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Lectures on D-instanton counting

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Introduction to D-brane instantons

Contents

1 Motivations

- **Gauge Theories**: Correlator are computed by path integrals dominated by the minima of the classical action and quantum fluctuations around them. Fluctuations around the trivial vacuum, i.e. where all fields are set to zero, give rise to loop corrections suppressed in powers of the gauge coupling. Instantons are non trivial solutions of the classical action and lead to exponentially suppressed corrections to the correlators. These corrections are important in theories like QCD where the gauge coupling get strong at low energies. Understanding of the instanton dynamics is then crucial in addressing the study of phenomena in the strong coupling regime like confinement or chiral symmetry breaking, etc.

- **D-brane instantons**: In string theory the situation is even more dramatic. The gauge coupling is given by the vev of a field and therefore weak coupling expansions are rather unnatural. Moreover, the discovery of strong weak dualities has revealed an impressive web of connections between the various string theories, gauge theories and gravity making clear that a better control of the non-perturbative dynamics is needed in order to address questions like, what is the string vacua ?, moduli stabilization, etc. In addition the structure of non-perturbative objects in string theory is very rich, and include beside the known gauge instanton effects a new class of instantons , named "stringy" or "exotic", of great phenomenological interest.

- **Applications of the instanton calculus**:
  I) String Phenomenology: Instantons generate interesting couplings like Majorana mass terms for neutrinos, Yukawa couplings, etc at scale not linked to the gauge theory scale (Hierarchy).
  II) Moduli stabilization: Instanton generated superpotentials ca be used to stabilized Kahler moduli
  III) Black hole physics: Black holes in d=4 can be built out of D4-D2-D0 systems wrapping divisors of a CY. The instanton partition function provides then a microscopic description of the corresponding black hole entropy
  IV) AGT conjecture: Instanton partition functions for N=2 SCFT's can be related to the conformal blocks of 2D integrable systems
  V) Topological string theories: Certain topological string amplitudes can be

extracted from the Instanton partition functions in presence of non-trivial gravitational backgrounds.

We will focus on $\mathcal{N} = 2, 4$ set ups.

2 Outline

- **Open strings**: D-branes, Brane intersections and quivers, $\mathcal{N} = 1, 2, 4$ gauge theories.
  hep-th/0007170 Johnson, hep-th/0512067 Bertolini, Billo, Lerda, Morales, Russo

- **D-brane instantons**: Instantons in gauge theories, ADHM construction, gauge and exotic instantons, D(-1)D3 systems
  Dorey, Hollowood, Khoze, Mattis hep-th/0206063.

- **Multi-instanton calculus**: Localization, Instanton moduli space symmetries, Instanton partition function.
  hep-th/0206161 Nekrasov, hep-th/0211108 Bruzzo, Fucito, Morales, Tanzini.

- **Explicit computations**: The $\mathcal{N} = 2$ prepotential. Black hole counting, Saddle point analysis.
  hep-th/0208176 Flume, Poghossian. hep-th/0610154 Fucito, Morales, Poghossian. hep-th/0306238 Nekrasov, Okounkov

3 Open strings

3.1 String action

In the conformal gauge the open string action can be written as

$$S = \frac{1}{2\pi\alpha'} \int d^2\sigma \, \left[2\, \partial_+ X^M \, \partial_- X_M + i\, \psi_+^M \partial_- \, \psi_{+M} + i\, \psi_-^M \partial_- \, \psi_{-M} \right] \tag{3.1}$$

with $\sigma \in [0, \pi]$, $\tau \in [-\infty, \infty]$, spacetime Lorentz indices raised and lowered with $\eta_{MN} = (- + + + + ..)$ and lightcone variables defined by

$$\sigma^\pm = \tau \pm \sigma \qquad \partial_\pm = \tfrac{1}{2}(\partial_\tau \pm \partial_\sigma) \qquad \eta^{+-} = \eta^{-+} = -2 \tag{3.2}$$

The action should be supplemented by the Super Virasoro constraints

$$\begin{aligned} T_{++} &= \partial_+ X^M \partial_+ X_M + \tfrac{i}{2}\psi_+^M \partial_+ \psi_{+M} = 0 \\ T_{--} &= \partial_- X^M \partial_- X_M + \tfrac{i}{2}\psi_-^M \partial_- \psi_{-M} = 0 \\ G_+ &= \psi_+^M \partial_+ X_M = 0 \\ G_- &= \psi_-^M \partial_- X_M = 0 \end{aligned} \tag{3.3}$$

3.2 Equations of motion and boundary conditions

The equations of motion following from (3.1) are

$$\begin{aligned} &\partial_-\partial_+ X^M = 0 \qquad \Rightarrow \qquad X^M = q^M + \tfrac{1}{2}\left[X_+^M(\sigma^+) + X_-^M(\sigma^-)\right] \\ &\partial_+ \psi_-^M = 0 \qquad \Rightarrow \qquad \psi_+^M = \psi_+^M(\sigma^+) \\ &\partial_- \psi_+^M = 0 \qquad \Rightarrow \qquad \psi_-^M = \psi_-^M(\sigma^-) \end{aligned} \tag{3.4}$$

Cancelation of the boundary terms requires:

$$\begin{aligned} &\delta X^M \partial_\sigma X_M \big|_{\sigma=0}^{\sigma=\pi} = 0 \\ &(\delta\psi_+^M \psi_{+M} - \delta\psi_-^M \psi_{-M})\big|_{\sigma=0}^{\sigma=\pi} = 0 \end{aligned} \tag{3.5}$$

There are two possible boundary conditions: Neumann (N) $\partial_\sigma X|_{\sigma=0,\pi} = 0$ or Dirichlet (D) $X|_{\sigma=0,\pi} =$ cost for the boson fields and $(\psi_+ \pm \psi_-)|_{0,\pi} = 0$ for the fermion fields. We can write then as

$$\begin{aligned} \partial_- X(0,\tau) &= \eta_0\, \partial_+ X(0,\tau) \qquad \partial_- X(\pi,\tau) = \eta_\pi\, \partial_+ X(\pi,\tau) \\ \psi_-^\mu(0,\tau) &= \eta_0\, \psi_+^\mu(0,\tau) \qquad \psi_-^\mu(\pi,\tau) = \eta_\pi\, \epsilon\, \psi_+^\mu(\pi,\tau) \end{aligned}$$

with

$$\eta_{0,\pi} = \begin{cases} +1 & \text{Neuman} \\ -1 & \text{Dirichlet} \end{cases} \qquad \epsilon = \begin{cases} +1 & \text{Ramond(R)} \\ -1 & \text{Neveu Schwarz(NS)} \end{cases} \tag{3.6}$$

A way to solve the boundary conditions is to think of $X_{L,R}$ and $\psi_\pm$ as fields on the whole complex plane satisfying:

$$\begin{aligned} X_-(\sigma,\tau) &= \eta_0 X_+(\sigma,\tau) \qquad X_+(\sigma+2\pi,\tau) = \eta_\pi^{-1}\, \eta_0\, X_+(\sigma,\tau) \\ \psi_-(\sigma,\tau) &= \eta_0\, \psi_+(\sigma,\tau) \qquad \psi_+(\sigma+2\pi,\tau) = \epsilon\eta_\pi^{-1}\, \eta_0\, \psi_+(\sigma,\tau) \end{aligned} \tag{3.7}$$

D-branes: Open strings can be thought as ending on $(p+1)$-dimensional hypersurfaces along which the open string end is taken with Neumann boundary conditions.

More precisely we take Neumann boundary conditions for the components X^μ with $\mu = 0, ..p$ and Dirichlet boundary conditions for X^i with $i = p+1, ..9$. In general an open string ends on two different Dp-branes with positions $X^i|_{\sigma=0,\pi}$.

In general for boundaries ending on a brane at the angles $\pi\theta_{0,\pi}$ one finds

$$\eta_{0,\pi} = e^{2\pi i\theta_{0,\pi}} \tag{3.8}$$

3.3 Mode expansions

The

$$\begin{aligned} X_+^M(\sigma,\tau) &= \sqrt{2\alpha'}\,(\sigma+\tau)\,\alpha_0^M + i\,\sqrt{2\alpha'}\sum_{n\neq 0}\frac{\alpha_n^M}{n}\,e^{-ni(\sigma+\tau)} \\ \psi_+^M(\sigma,\tau) &= \sqrt{\alpha'}\sum_{r\in\mathbb{Z}-a_0}\psi_r^M\,e^{-ir(\sigma+\tau)} \end{aligned} \tag{3.9}$$

with $a = 0$ and $a_0 = \frac{1}{2}$ in the Ramond and Neveu-Schwarz sectors respectively and

$$\alpha_0^M = \sqrt{2\alpha'}\,p_+^M \tag{3.10}$$

Quantization conditions:

$$\begin{aligned} [\dot{X}^M(\sigma,\tau), X^N(\sigma',\tau)] &= -2i\alpha'\,\eta^{MN}\,\delta(\sigma-\sigma') \\ \{\psi_\alpha^M(\sigma,\tau), \psi_\beta^N(\sigma',\tau)\} &= 2\pi\alpha'\,\eta^{MN}\,\delta_{\alpha\beta}\,\delta(\sigma-\sigma') \end{aligned} \tag{3.11}$$

