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Lectures on Kerr/CFT

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Kerr/CFT and (far) Beyond: Lecture 3

Hidden Conformal Symmetry

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Outline of Lecture 3

- A benchmark for black hole microstates: AdS/CFT correspondence: near horizon AdS_3 ***guarantees*** agreement for the entropy.

- Goal: acquire similar understanding for more general black holes.

- A hint: the massless scalar wave equation for a general background.

- Possible interpretation: ***hidden conformal symmetry***, a proposal for the structure of microstates for general black holes.

The BTZ Black Hole

Many black holes that permit detailed microstate counting have a ***near horizon AdS$_3$***.

The black holes in AdS$_3$ are the BTZ black holes with metric

$$ds^2 = -\frac{(r^2 - r_+^2)(r^2 - r_-^2)}{r^2} dT^2 + \frac{\ell^2 r^2 dr^2}{(r^2 - r_+^2)(r^2 - r_-^2)} + r^2 \left(d\phi + \frac{r_+ r_-}{r^2} dT \right)^2$$

where ℓ is the AdS$_3$ radius and $r_\pm$ are the outer and inner horizon coordinates (convention $r_+ > r_-$).

Important result: the BTZ black hole entropy

$$S = \frac{A_3}{4G_3} = \frac{2\pi r_+}{4G_3} ,$$

can be given a microscopic interpretation ***without any detailed assumptions***.

The AdS/CFT Correspondance

The BTZ black hole in standard (asymptotically AdS_3) Fefferman-Graham form is:

$$
\begin{aligned}
ds^2 \;&=\; \ell^2 d\eta^2 + \left(\ell^2 e^{2\eta} + \frac{1}{16\ell^2}(r_+^2 - r_-^2)^2 e^{-2\eta}\right) dw^+ dw^- \\
&+\; \frac{1}{4}(r_+ - r_-)^2 (dw^-)^2 + \frac{1}{4}(r_+ + r_-)^2 (dw^+)^2 \;,
\end{aligned}
$$

where the coordinates

$$
\begin{aligned}
r^2 \;&=\; r_+^2 \cosh^2(\eta - \eta_0) - r_-^2 \sinh^2(\eta - \eta_0) \;, \\
w^\pm \;&=\; \phi \pm T \;,
\end{aligned}
$$

with parameter

$$
e^{2\eta_0} = \frac{r_+^2 - r_-^2}{4\ell^2} \;.
$$

Global AdS$_3$

For comparison, global AdS$_3$ is

$$ds_3^2 = \ell^2 \left[d\eta'^2 + \sinh^2 \eta' d\phi'^2 - \cosh^2 \eta' dT'^2 \right] \ .$$

The global AdS$_3$ geometry is characterized by the fact that for small η' (at fixed T') the geometry is just R^2 in polar coordinates.

In other words: the ϕ-circle is ***contractible***.

In the black hole geometry it is the (Euclidian) time that is contractible, while the angular coordinate remains finite at the origin (the length of the corresponding circle gives the entropy).

Modular Parameters

Thermodynamic potentials (in 3D the chemical potential μ is the angular velocity of the horizon Ω):

$$\begin{aligned}\beta &= \pi\ell\left(\frac{1}{r_+ + r_-} + \frac{1}{r_+ - r_-}\right) ,\\ \mu &= \pi\ell\left(\frac{1}{r_+ + r_-} - \frac{1}{r_+ - r_-}\right) .\end{aligned}$$

Periodicity of Euclidean time and azimuthal angle determines a boundary torus with modular parameters (for μ imaginary):

$$\begin{aligned}\tau &= i\tfrac{\beta-\mu}{2\pi} = i\ell\tfrac{1}{r_+ - r_-} ,\\ \overline{\tau} &= -i\tfrac{\beta+\mu}{2\pi} = -i\ell\tfrac{1}{r_+ + r_-} .\end{aligned}$$

