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International Centre for Theoretical Physics



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**School and Workshop on D-brane Instantons, Wall Crossing and
Microstate Counting**

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M-theory, Liouville field and generalized matrix models

Alessandro TANZINI
SISSA
Via Bonomea, 265, 34100
TRIESTE
ITALY

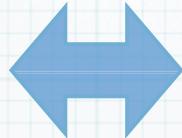
M-theory, Liouville field and generalised matrix models

Alessandro Tanzini, SISSA

Workshop on D branes, Wall crossing and
microstates counting, ICTP - 20 November 2010

Main theme:

S-duality in
4d gauge theories



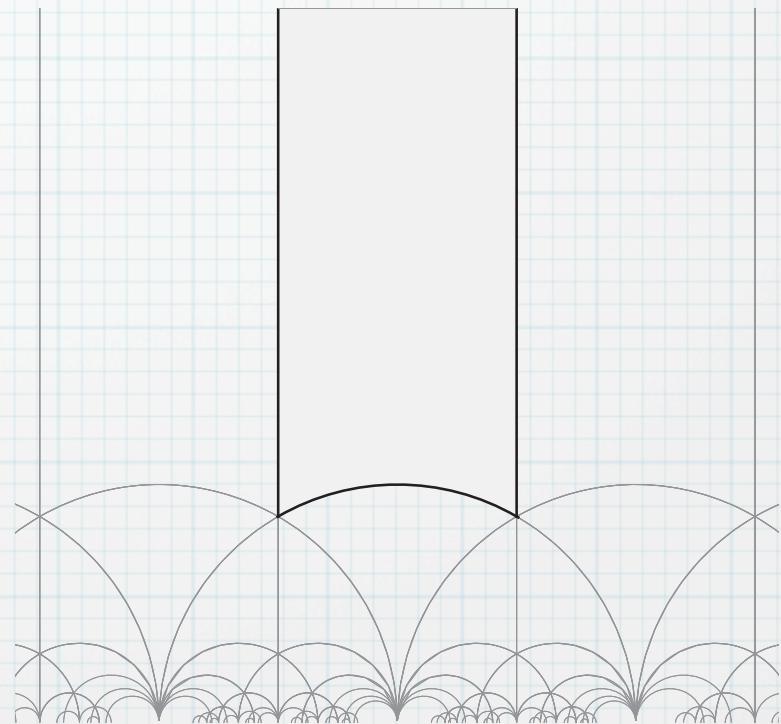
modular properties
of Riemann surfaces

Basic example:

N=4 SYM coupling

$$\tau \in \mathbb{H}/\mathrm{PSL}(2, \mathbb{Z})$$

moduli space of
complex structures
on torus



Plan:

- * 4d gauge th's and Riemann surfaces in M-theory
- * the AGT statement
- * Liouville/Toda 2d CFT as quantization of Hitchin integrable system
[Bonelli, A.T., 0909.4031]
- * generalised matrix models and check of AGT at all genera
[Bonelli, Maruyoshi, A.T., Yagi, in preparation]

N=2 superconformal gauge theories in 4d

global symmetry group

$$SU(2,2|2)$$

bosonic component:

$$SO(4,2) \times SU(2) \times U(1)$$

vacua: broken conformal symmetry \rightarrow Coulomb phase

low energy theory is abelian :