They imply the following commutation relations ($p = \frac{1}{2}p_+\frac{1}{2}p_-$)

$$\begin{aligned} [\alpha_m^M, \alpha_n^N] &= m\,\eta^{MN}\,\delta_{m+n,0} \qquad [q^M, p^N] = i\eta^{MN} \\ \{\psi_r^M, \psi_s^N\} &= \eta^{MN}\,\delta_{r+s,0} \end{aligned} \tag{3.12}$$

Virasoro generators:

$$\begin{aligned} T(z) &= -\tfrac{1}{2}(\partial X^M\partial X_M + \partial\psi^M\psi_M) \\ G(z) &= -\tfrac{1}{2}\psi^M\partial X_M \end{aligned} \tag{3.13}$$

In terms of the string modes

$$\begin{aligned} L_n - \tfrac{c}{24}\delta_{n,0} &= \frac{1}{2\alpha'}\oint\frac{dz}{2\pi i}\,z^{-n+1}\,T(z) \\ &= \tfrac{1}{2}\sum_{m\in\mathbb{Z}}\alpha_{n-m}^M\,\alpha_{mM} + \tfrac{1}{2}\sum_{r\in\mathbb{Z}-a}(r-\tfrac{1}{2}n)\,b_{n-r}^M\,b_{rM} \\ G_r &= \frac{1}{2\alpha'}\oint\frac{dz}{2\pi i}\,z^{-n+\frac{1}{2}}\,G(z) \\ &= \sum_{m\in\mathbb{Z}} b_{r-m}^M\,\alpha_{mM} \end{aligned} \tag{3.14}$$

with

$$
\begin{aligned}
L_0 - \tfrac{c}{24} &= \tfrac{1}{2}\sum_{m\in\mathbb{Z}} \alpha^M_{-m}\,\alpha_{mM} + \tfrac{1}{2}\sum_{r\in\mathbb{Z}-a_0} r\, b^M_{-r}\, b_{rM} \\
&= \alpha' p^2 + \sum_{m=1}^{\infty} :\alpha^i_{-m}\,\alpha^i_m: + \sum_{r=1-a_0}^{\infty} :r\, b^i_{-r}\, b^i_M: + \tfrac{D-2}{2}\left[\sum_{m=1}^{\infty} m - \sum_{m=1}^{\infty}(m-a_0)\right] \\
&= \alpha' p^2 + N + \tfrac{D-2}{2}\,[\zeta(-1,1) - \zeta(-1,1-a_0)] = \alpha' p^2 + (N-a) \qquad (3.15)
\end{aligned}
$$

with $D = 10$, $a = a_0 = 0$ for the Ramond and $a = a_0 = \frac{1}{2}$ for the Neveu-Schwarz. We have used the identity[1]

$$
\zeta(-1,1-a_0) = \sum_{m=1}^{\infty}(m-a_0) = -\tfrac{1}{12} - \tfrac{1}{2}a_0(a_0-1) \qquad (3.16)
$$

Acting on the superstring vacuum and equating the two sides of this equation one finds

$$
\begin{aligned}
(L_0 - \tfrac{c}{24})|0\rangle_{\rm NS} = -\tfrac{1}{2}|0\rangle_{NS} \quad &\Rightarrow \quad L_0|0\rangle_{\rm NS} = 0 \\
(L_0 - \tfrac{c}{24})|0\rangle_{\rm R} = 0|0\rangle_{\rm R} \quad &\Rightarrow \quad L_0|0\rangle_{\rm R} = \tfrac{1}{2}|0\rangle_{\rm R}
\end{aligned}
\qquad (3.17)
$$

with $\frac{c}{24} = 8 \times \frac{3}{2} \times \frac{1}{24} = \frac{1}{2}$.

Physical states are defined by

$$
\begin{aligned}
L_n|\Phi_{\rm phys}\rangle = \bar{L}_n|\Phi_{\rm phys}\rangle = 0 \qquad & n > 0 \\
G_r|\Phi_{\rm phys}\rangle = \bar{G}_r|\Phi_{\rm phys}\rangle = 0 \qquad & r > 0 \\
(L_0 - \tfrac{c}{24})|\Phi_{\rm phys}\rangle = (\bar{L}_0 - \tfrac{\tilde{c}}{24})|\Phi_{\rm phys}\rangle = 0 & \\
\left(\frac{1+(-)^F}{2}\right)|\Phi_{\rm phys}\rangle = 0 &
\end{aligned}
\qquad (3.18)
$$

Physical states has to satisfied in addition the zero mode conditions in (3.18), the mass shell and the so called level matching condition

$$
\begin{aligned}
M^2 &= -p^2 = \frac{1}{\alpha'}(N-a) \\
N &= \sum_{n=1}^{\infty} \alpha^i_{-n}\,\alpha_{ni} + \sum_{r=1-a} r b^i_{-r}\, b_{ri}
\end{aligned}
\qquad (3.19)
$$

with $a = 0, \frac{1}{2}$ for the R,NS sectors .

[1]The zeta di Hurwitz is defined as $\zeta(s,a) = \sum_{n=0}^{\infty}(n+a)^{-s}$.

3.4 Twisted open strings

Open string connecting branes at an angle $\pi\theta$ satisfied twisted boundary conditions (3.7) with

$$e^{2\pi i\,\theta} = \eta_\pi^{-1}\,\eta_0 \tag{3.20}$$

The mode expansion can be written as

$$\begin{aligned} X_+^i(\sigma,\tau) &= i\sqrt{2\alpha'}\sum_{n\in\mathbb{Z}-\theta_i}\frac{\alpha_n^M}{n}\,e^{-in(\sigma+\tau)} \\ \psi_+^i(\sigma,\tau) &= \sqrt{\alpha'}\sum_{r\in\mathbb{Z}-\theta_i-a_0}\psi_r^M\,e^{-ir(\sigma+\tau)} \end{aligned} \tag{3.21}$$

Quantization conditions:

$$[\alpha_m^M,\alpha_n^N] = m\,\eta^{MN}\,\delta_{m+n,0} \qquad \{\psi_r^M,\psi_s^N\} = \eta^{MN}\,\delta_{r+s,0} \tag{3.22}$$

Mass condition

$$\begin{aligned} M^2 &= \tfrac{1}{2}\sum_{i,m\in\mathbb{Z}-\theta_i}\alpha_{-m}^i\,\alpha_{mi} + \tfrac{1}{2}\sum_{i,r\in\mathbb{Z}-\theta_i-a_0} r\,b_{-r}^i\,b_{ri} + \text{h.c.} \\ &= N + N_0 \end{aligned} \tag{3.23}$$

with

$$N = \sum_{i,m=1-\theta_i}^{\infty} :\alpha_{-m}^i\,\alpha_{mi}: + \sum_{i,r=1-\theta_i-a_0}^{\infty} :r\,b_{-r}^i\,b_{ri}: \tag{3.24}$$

$$\begin{aligned} N_0 &= \sum_{i,m=1}^{\infty}(m-\theta_i) - \sum_{i,m=1}^{\infty}(m-\theta_i-a_0) = \sum_i\left[\zeta(-1,1-\theta_i)-\zeta(-1,1-a_0-\theta_i)\right] \\ &= \sum_i\left[-\tfrac{1}{2}\theta(\theta_i-1)+\tfrac{1}{2}(\theta_i+a_0)(\theta_i+a_0-1)\right] = \delta_{a_0,\frac{1}{2}}(-\tfrac{1}{2}+\tfrac{1}{2}\sum_i\theta_i) \end{aligned} \tag{3.25}$$

with $a = a_0 = 0$ for the Ramond and $a = a_0 = \frac{1}{2}$ for the Neveu-Schwarz. We have used the identity[2]

$$\zeta(-1,1-a_0) = \sum_{m=1}^{\infty}(m-a_0) = -\tfrac{1}{12} - \tfrac{1}{2}a_0(a_0-1) \tag{3.26}$$

[2]The zeta di Hurwitz is defined as $\zeta(s,a) = \sum_{n=0}^{\infty}(n+a)^{-s}$.