BTZ from Global AdS$_3$

The BTZ black hole in terms of modular parameters

$$\begin{aligned} ds^2 &= \ell^2 d\eta^2 + \ell^2 \left(e^{2\eta} + \frac{1}{(4\tau\overline{\tau})^2} e^{-2\eta} \right) dw^+ dw^- \\ &- \frac{\ell^2}{4\tau^2}(dw^-)^2 - \frac{\ell^2}{4\overline{\tau}^2}(dw^+)^2 \\ &= \ell^2 \left[d\eta'^2 + \sinh^2\eta' d\phi'^2 - \cosh^2\eta' dT'^2 \right] \ . \end{aligned}$$

We took $w'^- = -w^-/\tau$, $w'^+ = -w^+/\bar{\tau}$, $e^{2\eta} = e^{2\eta'}/(4\tau\bar{\tau})$.

The original coordinate were BTZ, while the primed coordinate are global AdS$_3$.

So BTZ is related to global AdS$_3$ by a coordinate transformation. (Both are locally AdS$_3$).

Reiteration: BTZ from Global AdS$_3$

The BTZ boundary conditions:

$$\begin{aligned} w^- &\equiv w^- + 2\pi \equiv w^- + 2\pi\tau \ , \\ w^+ &\equiv w^+ + 2\pi \equiv w^+ + 2\pi\bar{\tau} \ . \end{aligned}$$

The global AdS$_3$ boundary coordinates:

$$\begin{aligned} w'^- &\equiv w'^- + 2\pi \equiv w'^- - \frac{2\pi}{\tau} \ , \\ w'^+ &\equiv w'^+ + 2\pi \equiv w'^+ - \frac{2\pi}{\bar{\tau}} \ . \end{aligned}$$

The lesson: the BTZ black hole is related to global AdS$_3$ by a ***global*** coordinate transformation (a modular transformation in the boundary theory).

Brown-Henneaux Central Charge

According to AdS/CFT correspondance, asymptotically AdS$_3$ spacetime

$$ds^2 = d\eta^2 + \gamma_{ab} dy^a dy^b \;\;,\;\; \gamma_{ab} = e^{2\eta/\ell}\gamma_{ab}(0) + \gamma_{ab}^{(2)} + \ldots$$

is assigned a boundary energy momentum tensor

$$T_{ab} = -\frac{1}{8\pi G_3}\left(\mathcal{K}_{ab} - \mathcal{K}\gamma_{ab} + \frac{1}{\ell}\gamma_{ab}\right) \;,$$

where $\mathcal{K}_{ab}$ is the extrinsic curvature of the boundary.

Fixing conformal frame to $w^{\pm}$ and performing an infinitesimal local diffeomorphism $w^{\pm} \to w^{\pm} + \xi^{\pm}$, the boundary energy momentum tensor transforms as

$$T_{\pm\pm} \to T_{\pm\pm} + \frac{\ell}{8G_3}\partial_{\pm}^3 \xi \;.$$

This means the boundary theory furnishes an affine representation of the conformal algebra, with central charge determined by the Brown-Henneaux formula

$$c = \frac{3\ell}{2G_3} \, .$$

The value of the central charge is computed by c-extremization (analogous to entropy extremization), or by anomalies.

Remark: the “automatic agreement” means the ***value is not needed***.

Strategy: the conformal algebra determines the ground state energy of AdS$_3$, and then modular transformation will determine the BTZ entropy.

Conformal Weights

Physical parameters of the BTZ black hole:

$$M = \frac{r_+^2 + r_-^2}{8G_3\ell^2} , \quad J = \frac{r_+ r_-}{4G_3\ell} .$$

Conformal weights:

$$\begin{aligned} L_0 - \frac{c}{24} &= \frac{M\ell - J}{2} = \frac{1}{16G_3\ell}(r_+ - r_-)^2 , \\ \overline{L}_0 - \frac{c}{24} &= \frac{M\ell + J}{2} = \frac{1}{16G_3\ell}(r_+ + r_-)^2 . \end{aligned}$$

Global AdS$_3$ is $SL(2,R)$ invariant so $L_0 = 0, \overline{L}_0 = 0$:

$$\begin{aligned} M_{AdS} &= -\tfrac{1}{8G_3} , \\ J_{AdS} &= 0 . \end{aligned}$$

Remark: global AdS=negative BH mass = NS ground state.