S-duality



e.m. duality

$$F_D = \star F$$

Seiberg-Witten curve

v.e.v's = periods of meromorphic differential

$$a = \int_A \lambda_{SW}$$

$$a_D = \int_B \lambda_{SW}$$

e.-m. duality

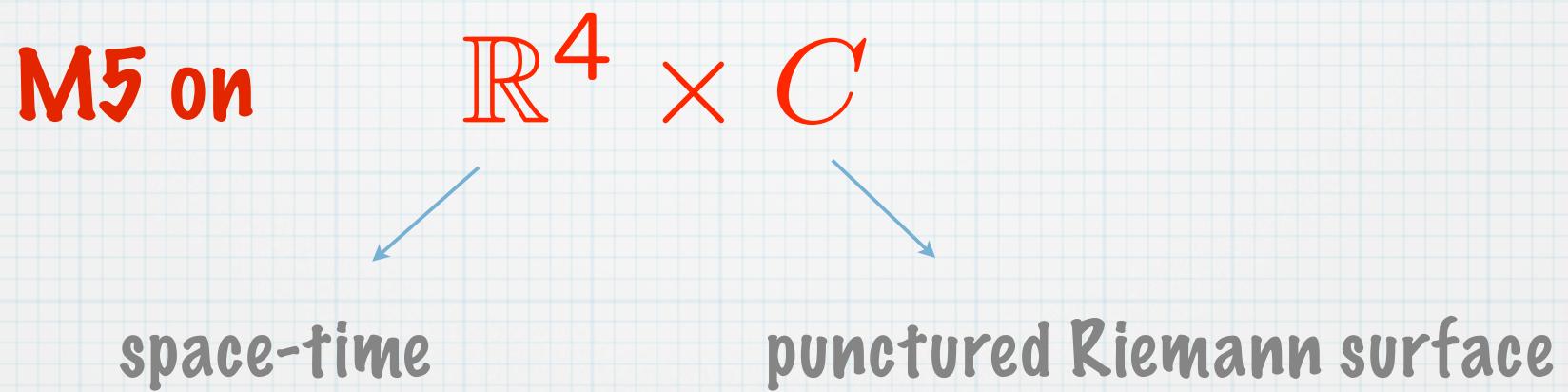


symplectic
change of basis

$$a_D = \frac{\partial \mathcal{F}}{\partial a}$$

The M-theory view:

natural framework:
fund. objects M2 & M5-branes



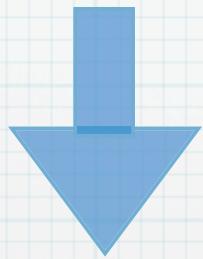
M2 provide observables in 4d and 2d theories

basic example:

M5 on $\mathbb{R}^4 \times T^2$

effective 4d theory N=4 SYM

I.-R. physics depends only on cplx. structure of torus



prediction of S-duality for N=4 SYM

A single M5 on

$$\mathbb{R}^4 \times \Sigma$$

6d theory: $T = F \wedge \Lambda + \star_4 F \wedge \star_2 \Lambda + V \wedge \omega + \star_4 V \wedge \star_2 \omega$

e.o.m.