3.5 Braneworlds

$\mathcal{N}=1$ gauge theories

$$L = \mathrm{tr}\int d^2\theta d^2\bar\theta\,\Phi^\dagger\, e^V\,\Phi + \mathrm{tr}\left(\int d^2\theta\left[\tfrac{\tau}{16\pi}\,W^\alpha W_\alpha + W(\Phi)\right] + \mathrm{h.c.}\right) \tag{3.27}$$

with

$$\begin{aligned} V &= -\theta\sigma^\mu\bar\theta\,A_\mu(x) + i\theta\theta\bar\theta\bar\lambda(x) + \tfrac{1}{2}\theta\theta\bar\theta\bar\theta D(x) \\ \Phi &= \phi(y) + \sqrt{2}\theta\psi(y) + \theta\theta F(y) \\ W_\alpha &= -\tfrac{1}{4}\bar D\bar D\,D_\alpha V \\ W^\alpha W_\alpha|_{\theta^2} &= -\tfrac{1}{2}F_{\mu\nu}F^{\mu\nu} + \tfrac{i}{4}\epsilon_{\mu\nu\sigma\rho}F^{\mu\nu}F^{\sigma\rho} + D^2 - 2i\bar\lambda\sigma^m\partial_m\lambda \end{aligned} \tag{3.28}$$

and

$$\begin{aligned} y^\mu &= x^\mu - i\theta\sigma^\mu\bar\theta \qquad\qquad D_\alpha = \frac{\partial}{\partial\theta^\alpha} + 2i\sigma^\mu_{\alpha\dot\alpha}\bar\theta^{\dot\alpha}\,\frac{\partial}{\partial y^\mu} \\ \tau &= \frac{\theta}{2\pi} + \mathrm{i}\frac{4\pi}{g_{YM}^2} \end{aligned} \tag{3.29}$$

Gauge theories with extended supersymmetry

$$\begin{aligned} &\mathcal{N}=1: && \mathbf{V} = (A_\mu, \lambda_\alpha, \bar\lambda_{\dot\alpha}, D)_{\mathbf{Adj}} && \mathbf{C} = (\phi, \psi_\alpha, F)_{\mathbf{rep}} \\ &\mathcal{N}=2: && \mathbf{V}_{\mathcal{N}=2} = (\mathbf{V}+\mathbf{C})_{\mathbf{Adj}} && \mathbf{H} = (\mathbf{C}+\bar{\mathbf{C}})_{\mathbf{rep}} \\ &\mathcal{N}=4: && \mathbf{V}_{\mathcal{N}=4} = (\mathbf{V}+3\mathbf{C})_{\mathbf{Adj}} \end{aligned} \tag{3.30}$$

$\mathcal{N}=4$ gauge theories

$$\begin{aligned} U(N) &\quad N\ D3-\text{branes} \\ SO(N) &\quad N\ D3-\text{branes} + 1\ O3^- - \text{plane} \\ Sp(N) &\quad N\ D3-\text{branes} + 1\ O3^+ - \text{plane} \end{aligned} \tag{3.31}$$

Brane intersections

- $\mathcal{N}=4$ vector multiplet D3D3,D(-1)D(-1): $\theta_i = 0$

  There is a ten dimensional vector $\psi^i_{-\frac{1}{2}}|0\rangle_{NS}, \bar\psi^i_{-\frac{1}{2}}|0\rangle_{NS}$ coming from the NS sector and a ten-dimensional massless fermion $\{|0\rangle_R, \psi^i_0\psi^j_0|0\rangle_R, \psi^i_0\psi^j_0\psi^k_0\psi^l_0|0\rangle_R\}$ (16 states) from the R-sector $i=1,..5$. On-shell degrees of freedom can be found by restricting to $i=1,..4$ leading to 8 states from the NS sector and 8 states from the R-sector.

- $\mathcal{N}=2$ hypermultiplet, D3D7,D3D(-1): $\theta_2=\theta_3=\frac{1}{2}$

  There are two massless scalars: $|0\rangle_{NS}, \psi_0^1\psi_0^2|0\rangle_{NS}$ coming from the NS sector and 4 fermionic degrees of freedom $\psi_0^i|0\rangle_R, (\psi_0^i\psi_0^j\psi_0^k)|0\rangle_R$, $i=3,4,5$, from the R-sector. On-shell degrees of freedom can be found by restricting to $i=3,4$ leading to 2 states from the NS sector and 2 states from the R-sector.

- $\mathcal{N}=1$ chiral multiplet D6D6,ED2D6: $\theta_1+\theta_2+\theta_3=1$

  There is one massless scalar $|0\rangle_{NS}$ from the NS sector and one fermion $|0\rangle_R$ from the R-sector.

- Unpaired Fermions D(-1)D7: $\theta_1=\theta_2=\theta_3=\theta_4=\frac{1}{2}$

  A single complex fermion $|0\rangle_R$ from the R-sector.

$\mathcal{N}=2$ quiver gauge theories

We consider the D3-brane system at a $\mathbb{R}^4/\mathbb{Z}_2$ singularity. At the singularity the N D3-branes group into stacks of N_n fractional branes with $n=0,1$ labelling the conjugacy classes of $\mathbb{Z}_2$. The gauge theory $U(N)$ decomposes as $U(N_0)\times U(N_1)$. More precisely, denoting by $\gamma_{\mathbb{Z}_2}$ the projective embedding of the orbifold group in the Chan-Paton group and imposing $\gamma_{\mathbb{Z}_2}^2=1$ one can write

$$\gamma_{\mathbb{Z}_2}=(\mathbf{1}_{N_0\times N_0},-\mathbf{1}_{N_1\times N_1})$$

with $N=\sum_n N_n$. The resulting gauge theory can be found by projecting the $\mathcal{N}=4$ $U(N)$ gauge theory under the $\mathbb{Z}_2$ orbifold group action:

$$V\to\gamma_{\mathbb{Z}_2}\,V\,\gamma_{\mathbb{Z}_2}^{-1}\qquad \Phi^I\to-\gamma_{\mathbb{Z}_2}\,\Phi^I\,\gamma_{\mathbb{Z}_2}^{-1} \tag{3.32}$$

Keeping only invariant components under (3.32) one finds the $\mathcal{N}=1$ quiver gauge theory

$$\begin{aligned}\mathbf{V}+\mathbf{C}:&\quad \mathbf{N_0\bar{N}_0}+\mathbf{N_1\bar{N}_1}\\ 2\mathbf{C}:&\quad \left[\mathbf{N_0\bar{N}_1}+\mathbf{N_1\bar{N}_0}\right]\end{aligned} \tag{3.33}$$

with gauge group $\prod_n U(N_n)$ andhypermultiplets in the bifundamentals.

$\mathcal{N}=1$ quiver gauge theories

We consider the D3-brane system at a $\mathbb{R}^6/\mathbb{Z}_3$ singularity. At the singularity the N D3-branes group into stacks of N_n fractional branes with $n=0,1,2$ labelling the conjugacy classes of $\mathbb{Z}_3$. The gauge theory $U(N)$ decomposes as $\prod_{n=0}^{2} U(N_n)$. More precisely, denoting by $\gamma_{\mathbb{Z}_3}$ the projective embedding of the orbifold group in the Chan-Paton group and imposing $\gamma_{\mathbb{Z}_3}^3=1$ and $\gamma_{\mathbb{Z}_3}^\dagger=\gamma_{\mathbb{Z}_3}^{-1}$ one can write

$$\gamma_{\mathbb{Z}_3}=(\mathbf{1}_{N_0\times N_0},\omega\mathbf{1}_{N_1\times N_1},\bar{\omega}\mathbf{1}_{\bar{N}_2\times N_2})$$

with $N=\sum_n N_n$. The resulting gauge theory can be found by projecting the $\mathcal{N}=4$ $U(N)$ gauge theory under the $\mathbb{Z}_3$ orbifold group action:

$$V\to\gamma_{\mathbb{Z}_3}\,V\,\gamma_{\mathbb{Z}_3}^{-1}\qquad \Phi^I\to\omega\,\gamma_{\mathbb{Z}_3}\,\Phi^I\,\gamma_{\mathbb{Z}_3}^{-1}\qquad \omega=e^{2\pi i/3} \tag{3.34}$$

Keeping only invariant components under (3.34) one finds the $\mathcal{N}=1$ quiver gauge theory

$$\begin{aligned} V:&\quad \mathbf{N_0\bar{N}_0}+\mathbf{N_1\bar{N}_1}+\mathbf{N_2\bar{N}_2}\\ \Phi^I:&\quad 3\times\left[\mathbf{N_0\bar{N}_1}+\mathbf{N_1\bar{N}_2}+\mathbf{N_2\bar{N}_0}\right]\end{aligned} \tag{3.35}$$

with gauge group $\prod_n U(N_n)$ and three generations of bifundamentals.