BTZ Entropy from Cardy Formula

Global AdS_3 has on-shell action:

$$I_{\text{thermal}} = \beta H + \mu J = \frac{i\pi}{12}(c\tau - \overline{c\tau}) \; .$$

The Cardy formula in CFT: relate high T behavior to ground state by a modular transformation.

The bulk version: the BTZ black hole entropy is related to the Casimir energy of global AdS_3 by a modular transformation:

$$\begin{aligned} I_{\text{BTZ}} &= \beta H + \mu J - S = -\frac{i\pi}{12}(\frac{c}{\tau} - \frac{\bar{c}}{\bar{\tau}}) \; , \\ \Rightarrow S_{\text{BTZ}} &= 2\pi \left(\sqrt{\frac{c}{6} h_L} + \sqrt{\frac{c}{6} h_R} \right) = \frac{2\pi r_+}{4G_3} \; . \end{aligned}$$

So: the ***black hole entropy of BTZ is accounted for automatically*** in a theory that implements diffeomorphism invariance, including ***global*** diffeomorphism invariance.

Hidden Conformal Symmetry

So far: considered black holes in AdS$_3$.

Corollary: extremal and near extremal black holes in $D = 4, 5$ have near horizon AdS$_3$ geometry so this discussion applies.

A goal (not yet realized): a similar line of reasoning for general black holes, with no SUSY at all.

The working assumption: such black holes are excitations of superconformal symmetry as well, it is just that the state breaks the symmetry.

Nonextremal Charged Kerr in $D = 5$

The geometry:

$$
\begin{aligned}
ds_5^2 &= (H_1 H_2 H_3)^{1/3}(x+y)[-\Phi(dt+\mathcal{A})^2 + ds_4^2] \ , \\
ds_4^2 &= \left(\frac{dx^2}{4X} + \frac{dy^2}{4Y}\right) + \frac{U}{G}(d\chi - \frac{Z}{U}d\sigma)^2 + \frac{XY}{U}d\sigma^2 \ ,
\end{aligned}
$$

where x, y are radial/polar coordinates, χ, σ are angular coordinates and

$$
\begin{aligned}
X &= (x+a^2)(x+b^2) - \mu x \ , \\
Y &= -(a^2-y)(b^2-y) \ , \\
G &= (x+y)(x+y-\mu) \ , \\
U &= yX - xY \ , \\
Z &= ab(X+Y) \ , \\
\Phi &= \frac{G}{(x+y)^3 H_1 H_2 H_3} \ ,
\end{aligned}
$$

$$
\begin{aligned}
H_i &= 1 + \frac{\mu \sinh^2 \delta_i}{x+y} \,, \\
\mathcal{A} &= \frac{\mu \prod_i \cosh \delta_i}{x+y-\mu}[(a^2+b^2-y)d\sigma - abd\chi] - \frac{\prod_i \sinh \delta_i}{x+y}(abd\sigma - yd\chi)
\end{aligned}
$$

This is just the geometry, there is also matter: gauge fields and scalar fields.

This full geometry has surprising simplifying features that have not yet been fully understood.

For example, the full wave equation is separable: angular and radial equations are independent.

This structure is surprising since angular momentum breaks spherical symmetry.

A Probe Scalar Field

Consider a massless scalar field Φ propagating in the background of the general black hole.

The radial part of the wave function satisfies

$$\frac{\partial}{\partial x}(x^2-\frac{1}{4})\frac{\partial}{\partial x}\Phi+\left[\frac{1}{x-\frac{1}{2}}\frac{\omega^2}{4\kappa_+^2}-\frac{1}{x+\frac{1}{2}}\frac{\omega^2}{4\kappa_-^2}+V(x)-l(l+1)\right]\Phi=0\,,$$

where a linearly redefined coordinate locates the horizons at $x=\pm\frac{1}{2}$:

$$x=\frac{r-\frac{1}{2}(r_++r_-)}{r_+-r_-}\,.$$

and the potential $V(x)$ is a quadratic function in x (in particular the equation is smooth at the horizons).