$$dT = 0$$

$$\begin{aligned} d\star F &= 0 \\ dF &= 0 \end{aligned}$$

$U(1)^g$ gauge theory in 4d

e.-m. duality

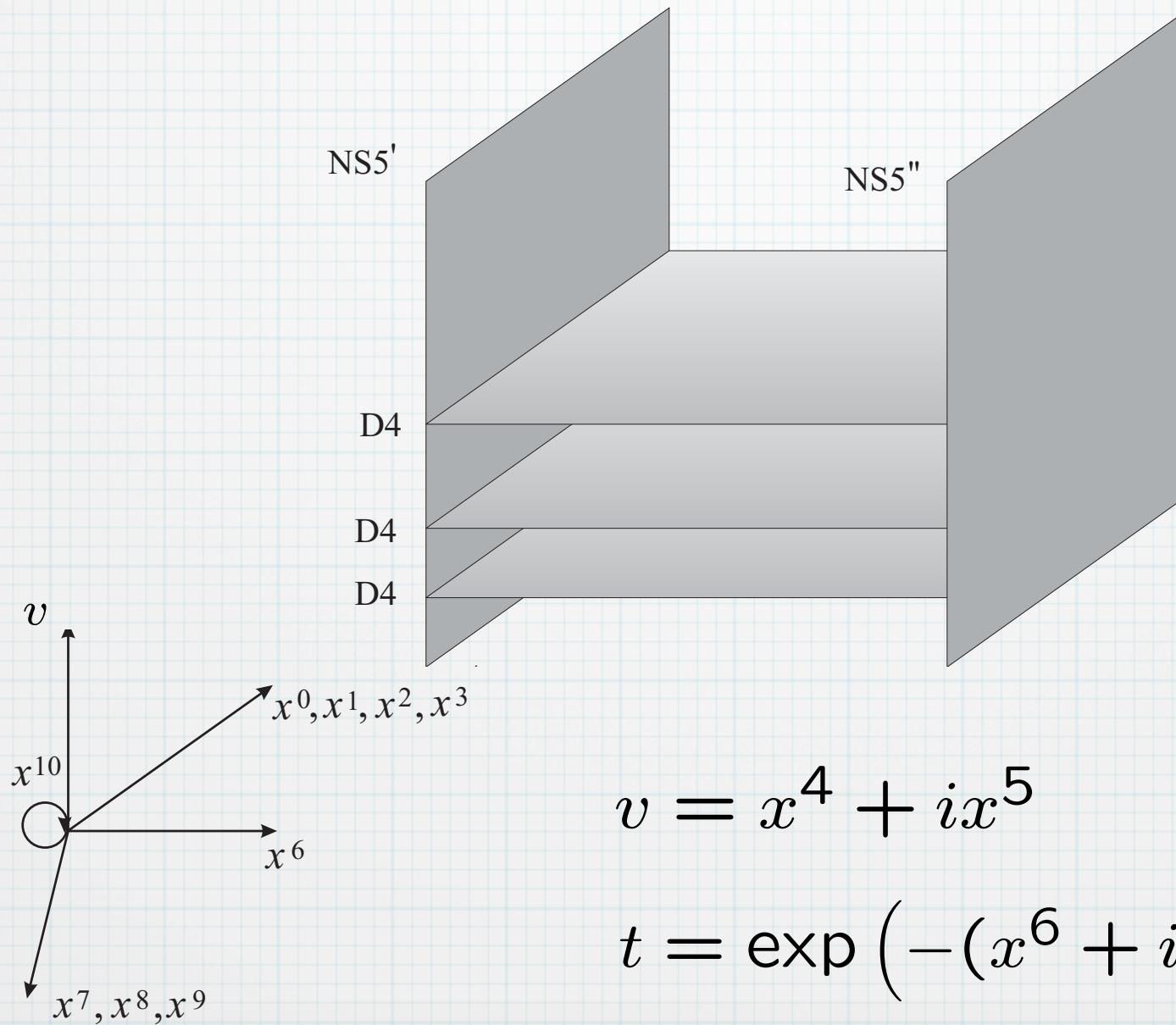


$$\Lambda \leftrightarrow \star_2 \Lambda$$

$$\Sigma \equiv \sum_{SW}$$

[Witten, 1997]

lift from type IIA :



\sum polynomial eq. in $(v, t) \in \mathbb{C} \times \mathbb{C}^*$

$$A(v)t^2 + B(v)t + C(v) = 0$$

$$A(v) = \prod_{i=1}^{N_f} (v - m_i)$$

$$B(v) = v^N + u_2 v^{N-2} + \dots + u_N$$

masses

Coulomb branch moduli

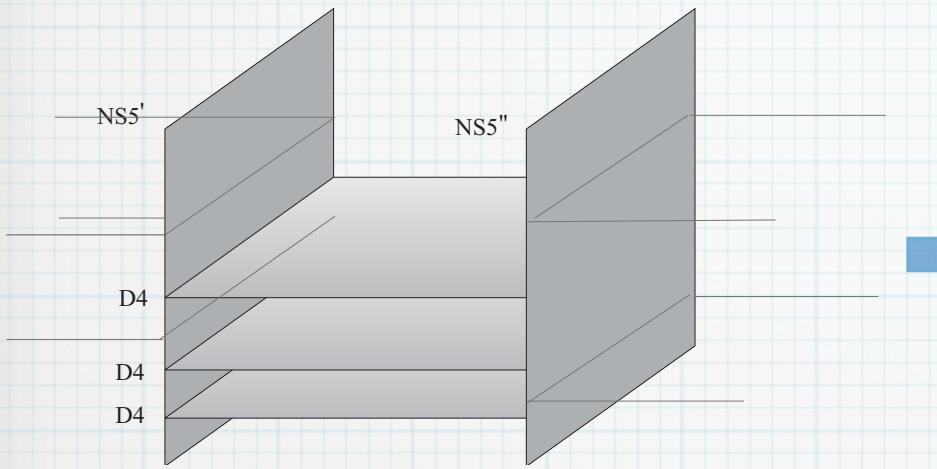
SW curve for $SU(N)$ SQCD $N_f=2N$

remark: degree $k+1$ pol. in t

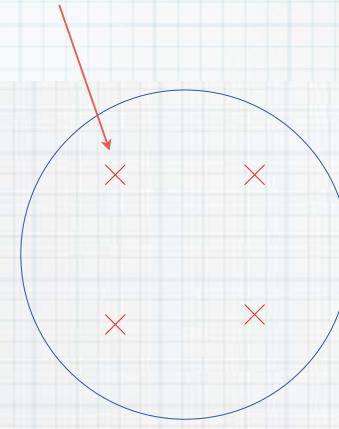


linear quiver
with k nodes

More general construction [Gaiotto, 2009] :



transversal M5s



N-M5 branes on punctured Riemann surface C :

A_{N-1} superconformal (2,0) theory on $\mathbb{R}^4 \times C$

The (2,0) theory is partially twisted along $C \rightarrow$
the scalars of the 4d vector multiplets become
holomorphic one forms

local M theory coordinates: $\mathbb{R}^4 \times T^*C$

fiber: $x = \frac{v}{z}$ base: z

\sum N-sheeted cover of C

natural SW differential as potential of
holom. symplectic form on T^*C

$$\lambda_{SW} = x dz$$

remark: for $z=t$ i.e. C a cylinder one recovers Witten's construction.