4 D-brane instantons

4.1 Gauge Instantons

The YM action reads

$$S_{YM}=-\frac{\operatorname{Im}\tau}{8\pi}\int d^4x\,\operatorname{Tr}F_{\mu\nu}F^{\mu\nu}+\frac{\operatorname{Re}\tau}{8\pi}\int d^4x\,\operatorname{Tr}F_{\mu\nu}\tilde{F}^{\mu\nu} \tag{4.1}$$

with

$$\begin{aligned} F_{\mu\nu}&=\partial_\mu A_\nu-\partial_\nu A_\mu+[A_\mu,A_\nu]\\ \tilde{F}_{\mu\nu}&=\tfrac{1}{2}\epsilon_{\mu\nu\sigma\rho}F^{\sigma\rho}\end{aligned} \tag{4.2}$$

and

$$\tau=\frac{\theta}{2\pi}+\mathrm{i}\frac{4\pi}{g_{YM}^2} \tag{4.3}$$

The equation of motion and Bianchi identity are

$$\begin{aligned} \text{YM equation}: \quad & D^\mu F_{\mu\nu} = 0 \\ \text{Bianchi identity}: \quad & D^\mu \tilde{F}_{\mu\nu} = 0 \end{aligned} \tag{4.4}$$

The second equation follows from the definition of $\tilde{F}_{\mu\nu}$, the first from the variation of the YM action. Combining the two equations one finds that self or anti-self dual connections

$$F = \pm \tilde{F} \tag{4.5}$$

are solutions of the YM equations. A connection satisfying (4.5) is called a *Yang Mills instanton*. It is important to remark that this equation has solutions only in the Euclidean since $\tilde{\tilde{F}} = -F$ in the Minkowskian. Yang-Mills instantons are classified by the topological integer

$$k = \frac{1}{16\pi^2} \int d^4x \mathrm{Tr}\, F_{\mu\nu} \tilde{F}^{\mu\nu} \tag{4.6}$$

called, the *instanton number*. Instantons minimize the Euclidean Yang-Mills action. To see this we start from the identity

$$\int d^4x \mathrm{Tr}\, (F \pm \tilde{F})^2 \geq 0 \tag{4.7}$$

and use $\mathrm{Tr} F^2 = \mathrm{Tr} \tilde{F}^2$ to write

$$\tfrac{1}{2g_{YM}^2} \int d^4x \mathrm{Tr}\, F^2 \geq \tfrac{1}{2g_{YM}^2} \left| \int d^4x\, \mathrm{Tr}\, F \tilde{F} \right| = \frac{8\pi^2 |k|}{g_{YM}^2} \tag{4.8}$$

The YM action for anti-selfdual connections can then be written in the Euclidean ($x_0 \to i x_4$, $F_{0i} \to -i F_{0i}$) as

$$S_{YM} = 2\pi k \tau \tag{4.9}$$

Non-perturbative corrections

$$\langle \mathcal{O} \rangle = \int DA\, e^{-S}\, \mathcal{O} = \sum c_k g_{\mathrm{YM}}^k + \sum d_k\, e^{-\frac{8\pi^2 k}{g_{YM}^2}} \tag{4.10}$$

4.2 ADHM construction

Here we review the ADHM construction of instantons in $\mathbb{R}^4$.
A self-dual connection A_μ can be construct as follows. Start from a matrix $\Delta_{[(N+2k)\times 2k]}$

$$\Delta = \begin{pmatrix} w_{\dot{\alpha}} \\ a_{\alpha\dot{\alpha}} - x_{\alpha\dot{\alpha}} \end{pmatrix} \tag{4.11}$$

with $a = a_m\sigma^m$, $x = x_m\sigma^m \otimes \mathbb{1}_{[k\times k]}$. The matrices $w_{[N\times 2k]}$, $a_{[2k\times 2k]}$ contains the instanton moduli. The connection can be written as

$$A_\mu = \bar{U}\partial_\mu U \tag{4.12}$$

with $U_{[(N+2k)\times N]}$ the normalized kernel of Δ i.e.

$$\bar{\Delta}U = \bar{U}\Delta = 0 \qquad \bar{U}\,U = \mathbb{1}_{[N\times N]} \tag{4.13}$$

By bars we will always mean hermitian conjugates. The connection (4.12) is self-dual if Δ satisfy

$$\bar{\Delta}\,\Delta = f^{-1} = f^{-1}_{[k\times k]} \otimes \mathbb{1}_{[2\times 2]} \tag{4.14}$$

In components one finds the ADHM constraints [3]

$$\tau^{c\dot{\beta}}_{\dot{\alpha}}(\bar{w}^{\dot{\alpha}}w_{\dot{\beta}} + \bar{a}^{\dot{\alpha}\alpha}a_{\alpha\dot{\beta}}) = \bar{w}\tau^c w - i\bar{\eta}^c_{mn}[a_m, a_n] = 0 \tag{4.15}$$

Notice that the resulting connection is invariant under $U(k)$ rotations

$$a_m \to U a_m U^\dagger \qquad w_{\dot{\alpha}} \to U w_{\dot{\alpha}} \tag{4.16}$$

The moduli space of instantons is then defined by the $U(k)$ quotient of the hypersurface defined by (5.33) and has dimension

$$\dim_{\mathbb{R}}\mathfrak{M}_k = 4k(N+2k) - 3k^2 - k^2 = 4kN \tag{4.17}$$

Notice that equation (4.14) implies

$$\mathbb{1} = U\bar{U} + \Delta\, f\, \bar{\Delta} \tag{4.18}$$

To see that the gauge connection constructed in this way is self-dual let us compute $F_{\mu\nu}$

$$\begin{aligned} F_{\mu\nu} &= \partial_\mu A_\nu - \partial_\nu A_\mu - [A_\mu A_\nu] \\ &= 2\partial_{[\mu}\bar{U}\partial_{\nu]}U - [\bar{U}\partial_\mu U, \bar{U}\partial_\nu U] \end{aligned} \tag{4.19}$$

[3] In the mathematical literature these equations are often written as $[B_1, B_2] + IJ = 0$, $[B_1, B_1^\dagger] + [B_2, B_2^\dagger] + II^\dagger - J^\dagger J = \xi\,\mathbf{1}_{k\times k}$. These equations follows from the identifications $B_\ell = \frac{1}{\sqrt{2}}(a_{2\ell} + ia_{2\ell-1})$ and $w = (J \;\; I^\dagger)$.

Inserting the identity (4.18) into the first term in (4.19), rewriting derivatives on U's as derivatives on Δ's and using (4.13) one finds

$$\begin{aligned} F_{\mu\nu} &= 2\partial_{[\mu}\bar{U}\Delta\, f\,\bar{\Delta}\,\partial_{\nu]}U = 2\bar{U}\partial_{[\mu}\Delta\, f\,\partial_{\nu]}\bar{\Delta}\,U \\ &= 2\bar{U}\begin{pmatrix} 0 \\ \sigma_{[\mu}\otimes \mathbb{1}_{[k\times k]} \end{pmatrix} f \begin{pmatrix} 0 & \sigma^{\dagger}_{\nu]}\otimes \mathbb{1}_{[k\times k]} \end{pmatrix} U \\ &= 4\bar{U}\begin{pmatrix} 0 & 0 \\ 0 & \sigma_{\mu\nu}\otimes f_{[k\times k]} \end{pmatrix} U \sim \sigma_{\mu\nu} \end{aligned} \tag{4.20}$$

There is a nice D-brane description of this system. In this formalism a $U(N)$ instanton with instanton number k is viewed as bound states of k $D(-1)$ and N $D3$-branes. The instanton moduli $a_\mu, w_{\dot\alpha}$ represent the lowest modes of open strings connecting the various branes. The ADHM constraints are identified with the F- and D- term flat conditions in the effective 0-dimensional theory.

Explicit solutions

The simplest solution: $k=1$, $N=2$

$$\begin{aligned} \Delta &= \begin{pmatrix} \rho\,\mathbb{1}_{[2\times 2]} \\ -x_{[2\times 2]} \end{pmatrix} \qquad U = \frac{1}{(\rho^2+r^2)^{\frac{1}{2}}}\begin{pmatrix} x_{[2\times 2]} \\ \rho\,\mathbb{1}_{[2\times 2]} \end{pmatrix} \\ \bar{\Delta}\,\Delta &= (\rho^2+r^2)\,\mathbb{1}_{[2\times 2]} \qquad \Rightarrow f = \frac{1}{\rho^2+r^2} \qquad r^2 = x_\mu x^\mu \\ F_{\mu\nu} &= 4\bar{U}\begin{pmatrix} 0 & 0 \\ 0 & \frac{\sigma_{\mu\nu}}{(\rho^2+r^2)} \end{pmatrix} U = \frac{4\,\rho^2\,\sigma_{\mu\nu}}{(\rho^2+r^2)^2} \end{aligned} \tag{4.21}$$

4.3 Classical instanton actions

Let us consider a Dp-brane wrapping a $(p-3)$-cycle $\mathcal{C}_A$ and denote by τ the complexified gauge coupling of the resulting four-dimensional super Yang-Mills theory. The world-volume action of wrapped Dp-brane in Euclidean signature is[4]

$$S^{\mathrm{D}p} = \mu_p\,\mathrm{Tr}\left[\int_{\mathbb{R}^4\times\mathcal{C}} d^4x\, e^{-\varphi}\sqrt{\det\,\big(g+2\pi\alpha' F\big)} \;-\; \mathrm{i}\int_{\mathbb{R}^4\times\mathcal{C}}\sum_n C_{2n}\, e^{2\pi\alpha' F}\right], \tag{4.22}$$

[4]Here we assume $\big[F_{\mu\nu}, F_{\sigma\rho}\big]=0$ and take $F=F_iT^j$ with $\mathrm{Tr}\big(T^iT^j\big)=\frac{1}{2}\delta^{ij}$ and i,j running in the adjoint of the gauge.

where $\mu_p = (2\pi)^{-p}(\alpha')^{-(p+1)/2}$ is the Dp-brane tension, φ the dilaton, g the string frame metric and C_{2n} the R-R $2n$-form potentials. Expanding (4.24) to quadratic order in F and comparing with the standard form of the Yang-Mills action (4.1) in Euclidean signature, we find that the complexified four-dimensional gauge coupling is

$$\tau = 2\pi(2\pi\alpha')^2\,\mu_p \int_{\mathcal{C}} \left[\mathrm{i}\, e^{-\varphi}\,\sqrt{\det\, g} + C_{p-3}\right] \ . \tag{4.23}$$