The Near Horizon Wave Function

The radial wave equation with $V(x) = 0$ has regular poles at the inner and outer horizons, and at infinity: ***it is the hypergeometric equation.***

The S-wave ($l = 0$) radial wave function (with outgoing boundary conditions at the horizon):

$$\Phi = \left(\frac{x - \frac{1}{2}}{x + \frac{1}{2}}\right)^{i\beta_H \omega / 4\pi} \cdot F(1 + i\frac{\beta_R \omega}{4\pi}, 1 + i\frac{\beta_L \omega}{4\pi} + 1 + i\frac{\beta_H \omega}{2\pi}, \frac{x - \frac{1}{2}}{x + \frac{1}{2}}) \,.$$

The corresponding Hawking emission spectrum suggests ***collisions between Right- and Left-moving excitations***:

$$\Gamma_{\rm em}(\omega)\frac{d^3k}{(2\pi)^3} = A\frac{1}{\beta_H \omega}\frac{\beta_L \omega/2}{(e^{\beta_L \omega/2} - 1)}\frac{\beta_R \omega/2}{(e^{\beta_R \omega/2} - 1)}\frac{d^3k}{(2\pi)^3} \,.$$

This picture can be pursued to a ***quantitative*** agreement.

Symmetries with $V(x) = 0$

Ultimate goal: understand these features from a detailed microscopic model and/or general symmetries (we are not there yet!).

A clue: all solutions of the hypergeometric equation are characters of $SL(2, R)$.

A spacetime interpretation: the geometry with potential neglected effectively reduce to AdS$_3$.

This motivates the hope that there is a Virasoro algebra, and an underlying CFT, also for general black holes.

The $SL(2,R)_R \times SL(2,R)_L$ Symmetry

The wave equation in $SL(2,R)_R \times SL(2,R)_L$ invariant form

$$\vec{\mathcal{R}}^2\Phi = \vec{\mathcal{L}}^2\Phi = l(l+1)\Phi \ ,$$

where (one choice of) the $SL(2,R)_R$ generators are

$$\begin{aligned}
\mathcal{R}_1 &= i\partial_+ \ , \\
\mathcal{R}_0 &= i(w^+\partial_+ + \frac{1}{2}y\partial_y) \ , \\
\mathcal{R}_{-1} &= i(w^{+2}\partial_+ + w^+ y\partial_y) - y^2\partial_- \ ,
\end{aligned}$$

where

$$\begin{aligned}
w^+ &= \sqrt{\frac{r-r_+}{r-r_-}}e^{2\pi\tilde{T}_R\phi} \ , w^- = \sqrt{\frac{r-r_+}{r-r_-}}e^{2\pi\tilde{T}_L\phi-2\pi(\tilde{T}_L+\tilde{T}_R)\Omega} \ , \\
y &= \sqrt{\frac{r_+-r_-}{r-r_-}}e^{\pi(\tilde{T}_L+\tilde{T}_R)(\phi-\Omega t)} \ .
\end{aligned}$$

The $SL(2,R)_L$ commutes with $SL(2,R)_R$. It has generators $\vec{\mathcal{L}}$ obtained by $w^+ \leftrightarrow w^-$.

Symmetry Breaking

The general geometry does not have a genuine $SL(2,R)_R \times SL(2,R)_L$ symmetry. There are two aspects to this:

- The embedding into the ambient spacetime is encoded in the potential $V(x)$ that breaks the symmetry ***explicitly***.

- The boundary conditions $\phi \equiv \phi + 2\pi$ act as a quotient on the conformal coordinates ($w^{\pm}$, y):

$$w^- \equiv w^- e^{-4\pi^2 \tilde{T}_L} , \quad w^+ \equiv w^+ e^{4\pi^2 \tilde{T}_R} .$$

  This breaks the symmetry ***spontaneously*** as $SL(2,R)_R \times SL(2,R)_L \to U(1)_R \times U(1)_L$.