example: $SU(2)$ SQCD $N_f=4$

$$v^2(t-1)(t-t_1) = ut$$

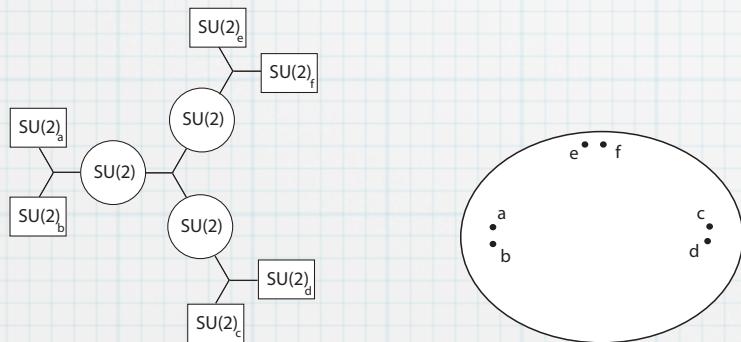
after change of variables:

$$x^2 = \frac{u}{t(t-1)(t-t_1)} = \phi_2(t)$$

$\phi_2(t)$ quadratic differential on C

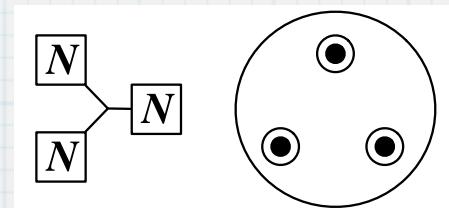
REMARKS:

- # residue of the simple pole: Coulomb branch parameter
- # in presence of masses: quadratic poles i.e. simple pole of SW differential
- # for higher rank groups i.e. N M5s : higher differentials w. richer polar structure described by partitions of N
- # generalised quivers with new building blocks e.g.



tri-fundamental hypers

strongly coupled sectors



The SW curve of this full class of N=2 theories is

$$x^N = \sum_{j=2}^N \phi_j(z) x^{N-j}$$

masses and
Coulomb moduli of
N=2 gauge th.s in 4d



extended
Teichmuller
space of C

marginal
deformations

simple poles of $\phi_j(z)$

relevant
deformations

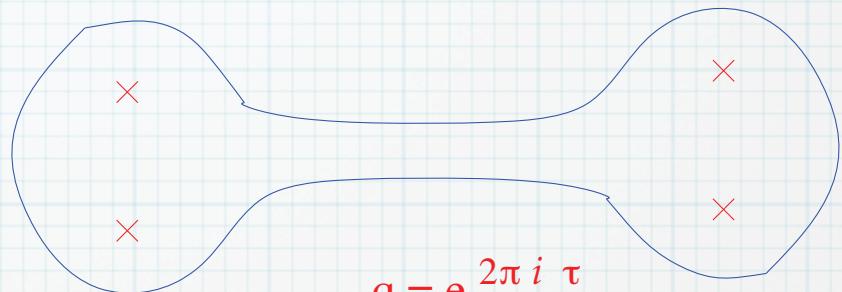
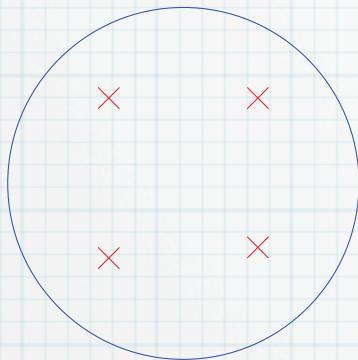
higher poles

weak coupling



nodal singularity on C

e.g.



$$q = e^{2\pi i \tau}$$

**S-dual weakly
coupled gauge th.s**



**different pant
decompositions of C**

e.g.

**three S-dual theories associated to the three
possible collisions of points**

4d observables should behave properly
under pant decomposition



modular properties

notable example:

instanton partition
functions of 4d
gauge th.s



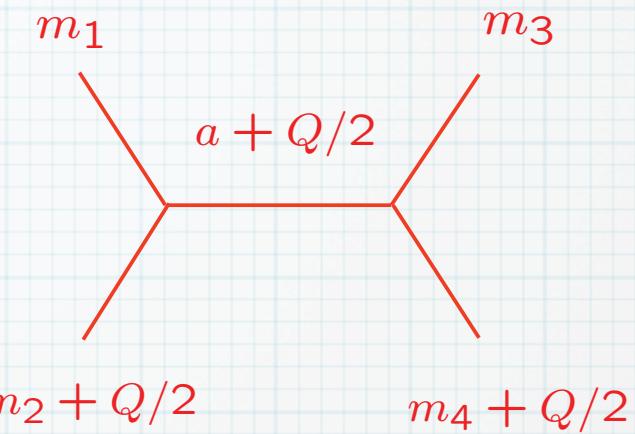
conformal blocks of
Liouville/Toda 2d CFT

AGT statement

AGT dictionary :

$$Z_{inst}^{U(2), N_f=4}(a, m_i, \epsilon_1, \epsilon_2; q) \propto$$

free boson factor for
U(1) center of mass



masses \leftrightarrow

external momenta

Coulomb moduli \leftrightarrow

internal momenta

gauge couplings \leftrightarrow

positions of vertex insertions

$$c = 1 + 6Q^2$$

$$Q = \frac{\epsilon_1 + \epsilon_2}{\sqrt{\epsilon_1 \epsilon_2}}$$

Why Liouville/Toda ?