In the background of a gauge instanton connection with instanton number k one finds

$$S^{\mathrm{D}p} = -\mathrm{i}\, N\,\mu_p \int_{\mathbb{R}^4\times\mathcal{C}} C_{p+1} - \mathrm{i}\, k\,\mu_{p-4} \int_{\mathcal{C}} C_{p-3} + \dots \ , \tag{4.24}$$

This suggests that a gauge instanton can be described in terms of a bond state of N Dp-branes and k Euclidean $(p-4)$-brane wrapping the same $(p-3)$-cycle $\mathcal{C}$. Indeed, the action for a Euclidean $(p-4)$-brane wrapping $\mathcal{C}$ is given by

$$S^{\mathrm{E}(p-4)} = \mu_{p-4}\left[\mathrm{i}\int_{\mathcal{C}} \epsilon^{-\varphi}\,\sqrt{\det\, g} \ + \ \int_{\mathcal{C}} C_{p-3}\right] = 2\pi\tau \tag{4.25}$$

matching the instanton action. On the other hand, if the instanton wraps a cycle $\mathcal{C}_E$ different from $\mathcal{C}$, the action $S^{\mathrm{E}(p-4)} = 2\pi\tau_E$, is given in terms of τ_E defined by (4.23) with $\mathcal{C} \to \mathcal{C}_E$. These instantons induce non-perturbative interactions weighted by $e^{2\pi i k\tau_E}$ with k being the number of instantonic branes and $e^{2\pi i\tau_E}$ a scale not directly link to the gauge theory scale $e^{2\pi i\tau}$.

5 D(-1)/D3 system

5.1 $\mathcal{N} = 4$ case

Let us consider the bound state of k D(-1) and N D3-branes in flat space. The dynamics of the bound state is described by a $U(k)\times U(N)$ matrix model describing the low energy interactions of the open strings connecting the various D-branes. In particular the dynamics of D(-1)D(-1) strings is described by the dimensional reduction to $0+0$ dimensions of an $\mathcal{N} = 1$ $U(k)$ gauge theory in ten dimensions. This results into

$$S_{D(-1)D(-1)} = \mathrm{tr}\left(-\tfrac{1}{2}F_{MN}F^{MN} + i\,\Psi\Gamma^M D_M\Psi\right) \tag{5.26}$$

Moduli	$U(k)\times U(N)$	$\mathrm{SU}(2)^2_{\alpha,\dot\alpha}\times \mathrm{SO}(6)_A$	Δ
χ_a	$\mathbf{k}\bar{\mathbf{k}}$	$(\mathbf{1},\mathbf{1},\mathbf{6})$	-1
a_m	$\mathbf{k}\bar{\mathbf{k}}$	$(\mathbf{2},\mathbf{2},\mathbf{1})$	-1
D^c	$\mathbf{k}\bar{\mathbf{k}}$	$(\mathbf{1},\mathbf{3},\mathbf{1})$	-2
$\mathcal{M}^{\alpha A}$	$\mathbf{k}\bar{\mathbf{k}}$	$(\mathbf{2},\mathbf{1},\mathbf{4})$	$\frac{1}{2}$
$\lambda_{\dot\alpha A}$	$\mathbf{k}\bar{\mathbf{k}}$	$(\mathbf{1},\mathbf{2},\bar{\mathbf{4}})$	$-\frac{3}{2}$
$w_{\dot\alpha}$	$\mathbf{k}\,\overline{\mathbf{N}}$	$(\mathbf{1},\mathbf{2},\mathbf{1})$	1
$\bar{w}_{\dot\alpha}$	$\bar{\mathbf{k}}\,\mathbf{N}$	$(\mathbf{1},\mathbf{2},\mathbf{1})$	1
μ^A	$\mathbf{k}\,\bar{\mathbf{N}}$	$(\mathbf{1},\mathbf{1},\mathbf{4})$	$\frac{1}{2}$
$\bar{\mu}^A$	$\bar{\mathbf{k}}\,\mathbf{N}$	$(\mathbf{1},\mathbf{1},\mathbf{4})$	$\frac{1}{2}$

Table 1: D(–1)/D3 instanton moduli for $\mathcal{N}=4$ theory. The second column lists the representations under the brane symmetry group, Third column displays the representations under the Lorentz symmetry group and fourth column the length dimension of the various fields.

with

$$\begin{aligned}
F_{MN} &= [A_m, A_N] \qquad A_M = (a_\mu, \chi_a) \\
D_M\Psi &= [A_M, \Psi] \qquad \Psi = \begin{pmatrix}1\\0\end{pmatrix}\otimes\begin{pmatrix}1\\0\end{pmatrix}\mathcal{M}^A_\beta + \begin{pmatrix}0\\1\end{pmatrix}\otimes\begin{pmatrix}0\\1\end{pmatrix}\lambda_{\dot\beta A} \\
\Gamma_\mu &= \mathbb{1}_{8\times 8}\otimes\gamma^\mu \qquad \Gamma_a = \begin{pmatrix}0 & \Sigma^{aAB}\\ \bar\Sigma^a_{AB} & 0\end{pmatrix}\otimes\gamma^5
\end{aligned} \tag{5.27}$$

In presence of D3-branes, the matrix model can be found by dimensional reduction of $\mathcal{N}=1$ SYM in $D=6$ with gauge group $U(k)$ an adjoint hypermultipet and N fundamental hypermultiplets. The field content is then

$$\begin{aligned}
V &= \{\chi_a, \lambda_{\dot\alpha A}, D_c\}_{ij} \qquad i=1,...k \quad a=1,..6 \quad c=1,..3 \\
H_{adj} &= \{a_m, \mathcal{M}^A_\alpha\}_{ij} \qquad m=1,..4 \quad \alpha,\dot\alpha=1,2 \quad A=1,..4 \\
H_{fund} &= \{w_{\dot\alpha}, \mu^A\}_{iu} \qquad u=1,..N
\end{aligned} \tag{5.28}$$

The action can be written as

$$S_k = S_{\text{gauge}} + S_{\text{adj}} + S_{\text{fund}} \tag{5.29}$$

with

$$
\begin{aligned}
S_{\text{gauge}} &= \frac{1}{g_0^2}\,\mathrm{tr}\left(-\tfrac{1}{2}F_{ab}^2 - \tfrac{i}{2}\Sigma_a^{AB}\lambda_A^{\dot\alpha}D_a\lambda_{\dot\alpha B} - \tfrac{1}{2}D_c^2\right) \\
S_{\text{adj}} &= \mathrm{tr}\left(-(D_a a_n)^2 - \tfrac{i}{2}\bar\Sigma_{AB}^a\mathcal{M}^{\alpha A}D_a\mathcal{M}_\alpha^B + i\lambda_A^{\dot\alpha}[\mathcal{M}^{\alpha A}, a_{\alpha\dot\alpha}] - iD^c\tau_{\dot\alpha}^{c\dot\beta}\bar a^{\dot\alpha\alpha}a_{\alpha\dot\beta}\right) \\
S_{\text{fund}} &= \mathrm{tr}\left(-D^a\bar w^{\dot\alpha}D_a w_{\dot\alpha} + i\bar\mu^A\bar\Sigma_{AB}^a D_a\mu^B + i(\bar\mu^A w_{\dot\alpha} + \bar w_{\dot\alpha}\mu^A)\lambda_A^{\dot\alpha} - iD_c\tau_{\dot\alpha}^{c\dot\beta}\,\bar w^{\dot\alpha}w_{\dot\beta}\right)
\end{aligned}
\tag{5.30}
$$

and $D_a\Phi = [\chi_a, \Phi]$ or $D_a\Phi = \chi_a\Phi$ for $U(k)$ adjoint or fundamental fields Φ. The Lorentz symmetry group is $SO(4)\times SO(6)$ and the domain of the various indices are: $\alpha, \dot\alpha = 1, 2$ (Left an Right moving spinors), $a = 1, ..6$ (vector of SO(6)), upper/lower $A = 1, ..4$, (Left/Right spinor of SO(6)), $c = 1, 2.3$ (Self-dual two-form of SO(4)). After reducing to 0+0 dimensions the action can be written as

$$
S_{k,N} = \frac{1}{g_0^2}S_G + S_K + S_D \tag{5.31}
$$

with

$$
\begin{aligned}
S_G &= \mathrm{tr}_k\big(-\tfrac{1}{2}[\chi_a, \chi_b]^2 - i\lambda_{\dot\alpha A}[\chi_{AB}^\dagger, \lambda_B^{\dot\alpha}] - \tfrac{1}{2}D_c^2\big) \\
S_K &= \mathrm{tr}_k\big(-[\chi_a, a_n]^2 + \chi_a\bar w^{\dot\alpha}w_{\dot\alpha}\chi_a - i\mathcal{M}^{\alpha A}[\chi_{AB}\mathcal{M}_\alpha^B] + i\chi_{AB}\bar\mu^A\mu^B\big) \\
S_D &= \mathrm{tr}_k\left(i\left([\mathcal{M}^{\alpha A}, a_{\alpha\dot\alpha}] + \bar\mu^A w_{\dot\alpha} + \bar w_{\dot\alpha}\mu^A\right)\lambda_A^{\dot\alpha} - iD_c\tau_{\dot\alpha}^{c\dot\beta}(\bar w^{\dot\alpha}w_{\dot\beta} + \bar a^{\dot\alpha\alpha}a_{\alpha\dot\beta})\right)
\end{aligned}
\tag{5.32}
$$

