can make that fits the data presented; but it is seems ***plausible that a better prescription for Kerr/CFT exists***.

Towards a General Counting

The BPS “counting functions” are indices. But they essentially count states.

Their ground states are the perturbative string states and their excited states are extremal black hole states.

The (very tentative) full partition functions: multiply holomorphic and anti-holomprhic invariant combinations of those BPS counting functions.

This principle is ***holomorphic factorization***.

If the resulting partition function is not already invariant under (generalized) modular invariance, sum over orbits so that it is.

The legitimate “counting functions” would be either the BPS type functions (like the symmetric product) or of non-BPS type (totally unknown!).

The precise procedure for the multiplication should be constrained by

- Generalized Level Matching: $N_R - N_L = \mathrm{integer}$.
- Some kind of generalized modular invariance, for which $Sp(2, Z)$ is the most obvious proposal.

Summary (of this Lecture)

The beginnings of precision counting for Kerr:

- Although Kerr is not supersymmetric, the ***non-BPS branch is not relevant*** for its description.

- A proposal: starting from the BPS theory, perform spectral flow on the sector that is in its supersymmetric ground state.

- This state breaks SUSY, but the states responsible for the entropy have not changed $\Rightarrow$ they can be counted precisely.

- Warning: the implementation with $k = 1$ is probably incomplete (since Kerr/CFT suggests $k = 2J$.