Recall that the SW curve describing M-theory vacua

$$x^N = \sum_{j=2}^N W_j(z) x^{N-j}$$

is the spectral curve of the Hitchin integrable system:

flat $SL(N, \mathbb{C})$ connection $\nabla = (\partial + X) + \bar{\partial}$

$$X = \begin{pmatrix} 0 & 1 & & & \\ & 0 & 1 & & \\ & & & \ddots & \\ & & & & 0 & 1 \\ W_N & W_{N-1} & \dots & W_2 & 0 \end{pmatrix} \quad \text{W}_j \text{ commuting Hamiltonians}$$

2d CFT quantizing these moduli ?

Bring to METRIC connection via conjugation by a real field $\Phi \in sl(N, \mathbb{R})$ taking values in the Cartan subalgebra.

The Hitchin equations are

$$F_A + [\beta, \beta^\dagger] = 0$$

$$\bar{\partial}_A \beta = 0$$

The conjugates of $\beta = \mathcal{E}_+ dz$ and $A_{\bar{z}} = 0$ are

$\beta^\dagger = e^{-\Phi} \mathcal{E}_- e^\Phi d\bar{z}$ and $A_z = \partial_z \Phi$, with \mathcal{E}_\pm the sum of positive (neg.) simple roots.

Hitchin eqs.  flatness of

$$\nabla = \partial_A + \beta + (\partial_A + \beta)^\dagger = \partial + \mathcal{A} + (\partial + \mathcal{A})^\dagger$$

We need to gauge transform with $g = e^{-\Phi/2}$ to a manifestly **UNITARY** spectral connection:

$$\begin{aligned}\mathcal{A}_z^g &= \frac{1}{2} \partial_z \Phi + \exp\left(\frac{1}{2} ad_{\Phi}\right) \mathcal{E}_+ \\ \mathcal{A}_{\bar{z}}^g &= -\frac{1}{2} \partial_{\bar{z}} \Phi + \exp\left(-\frac{1}{2} ad_{\Phi}\right) \mathcal{E}_-\end{aligned}$$

Flatness of \mathcal{A}^g reads now:

$$\partial_z \partial_{\bar{z}} \Phi = \sum_i h_i e^{\alpha_i(\Phi)}$$

h_i Cartan elements of the Lie algebra and α_i its simple roots.

Toda A_N-1 field equations !

Example: Liouville (N=2)

Drinfel'd - Sokolov gauge:

$$X = \begin{pmatrix} 0 & 1 \\ (\partial_z \varphi)^2 - \partial_z^2 \varphi & 0 \end{pmatrix} \quad \bar{X} = 0 \quad \text{on-shell}$$

$$W_2 = (\partial_z \varphi)^2 - \partial_z^2 \varphi \quad \text{Liouville stress-energy tensor}$$

Unitary gauge:

$$\mathcal{A}_z^g = \begin{pmatrix} \frac{1}{2} \partial_z \varphi & e^\varphi \\ 0 & -\frac{1}{2} \partial_z \varphi \end{pmatrix}, \quad \mathcal{A}_{\bar{z}}^g = \begin{pmatrix} -\frac{1}{2} \partial_{\bar{z}} \varphi & 0 \\ e^\varphi & \frac{1}{2} \partial_{\bar{z}} \varphi \end{pmatrix}$$

flatness



$$\partial_z \partial_{\bar{z}} \varphi = e^{2\varphi}$$

Liouville
equation

Check I:

local BPS operators in
4d gauge th.



integrated
currents in 2d CFT

$$\langle \text{Tr}(\phi^j) \dots \rangle_{\mathcal{N}=2} \sim \left\langle \int \mu_{\text{collar}}^{(-j+1,1)} W_j \dots \right\rangle_{CFT_2}$$

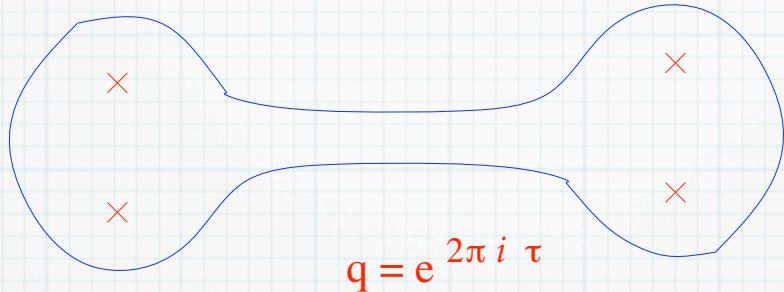
recall:

$$\langle \text{Tr}(\phi^j) \rangle_{\text{inst}} = \frac{1}{Z} \sum_Y \text{Tr}(\phi^j)_Y Z_Y q^{|Y|}$$

in particular, for $j=2$ $\text{Tr}(\phi^2)_Y = \epsilon_1 \epsilon_2 |Y|$ then:

$$\langle \text{Tr}(\phi^2) \rangle_{\text{inst}} = \epsilon_1 \epsilon_2 q \partial_q \ln Z$$

In the 2d CFT this corresponds to deriv. w.r.t. collar modulus:



which is realized precisely by the insertion of stress-energy tensor:

$$\int_{\Sigma} \mu_{collar}^{(-1,1)} T \quad T = W_2$$

$\mu_{collar}^{(-1,1)} = \partial_{\bar{z}} v^z$ is the Beltrami diff. supported on the collar,
 $v^z = \frac{\ln |z|}{\ln |\tau|}$.