Given the classical group isomorphism $SO(6)_{\mathcal{R}} \cong SU(4)_{\mathcal{R}}$, $SO(6)_{\mathcal{R}}$ vectors can also be written as $\chi_{AB} \equiv \frac{1}{2}\bar\Sigma_{AB}^a\chi_a$ with the $\bar\Sigma_{AB}^a = (\eta_{AB}^c, i\bar\eta_{AB}^c)$ given in terms of the t'Hooft symbols. In the limit $g_0 = 4\pi(4\pi^2\alpha')^{-2}g_s \to \infty$, gravity decouples from the gauge theory and the contributions coming from S_G are suppressed; then the fields $\lambda_{\dot\alpha}^A$, D_c become lagrangian multipliers implementing the ADHM constraints (5.33) in the form

$$
\begin{aligned}
&\bar\mu^A w_{\dot\alpha} + \bar w_{\dot\alpha}\mu^A - [a_{\alpha\dot\alpha}, \mathcal{M}'^{\alpha A}] = 0 \\
&\tau_{\dot\alpha}^{c\dot\beta}(\bar w^{\dot\alpha}w_{\dot\beta} + \bar a^{\dot\alpha\alpha}a_{\alpha\dot\beta}) = 0
\end{aligned}
\tag{5.33}
$$

In presence of a vev for the D3D3 fields $\langle\Phi\rangle = \mathrm{diag}(a_1, ...a_N)$ the D3D(-1) action is modified by replacing

$$
\chi \to \chi + q\langle\Phi\rangle \tag{5.34}
$$

with $q = 1$ when acting on $w_{\dot\alpha}, \mu^A$ charged fields and $q = 0$ otherwise.

Moduli	$U(k)\times U(N)$	$\mathrm{SU}(2)^3_{\alpha,\dot\alpha,\dot a}\times \mathrm{U}(1)$	Δ
χ	$\mathbf{k}\bar{\mathbf{k}}$	$(\mathbf{1},\mathbf{1},\mathbf{1})_+$	-1
$\bar\chi$	$\mathbf{k}\bar{\mathbf{k}}$	$(\mathbf{1},\mathbf{1},\mathbf{1})_-$	-1
a_m	$\mathbf{k}\bar{\mathbf{k}}$	$(\mathbf{2},\mathbf{2},\mathbf{1})_0$	-1
D^c	$\mathbf{k}\bar{\mathbf{k}}$	$(\mathbf{1},\mathbf{3},\mathbf{1})_0$	-2
$\mathcal{M}^{\alpha\dot a}$	$\mathbf{k}\bar{\mathbf{k}}$	$(\mathbf{2},\mathbf{1},\mathbf{2})_{\frac12}$	$\frac12$
$\lambda_{\dot\alpha\dot a}$	$\mathbf{k}\bar{\mathbf{k}}$	$(\mathbf{1},\mathbf{2},\mathbf{2})_{-\frac12}$	$-\frac32$
$w_{\dot\alpha}$	$\mathbf{k}\,\overline{\mathbf{N}}$	$(\mathbf{1},\mathbf{2},\mathbf{1})_0$	1
$\bar w_{\dot\alpha}$	$\bar{\mathbf{k}}\,\mathbf{N}$	$(\mathbf{1},\mathbf{2},\mathbf{1})_0$	1
$\mu^{\dot a}$	$\mathbf{k}\,\bar{\mathbf{N}}$	$(\mathbf{1},\mathbf{1},\mathbf{2})_{\frac12}$	$\frac12$
$\bar\mu^{\dot a}$	$\bar{\mathbf{k}}\,\mathbf{N}$	$(\mathbf{1},\mathbf{1},\mathbf{2})_{\frac12}$	$\frac12$
μ'	$\mathbf{k}\,\overline{\mathbf{N}}_{\mathbf{f}}$	$(\mathbf{1},\mathbf{1},\mathbf{1})_{-\frac12}$	$\frac12$

Table 2: D(–1)/D3/D7 instanton moduli for $\mathcal{N}=2$ theory. The second column lists the representations under the brane symmetry group, Third column displays the representations under the Lorentz symmetry group and fourth column the length dimension of the various fields.

5.2 $\mathcal{N}=2$ case

Pure $\mathcal{N}=2$ SYM theory is realized by placing N fractional D3-branes at a $\mathbb{R}^4/\mathbb{Z}_2$ singularity with $\mathbb{Z}_2$ a discrete subgroup of a $SU(2)\in SO(6)$. The D(-1)/D3 instanton moduli can be derived starting from those in flat spacetime after projecting out the non-invariant components under $\mathbb{Z}_2$. This can be achieved by restricting $SO(6)$ vector indices to $a=1,2$, and $SO(6)$ spinor indices to $A=\dot a=1,2$. More precisely we keep only the components $\chi_a=(\chi,\bar\chi)$ and $(\mathcal{M}^{\dot a}_{\alpha},\lambda_{\dot\alpha\dot a},\mu^{\dot a})$. The field content is summarized in table 5.2[5]. The D(-1)/D3 action is given again by (5.32) with SO(6) indices now running over $a=1,2$ and $A=\dot a=1,2$.

In addition fundamental matter can be realized by the introduction of N_f D7-

[5]The index $\alpha,\dot\alpha$ runs over the weights of the spinor left and right moving spinor of the $SO(4)$ Lorentz group acting on the ND plane, $\dot a$, the spinor left weights of the SO(4) acting on the DD four plane perpendicular to the χ-plane. Undotted index α stands for the weights $\frac12(+-)$,$\frac12(-+)$, dotted indices $\dot\alpha,\dot a$ denote the weights $\frac12(++)$,$\frac12(--)$, and $\pm\frac12$ the weights along the χ-plane. Finally $\frac12(-----)$ denotes the lowest spin weight of the Ramond D(-1)D(-1) open string.

branes. The D(-1)D7 interaction is described by the action

$$S_{\text{fund}} = \bar{\mu}'\chi\mu' \tag{5.35}$$

5.3 $\mathcal{N} = 1$ case

Pure $\mathcal{N} = 2$ SYM theory is realized by placing N fractional D3-branes at a $\mathbb{R}^6/\mathbb{Z}_3$ singularity with $\mathbb{Z}_3$ a discrete subgroup of a $SU(3) \in SO(6)$. The D(-1)/D3 instanton moduli can be derived starting from those in flat spacetime after projecting out the non-invariant components under $\mathbb{Z}_3$. This can be achieved by projecting out the χ_a fields and restricting $SO(6)$ spinor indices A to $A = 1$ (i.e. omitting the index A. The field content is summarized in table 6.2. The D(-1)/D3 action is given again by (5.32) with $\chi_a = 0$ and $A = 1$.

In addition fundamental matter can be realized by the introduction of N_f D7-branes.

6 Instanton partition function

In this section we will compute the instanton corrections to the prepotential of $\mathcal{N} = 2$ theories. The instanton corrections to the prepotential are given by the moduli space integral

$$\begin{aligned} S_{\text{eff}} &= \int d^4x d^4\theta \mathcal{F}_{\text{non-pert}}(\Phi) = \sum q^k \int d\mathfrak{M}_k e^{S_{\text{mod}}(\Phi)} \\ &= \sum q^k \int d^4x d^4\theta \, d\widehat{\mathfrak{M}}_k e^{S_{\text{mod}}(\Phi)} \end{aligned} \tag{6.36}$$

i.e.

$$\mathcal{F}_{\text{non-pert}}(a_u) = \sum q^k \int d\widehat{\mathfrak{M}}_k e^{S_{\text{mod}}(a_u)} \tag{6.37}$$

Here we denote by $q = \mu^\beta e^{2\pi i\tau}$, with μ^β compensating for the length dimension of instanton moduli space measure. We will regularize the volume factor by introducing some $\epsilon_{1,2}$-deformations of the four-dimensional geometry and recover the flat space result from the limit $\epsilon_{1,2} \to 0$. More precisely we will find

$$\mathcal{F}_{\text{non-pert}}(a_u, q) = -\lim_{\epsilon_\ell \to 0} \epsilon_1\epsilon_2 \ln Z(\epsilon_\ell, a_u, q) \tag{6.38}$$

with

$$Z(\epsilon_\ell, a_u, q) = \sum q^k \int d\mathfrak{M}_k e^{S_{\text{mod}}(a_u, \epsilon_\ell)} \tag{6.39}$$

The factor $\epsilon_1\epsilon_2$ in (6.38) takes care of the volume factor $\int d^4x d^4\theta \sim \frac{1}{\epsilon_1\epsilon_2}$.