The manifest $U(1)_R \times U(1)_L$ symmetries are those generated by the Killing vectors $\mathcal{R}_1 = i\partial_{t^+}$ and $\mathcal{L}_1 = i\partial_{t^-}$ where

$$\begin{aligned} t^- &= 2\pi \tilde{T}_R \Omega t - 2\pi \tilde{T}_L(\phi - \Omega t) \; , \\ t^+ &= 2\pi \tilde{T}_R \phi \; . \end{aligned}$$

The conformal coordinates $w^\pm$ and the CFT coordinates $t^\pm$ are related as

$$w^\pm = e^{\pm t^\pm} \; ,$$

which is the same as the relation between Minkowski and Rindler space.

We can thus characterize the symmetry breaking state: the CFT is thermal state CFT temperatures $\tilde{T}_R$ and $\tilde{T}_L$.

Length Scale

An important point: since $SL(2)$ is non-abelian, the construction determines the ***normalization*** of $\mathcal{R}_1 = i\partial_{t^+}$ and $\mathcal{L}_1 = i\partial_{t^-}$.

These operators are identified with L_0, $\overline{L}_0$ in the Virasoro algebra, so this normalizes the conformal weights in the conjectured dual CFT.

Concretely, the dimensionless CFT temperatures $\tilde{T}_R$ and $\tilde{T}_L$ are related to the dimensionful spacetime temperatures T_R, T_L by the length scale

$$L = \frac{\mu^2}{l}\left(\prod_{i=0}^{3}\cosh\delta_i + \prod_{i=0}^{3}\sinh\delta_i\right) .$$

Remark: any CFT description of black holes must specify a length scale for the physical theory, in this concrete manner.

Warning: other approaches give different values so the scale has not been settled yet. ***Establishing this effective scale is essential***.

Black Hole Entropy

There is not yet a persuasive derivation of black hole entropy from hidden conformal symmetry.

We proceed by assuming

- The dimensionless CFT temperatures have been correctly determined.

- The general black hole entropy is accounted for by the weakly coupled gas expression

$$S = \frac{\pi^2}{3} c(\tilde{T}_L + \tilde{T}_R) \ .$$

Comparison gives the central charge $c = 12J$.

The value $c = 12J$ was previously determined in Kerr/CFT using different methods.

In the hidden conformal symmetry approach, a theory with this central charge is proposed as the master theory for all black holes.

Explicit Breaking: the Potential $V(x)$

The full wave equation has a nontrivial angular equation (with eigenvalues l deformed from their habitual integral values in rotating backgrounds).

More importantly (in the present discussion) is the potential in the radial equation:

$$V(x) = x^2\omega^2(r_+ - r_-)^2 + \frac{1}{2}xM\omega^2(r_+ - r_-)\ .$$

Some circumstances when the potential be neglected are:

- Near extremality (large charges and/or nearly Kerr): $r_+ \sim r_-$.
- Low energy: $\omega(r_+ - r_-) \ll 1$.

In general it is simply incorrect to neglect the potential!

Interpretation:

- We need to to decouple the black hole states from the “far away” states.
- In the cases with an AdS_3 near horizon symmetry, the energy scales guarantee such a decoupling.
- In other near extreme cases, like near extreme Kerr, the energy scales similarly guarantee decoupling.
- But in general, they do not.

A Universal Box of AdS-Type

The scalar wave equation is essentially the inverse metric.

Procedure: throw away the potential term in the wave equation, then invert back to an effective geometry.

The resulting geometry has large metric factors far from the black hole, effectively producing a box.

In fact, the geometry is precisely (a projection of) AdS$_3$ with S^2 fiber!

A feature: the ***size*** of the AdS$_3$ constructed in this way is ambiguous, corresponding to a box of flexible size.

Accordingly, the central charge of the CFT in which the general black holes are excitations, does not have a definite central charge.

Summary

This lecture reviewed:

- The BTZ entropy from symmetries of AdS_3.
- The global diffeomorphism (modular transformation) is a central aspect.
- This approach is solid, well-established, and it generalizes to precision counting.
- A recent attempt to find a CFT description of general black holes: hidden conformal symmetry.
- My judgement: spontaneously broken conformal symmetry is a promising idea, but details are incomplete and/or suspicious.