OPE of local BPS observables:

$$\begin{aligned}\langle Tr(\phi^2)Tr(\phi^j)\rangle &= \frac{\epsilon_1\epsilon_2}{Z} \sum_Y Tr(\phi^j)_Y Z_Y |Y| q^{|Y|} \\ &= \epsilon_1\epsilon_2 q \partial_q \langle Tr(\phi^j) \rangle + \epsilon_1\epsilon_2 \frac{Z'}{Z} \langle Tr(\phi^j) \rangle\end{aligned}$$

gives

$$\left\langle \frac{1}{2} Tr\phi^2 Tr\phi^j \right\rangle - \left\langle \frac{1}{2} Tr\phi^2 \right\rangle \langle Tr\phi^j \rangle = -q \partial_q \langle Tr\phi^j \rangle$$

which corresponds to the **W-algebra of integrated currents**

$$-q \partial_q \left\langle \int_{\Sigma} \mu_{collar}^{(-j+1,1)} W_j \right\rangle = \left\langle \left(\int \mu_{collar}^{(-1,1)} W_2 \right) \left(\int \mu_{collar}^{(-j+1,1)} W_j \right) \right\rangle_{conn}$$

Check II:

The anomaly polynomial of N - M5 branes

[Harvey, Minasian, Moore 1998]

$$I[A_{N-1}] = (N-1) \left[I[1] + N(N+1) \frac{p_2(\mathcal{N}_Y)}{24} \right],$$

reproduces the Toda 2d CFT central charge

$$c = (N-1) (1 + Q^2 N(N+1))$$

by dimensional reduction.

Detailed check via equivariant integration over \mathbb{R}^4 by Alday,
Benini, Tachikawa 0909.4776

Can we recover M-theory geometry from Liouville CFT ?

Matrix models as a link between 4d gauge theories and 2d CFTs [Dijkgraaf-Vafa, 2009]

Liouville field emerging as collective large N excitation of matrix model:

- # Liouville correlators expanded in free-field correlators on \mathbb{C}
- # holomorphic factorization \rightarrow generalised matrix model - eigenvalues living on \mathbb{C}

Liouville correlator $\left\langle \prod_{k=1}^n e^{-2m_k \phi(w_k, \bar{w}_k)} \right\rangle$ in terms of free fields

residue at the N
pole

$$A_N = \frac{(-\mu)^N}{2bN!} \left\langle \prod_{I=1}^{2g-2} e^{Q\tilde{\phi}(\xi_I)} \int \prod_{i=1}^N d^2 z_i |\omega(z_i)|^2 e^{2b\tilde{\phi}(z_i)} \prod_{k=1}^n e^{-2m_k \tilde{\phi}(w_k)} \right\rangle_{\text{free on } \mathcal{C}}$$

with $N \equiv \frac{1}{b} \sum_k m_k + \frac{Q}{b}(1-g)$. Explicit expression :

$$\begin{aligned} A_N \propto & \delta(N) \prod_{a=1}^g \int_{-\infty}^{+\infty} dp_a \left| \exp \left(2\pi i \sum_{a,b} p_a p_b \tau_{ab} + 2\pi \sum_a p_a \left(Q \sum_I \int_{\xi_I}^{z_I} \omega_a - 2 \sum_k m_k \int_{w_k}^{z_k} \omega_a \right) \right) \right|^2 \\ & \prod_{i=1}^N \int d^2 z_i |\omega(z_i)|^{2+2b^2} \left| \exp \left(4\pi b \sum_{a,i} p_a \int_{w_k}^{z_i} \omega_a \right) \right|^2 \\ & \left| \prod_{i < j} E(z_i, z_j)^{-2b^2} \prod_{i,k} E(z_i, w_k)^{2bm_k} \prod_{I,i} E(\xi_I, z_i)^{-1-b^2} \right|^2 \end{aligned}$$

Large N factorization in holomorphic x antiholomorphic integral:

saddle point equations

$$b \sum_{j \neq i} \frac{E'(z_i, z_j)}{E(z_i, z_j)} dz_i - \sum_{k=1}^n m_k \frac{E'(z_i, w_k)}{E(z_i, w_k)} dz_i - 2\pi \sum_{a=1}^g p_a \omega_a(z_i) = 0$$

with $E'(z_1, z_2) \equiv \partial_{z_1} E(z_1, z_2)$.