6.1 Localization formula

Here we specify to $\mathfrak{g} = U(1)^r$ group action on a manifold M of complex dimension ℓ specified by the vector field

$$\xi = \xi^i(x)\frac{\partial}{\partial x^i} \qquad \delta_\xi x^i = \xi^i(x)$$

We introduce the equivariant derivative

$$Q_\xi \equiv d + i_\xi \qquad Q_\xi^2 = di_\xi + i_\xi d = \delta_\xi$$

with d the exterior derivative, $i_\xi dx^i \equiv \delta_\xi x^i$ the contraction with ξ and δ_ξ the Lie derivative along ξ.

Let $\Omega(M)$ the spaces of forms in M. A form $\alpha(\xi) : \mathfrak{g} \to \Omega(M)$ satisfying

$$Q_\xi\, \alpha = 0 \tag{6.40}$$

is said to be equivariantly closed.

If critical points x_0^s's of the group action ξ, i.e. points where $\xi^i(x_0^s) = 0\ \forall i$, are isolated the integral of an equivariantly closed form is given by the localization formula:

$$\int_M \alpha = (-2\pi)^\ell \sum_s \frac{\alpha_0(x_0^s)}{\det^{\frac{1}{2}} Q_\xi^2(x_0^s)} \tag{6.41}$$

with $Q^2{}_i^j = \partial_i \xi^j : T_{x_0}M \to T_{x_0}M$ the tangent space map induced by the vector field ξ.

Example: Gaussian integral via $U(1)$ localization on $\mathbb{R}^2$.

$$\begin{aligned}
&\xi = \hbar\,(x\,\frac{\partial}{\partial y} - y\,\frac{\partial}{\partial x}) \\
&\alpha = e^{-a(x^2+y^2)}\,dxdy - \frac{\hbar}{2a}e^{-a(x^2+y^2)} \\
&\int_{\mathbb{R}^2} e^{-a(x^2+y^2)}\,dxdy = \int_{\mathbb{R}^2} \alpha = 2\pi\,\frac{\hbar\, e^{-a(x_0^2+y_0^2)}}{2a\hbar} = \frac{\pi}{a}
\end{aligned} \tag{6.42}$$

with $x_0 = y_0 = 0$ the critical point. Notice that the right hand side does not depend on $\hbar$.

6.2 The BRST charge

The localization procedure is based on the cohomological structure of the instanton moduli action which is exact with respect to a suitable BRST charge Q_0:

$$S_{\text{mod}} = Q_0 \Xi\ . \tag{6.43}$$

Q_0 can be obtained by choosing any component of the supersymmetry charges $Q_{\alpha A}, Q^A_{\dot\alpha}$ preserved on the brane system. Supersymmetry charges are invariant under $\mathrm{U}(k) \times \mathrm{U}(N_c) \times \mathrm{U}(N_f)$ but transform as a spinor of $\mathrm{SO}(4)^2$, so that the choice of Q_0 breaks this symmetry to the $\mathrm{SU}(2)^3$ subgroup which preserves this spinor. In our case we take

$$\mathrm{SU}(2)_1 \times \mathrm{SU}(2)_2 \times \mathrm{SU}(2)_3 = \mathrm{SU}(2)_+ \times \left[\mathrm{SU}(2)_- \times \widehat{\mathrm{SU}}(2)_-\right]_{\mathrm{diag}} \times \widehat{\mathrm{SU}}(2)_+ \ . \quad (6.44)$$

This reduction is achieved by identifying the spinor indices "$\dot\alpha$" and "$\dot a$" of the Left moving SU(2)'s in the two four-dimensional planes. More precisely, we decompose $SO(6)$ spinor indices as $A = (a, \dot a)$ with $a, \dot a = 1, 2$ and identify

$$Q_0 = \tfrac{1}{2}\epsilon^{\dot\alpha\dot a} Q_{\dot\alpha\dot a} \quad (6.45)$$

After this identification is made, the fermionic moduli $\mathcal{M}_{\alpha\dot a}$ and $\lambda_{\dot\alpha a}$ can be renamed as $M_{\ell=\alpha\dot\alpha}$ and $M_{\dot\ell=\dot a a}$, $\ell = 1, 2$, $\dot\ell = 3, 4$ and paired with a_ℓ and $B_{\dot\ell}$ into BRST multiplets. Similarly, the singlet component $\eta \equiv \frac{1}{2}\lambda_{\dot\alpha\dot a}\epsilon^{\dot\alpha\dot a}$ and the $(-1)/3$ fermionic moduli $\mu_{\dot\alpha=\dot a}$ have the right transformation properties to qualify for the BRST partners of $\bar\chi$ and $w_{\dot\alpha}$ respectively.

The remaining fields $N_c \equiv \frac{1}{2}\sigma_c^{\dot\alpha\dot a}\lambda_{\dot\alpha\dot a}$, $N_{\alpha a} = \mathcal{M}_{\alpha a}$ and μ_a are unpaired, and should be supplemented with auxiliary fields having identical transformation properties. We denote such fields as D_c, $d_{\alpha a}$ and h_a respectively. The seven auxiliary moduli D_c, $d_{\alpha a}$, of dimension L^2, linearize the quartic interactions among the scalars B_ℓ and $B_{\dot\ell}$. In particular, the triplet D_c disentangles the quartic interactions of a_ℓ and $B_{\dot\ell}$ among themselves, while the quartet $d_{\alpha a}$ decouples the quartic interactions between a_ℓ and $B_{\dot\ell}$. Likewise, the dimensionless auxiliary $(-1)/3$ moduli h_a disentangle the quartic interactions between $B_{\dot\ell}$ and $w_{\dot\alpha}$. In the end, χ remains unpaired and therefore $Q_0\chi = 0$.

The BRST transformations read

$$\begin{aligned} Q_0\vec\Phi &= \Psi \qquad & Q\vec\Psi &= \chi\cdot\vec\Phi \\ Q_0\vec N &= \vec D \qquad & Q\vec D &= \chi\cdot\vec N \end{aligned} \quad (6.46)$$

with

$$\begin{aligned} \vec\Phi &= (a_\ell, B_{\dot\ell}, w_{\dot\alpha}, \bar\chi) \qquad & \vec N &= (N_c, N_{\alpha a}, \mu_a) \\ \vec\Psi &= (M_\ell, M_{\dot\ell}, \mu_{\dot\alpha}, \eta) \qquad & \vec D &= (D_c, d_{\alpha a}, h_a) \end{aligned} \quad (6.47)$$

(ϕ,ψ)	$(-)^{F_\phi}$	$\mathbf{R}_G$	$\mathrm{SU}(2)^3_{\alpha,\dot\alpha,a}$	Ω_G	$\Omega_{SU(2)^3}$
(a_ℓ, M_ℓ)	$+$	$\mathbf{k}\bar{\mathbf{k}}$	$(\mathbf{2},\mathbf{2},\mathbf{1})$	$\chi_i-\chi_j$	ϵ_1,ϵ_2
(N_c, D_c)	$-$	$\mathbf{k}\bar{\mathbf{k}}$	$(\mathbf{1},\mathbf{3},\mathbf{1})$	$\chi_i-\chi_j$	$0_{\mathbb{R}},\epsilon_1+\epsilon_2$
$(\bar\chi,\eta)$	$+$	$\mathbf{k}\bar{\mathbf{k}}$	$(\mathbf{1},\mathbf{1},\mathbf{1})$	$\chi_i-\chi_j$	$0_{\mathbb{R}}$
$(w_{\dot\alpha},\mu_{\dot\alpha})$	$+$	$\mathbf{k}\,\overline{\mathbf{N}}_{\mathbf{c}}$	$(\mathbf{1},\mathbf{2},\mathbf{1})$	χ_i-a_u	$\frac{1}{2}(\epsilon_1+\epsilon_2)$
$(\bar w_{\dot\alpha},\bar\mu_{\dot\alpha})$	$+$	$\bar{\mathbf{k}}\,\mathbf{N}_{\mathbf{c}}$	$(\mathbf{1},\mathbf{2},\mathbf{1})$	$a_u-\chi_i$	$\frac{1}{2}(\epsilon_1+\epsilon_2)$
$(B_{\dot\ell},M_{\dot\ell})$	$+$	$\mathbf{k}\bar{\mathbf{k}}$	$(\mathbf{1},\mathbf{2},\mathbf{2})$	$\chi_i-\chi_j$	ϵ_3,ϵ_4
$(N_{\alpha a},d_{\alpha a})$	$-$	$\mathbf{k}\bar{\mathbf{k}}$	$(\mathbf{2},\mathbf{1},\mathbf{2})$	$\chi_i-\chi_j$	$\epsilon_2+\epsilon_3,\epsilon_1+\epsilon_3$
(μ_a,h_a)	$-$	$\mathbf{k}\,\bar{\mathbf{N}}_{\mathbf{c}}$	$(\mathbf{1},\mathbf{1},\mathbf{2})$	χ_i-a_u	$\frac{1}{2}(\epsilon_3-\epsilon_4)$
$(\bar\mu_a,\bar h_a)$	$-$	$\bar{\mathbf{k}}\,\mathbf{N}_{\mathbf{c}}$	$(\mathbf{1},\mathbf{1},\mathbf{2})$	$a_u-\chi_i$	$\frac{1}{2}(\epsilon_3-\epsilon_4)$
(μ',h')	$-$	$\mathbf{k}\,\overline{\mathbf{N}}_{\mathbf{f}}$	$(\mathbf{1},\mathbf{1},\mathbf{1})$	χ_i-m_f	0

Table 3: D(–1)/D3/D7 instanton moduli. The first and second columns dispaly the Q-multiplets and the spin statistics of their lowest component. The third and fourth columns report the transformation properties under the symmetry groups $G = \mathrm{U}(k)\times\mathrm{U}(N_c)\times\mathrm{U}(N_\mathrm{f})$ and $SU(2)^3$ respectively and the fifth and sixth columns the corresponding eigenvalues. The table is divided into three blocks corresponding to the contributions of the gauge, adjoint matter and fundamental matter respectively.