The derivative w.r.t. \bar{z}_i give just the complex conjugate equations.

Holomorphic integral as a generalised matrix model :

conformal inv. factor

$$Z \equiv \int \prod_{i=1}^N dz_i \left[\omega(z_i)^{1+b^2} \prod_{i,I} E(z_i, \xi_I)^{-1-b^2} \right] \prod_{1 \leq i < j \leq N} E(z_i, z_j)^{-2b^2} \prod_i E(z_i, z^*)^{2b \sum_k m_k / g_s}$$

generalised
Van der Monde

reference point
for the potential

$$\times \exp \left(\frac{b}{g_s} \sum_{i=1}^N \left(\sum_{k=1}^n 2m_k \log \frac{E(z_i, w_k)}{E(z_i, z^*)} + 4\pi \sum_{a=1}^g p_a \int^{z_i} \omega_a \right) \right)$$

matrix model potential

$$N \equiv \frac{1}{b} \sum_k m_k + \frac{Q}{b} (1 - g)$$

momentum conservation ensures well-def. of
the measure as holomorphic one-form

Counting moduli :

masses are explicitly parametrized by momenta of
Liouville vertex operators

Coulomb moduli are parameterized by the g-loop momenta and filling fractions :

$$g + \text{[} 2g - 2 + n + 1 \text{]} - 1 - 1 = 3g - 3 + n$$


 A blue speech bubble contains the term $2g - 2 + n + 1$. Three arrows point downwards from this term to three labels below: "loop momenta", "critical points of the matrix model potential", and "base point". Another arrow points from the term -1 to the label "momentum conservation".

loop momenta

critical points of the matrix model potential

base point

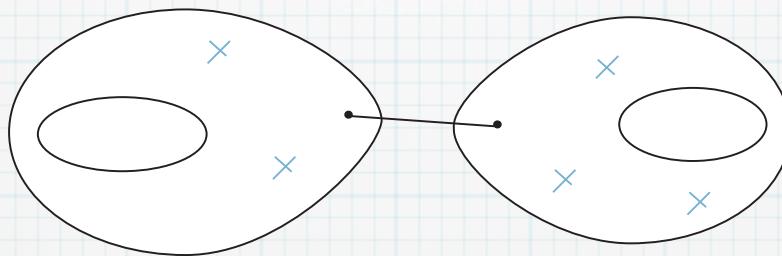
momentum conservation:

Degenerations :

$$Z_N^{\mathcal{C}_{g,n}}(\{w_k\}, \{m_k\}, p_a, \nu)$$

dividing

$$E(z', z'') \sim E_1(z', p_1) E_2(p_2, z'') t^{-1/2}$$



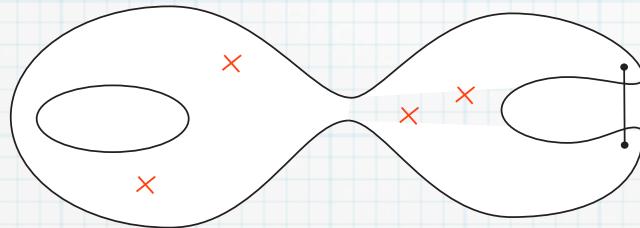
$$Z_{N'}^{\mathcal{C}_{g_1, n_1+1}}(\{w_{k'}\} \cup P_1, \{m_{k'}\} \cup m_1^*, p_{a'}, \nu') Z_{N''}^{\mathcal{C}_{g_2, n_2+1}}(\{w_{k''}\} \cup P_2, \{m_{k''}\} \cup m_2^*, p_{a''}, \nu'')$$

the Coulomb modulus of the degenerating collar is traded for a mass:

$$m_1^* = bN' - \sum_{k'} m_{k'} + Q(g_1 - 1) \quad m_1^* + m_2^* = -Q$$

pinching:

$$\omega_g(z) \sim \partial_z \log \frac{E(z, P_1)}{E(z, P_2)}$$



$$Z_N^{\mathcal{C}_{g-1,n+2}} (\{w_k\} \cup P_1 \cup P_2, \{m_k\} \cup m_+^* \cup m_-^*, \hat{p}_a, \nu)$$

the **Coulomb** modulus of the degenerating handle is traded for a **mass**:

$$m_{\pm}^* = -\frac{Q}{2} \pm 2\pi p_g$$

Spectral curve :

critical points

$$\frac{b}{g_s} dW(z_i) - 2b^2 \sum_{j \neq i} d_{z_i} \log \left(\frac{E(z_i, z_j)}{E(z_i, z_*)} \right) = 0$$

N_k eigenvalues distributed on line segments C_α around the critical points, $\alpha = 1, \dots, n + 2g - 2$.

Introduce the spectral density of eigenvalues, normalised as

$$\int_{C_\alpha} \rho(z) = bg_s N_\alpha \equiv \nu_\alpha . \text{ Then}$$

$$dW(z) - 2 \int_{\sum_\alpha C_\alpha} \rho(z') dz' \log \left(\frac{E(z, z')}{E(z, z^*)} \right) = 0$$

and the genus-zero free energy reads:

$$\mathcal{F} = \int_{\sum_\alpha C_\alpha} \rho(z) W(z) - \int_{\sum_\alpha C_\alpha} \int_{\sum_\alpha C_\alpha} \rho(z) \rho(z') \log \frac{E(z, z')}{E(z, z')}$$

Introduce the resolvent:

$$R(z) \equiv \int_{\sum_{\alpha} C_{\alpha}} \rho(z') d_z \log \left(\frac{E(z, z')}{E(z, z^*)} \right)$$

which behaves on the cuts C_{α} as

$$R(z + i\varepsilon e^{i\varphi(z)}) + R(z - i\varepsilon e^{i\varphi(z)}) = 2P \int_{\sum_{\alpha} C_{\alpha}} \rho(z') d_z \log \left(\frac{E(z, z')}{E(z, z^*)} \right) = dW(z),$$

$$R(z + i\varepsilon e^{i\varphi(z)}) - R(z - i\varepsilon e^{i\varphi(z)}) = \oint_{C_{\alpha}} \rho(z') d_z \log \left(\frac{E(z, z')}{E(z, z^*)} \right) = -2\pi i \rho(z)$$

we have:

$$R(z) = \frac{1}{2} dW(z) + R(z)_{\text{sing}}$$

with :

$$R(z + i\varepsilon e^{i\varphi(z)})_{\text{sing}} + R(z - i\varepsilon e^{i\varphi(z)})_{\text{sing}} = 0.$$

$$R(z + i\varepsilon e^{i\varphi(z)})_{\text{sing}} - R(z - i\varepsilon e^{i\varphi(z)})_{\text{sing}} = -2\pi i \rho(z).$$

plugging the explicit form of $W(z)$ one gets:

$$R_{\text{sing}}(z) = d_z \int_{\sum_\alpha C_\alpha} \rho(z') \log E(z, z') - \sum_{k=1}^n m_k d_z \log E(z, w_k) - 2\pi \sum_{a=1}^g p_a \omega_a(z)$$

- no base point dependence
- cuts in C_α , simple poles at w_k with residues m_k
- change sign across the cuts $\rightarrow R(z)_{\text{sing}}^2$ has only singularities in w_k with at most quadratic poles.

Spectral curve :

$$R_{\text{sing}}(z)^2 = \sum_{k=1}^n m_k^2 \eta(z, w_k) + \zeta(z)$$

$\eta(z, w_k)$ quadratic Strebel differential

$\zeta(z)$ quadratic differential w. at most simple poles at w_k

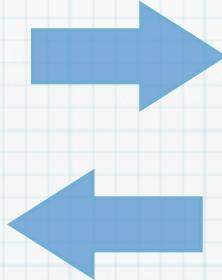
coincides with the Seiberg-Witten curve for $SU(2)$
gauge theory on $\mathcal{C}_{g,n}$

Summary

M theory
compactifications on

$$\mathbb{R}^4 \times \mathcal{C}_{g,n}$$

Hitchin system in
unitary gauge



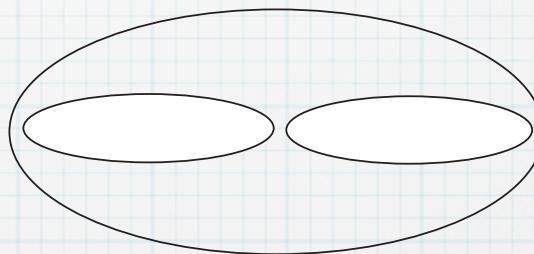
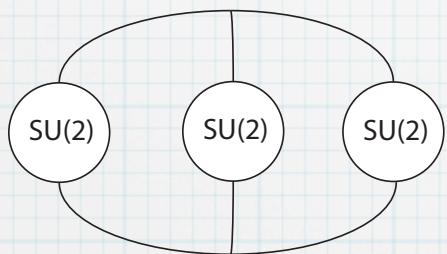
Liouville theory
on $\mathcal{C}_{g,n}$

generalised
matrix model at
large N

Open questions :

finite N factorization & contour
prescription for the matrix model

instanton counting for tri-fundamentals



Further directions

- # M theory/string theory description of instanton equivariant parameters [Antoniadis, Narain, Taylor 2009] and generalised HAEs for instanton partition functions.
- # surface operators & vortex counting
 - localisation on a lagrangian submanifold of the instanton moduli space
 - relation with topological strings via refined topological vertex [G. Bonelli, A.T., Z. Jian, work in progress]
- # application to non abelian statistics in FQH effect:
 $Z_{inst}^{b,1/b}$ = FQH N - body wave function [Santachiara, A.T. 2009]

Thanks !