Notice that the lowest component of the multiplet is a boson if the multiplet is built out of physical moduli, and is a fermion if instead the multiplet contains auxiliary fields. Indeed, the auxiliary fields, being related to D- and F-terms, can only appear as highest components in the BRST multiplets while the physical bosonic moduli enter as the lowest components of the pair. These statistical properties and transformation properties are listed in the second column of Tab. 5. It is also important to remark that $Q_0^2 = 0$ up to a $U(k)$ rotation.

With all these ingredients at hand one can write the D(-1)D3 action in the form

$$S = Q_0\,\mathrm{tr}_k\Big(\vec{\mathcal{E}}\,\vec{N} + \vec{\Phi}\cdot\bar\chi\vec{\Psi}\Big)\ , \tag{6.48}$$

with

$$\vec{\mathcal{E}} = (\mathcal{E}_c, \mathcal{E}_{\alpha a}, \mathcal{E}_a) \tag{6.49}$$

some bosonic bilinears realizing the generalized ADHM constraints. In particular for $\mathcal{N} = 2$, omitting states with $a, \dot{\ell}$ indices one finds

$$\vec{\mathcal{E}} = \{\mathcal{E}_c\} = \{\tau^{c\dot{\beta}}_{\dot{\alpha}}(\bar{w}^{\dot{\alpha}} w_{\dot{\beta}} + \bar{a}^{\dot{\alpha}\alpha} a_{\alpha\dot{\beta}})\} \tag{6.50}$$

6.3 Equivariant deformations

To localize the integral over moduli space, it is necesssary to make the charge Q equivariant with respect to all symmetries, which in our case are the gauge symmetry $\mathrm{U}(k) \times \mathrm{U}(N_c) \times \mathrm{U}(N_f)$, and the residual Lorentz symmetry $\mathrm{SU}(2)^3$. For our purposes it is enough to consider the Cartan directions of the various groups. We label the Cartan components of $\mathrm{U}(k)$ by $\vec{\chi}$, those of $\mathrm{U}(N_c)$ by $\vec{a}$ and those of $\mathrm{U}(N_f)$ by $\vec{m}$. From the string perspective $\vec{\chi}$, $\vec{a}$ and $\vec{m}$ parametrize, respectively, the positions of the D(–1), D3 and D7-branes along the overall transverse two-dimensional plane, and their appearance in the moduli action can be deduced from disk amplitudes with (part of) their boundary on the D-instantons and with insertion of (–1)/(–1), 3/3 or 7/7 fields. Thus, $\vec{a}$ can be interpreted as the vacuum expectation value of the chiral superfield Φ of gauge theory on the D3-branes, and $\vec{m}$ as the analogue for the gauge theory on the D7-branes. Finally, the Cartan directions of the residual Lorentz group $\mathrm{SU}(2)^3$ are parametrized by ϵ_I $(I = 1, \ldots, 4)$ subject to the constraint

$$\epsilon_1 + \epsilon_2 + \epsilon_3 + \epsilon_4 = 0 \ . \tag{6.51}$$

Although only three out of the four ϵ's are independent variables, it is convenient during the computation to keep all of them as independent variables and impose the relation (6.51) only at the very end.

After the equivariant deformation, the charge Q becomes nilpotent up to an element of the symmetry group. It is convenient to use the basis provided by the weights of this group, and thus we denote by ϕ_q and ψ_q the components of ϕ and ψ along a weight

$$\vec{q} \equiv \left(\vec{q}_{\mathrm{U}(k)}, \vec{q}_{\mathrm{U}(N)}, \vec{q}_{\mathrm{U}(m)}, \vec{q}_{\mathrm{SU}(2)^3}\right) \in \mathcal{W}(\phi) \ , \tag{6.52}$$

where $\mathcal{W}(\phi)$ is the set of weights of the representation under which ϕ transforms, which can be read from the third and fourth columns of Tab. 5. Then, in this basis

the charge Q acts diagonally as follows

$$Q\phi_q = \psi_q \ , \quad Q\psi_q = \Omega_q \phi_q \ , \tag{6.53}$$

where Ω_q parametrizes the equivariant deformation, i.e. the eigenvalues of Q^2. From the brane perspective, Ω_q specifies the distance in the overall two-dimensional transverse plane between the branes at the two endpoints of the open string. Explicitly, we have

$$\Omega_q = \vec{\chi} \cdot \vec{q}_{\mathrm{U}(k)} + \vec{a} \cdot \vec{q}_{\mathrm{U}(N_c)} + \vec{m} \cdot \vec{q}_{\mathrm{U}(N_f)} + \vec{\epsilon} \cdot \vec{q}_{\mathrm{SU}(2)^3} \ . \tag{6.54}$$

with

$$\vec{\epsilon} \cdot \vec{q}_{\mathrm{SU}(2)^3} = q_1(\epsilon_1 - \epsilon_2) + q_2(\epsilon_1 + \epsilon_2) + q_3(\epsilon_3 - \epsilon_4) \tag{6.55}$$

and $q_i = 0$ for states in the $\mathbf{1}$, $q_i = \pm\frac{1}{2}$ for states in the $\mathbf{2}$ and so on[6]. All this is summarized in the last column of Tab. 5, where we have displayed the positive eigenvalues of $\vec{\epsilon} \cdot \vec{q}_{\mathrm{SU}(2)^3}$ (assuming $\epsilon_1 > \epsilon_2 > \epsilon_3 > \epsilon_4$) corresponding to the holomorphic components of the various fields.

With all these ingredients at hand, one can show that the moduli action S_{mod} can be written in the form (6.43). The details of the fermion Ξ are irrelevant to the computation, since integrals are insensitive to Q-exact terms.

Since the length dimension of the BRST charge is $L^{-1/2}$, the length dimensions of the components (ϕ, ψ) of Q-multiplet are $\left(\Delta, \Delta - \frac{1}{2}\right)$. Thus, recalling that a fermionic variable and its differential have opposite dimensions, we find that the measure on the instanton moduli space

$$d\mathfrak{M}_k \equiv d\chi \prod_{(\phi,\psi)} d\phi \ d\psi \tag{6.56}$$

has the following scaling dimensions

$$L^{-k^2 + \frac{1}{2}(n_+ - n_-)} \tag{6.57}$$

Here, the first term in the exponent accounts for the unpaires k^2 bosonic moduli χ, of dimension L^{-1}, and $n_\pm$ denotes the number of Q-multiplets where the statistics

[6] To see this, associate to each modulus the $\mathrm{SU}(2)^4$ charges $q_\pm, \hat{q}_\pm$ and the eigenvalue $\vec{\epsilon} \cdot \vec{q}_{\mathrm{SU}(2)^4} = \epsilon_1(q_+ + q_-) + \epsilon_2(q_+ - q_-) + \epsilon_3(\hat{q}_+ + \hat{q}_-) + \epsilon_4(\hat{q}_+ - \hat{q}_-)$. Then, (6.55) follows after the identification $q_1 = q_-$, $q_3 = \hat{q}_-$, $q_2 = q_+ - \hat{q}_+$ and the use of (6.51). For example, $B_{1,2} \in (\mathbf{2}, \mathbf{1}, \mathbf{2})$ have $\mathrm{SU}(2)^3$ weights $(\pm\frac{1}{2}, 0, \pm\frac{1}{2})$ that once plugged into (6.55) lead to $\pm\epsilon_1$ and $\pm\epsilon_2$.

of the lowest component is $(-)^{F_\phi} = \pm$. One finds that the measure is dimensionless in the case of $\mathcal{N} = 2$ plus adjoint matter and

$$d\mathfrak{M}_k \sim L^{\beta\, k} \qquad \beta = 2N_c - N_f \tag{6.58}$$

for $\mathcal{N} = 2$ plus N_f fundamentals.

6.4 The integral

The k-instanton partition function Z_k is given by the moduli space integral

$$\begin{aligned} Z_k &= \int d\mathfrak{M}_k\; e^{-S_{\rm mod}} = \int \frac{d\chi}{{\rm vol}U(k)} \prod_{(\phi,\psi)} d\phi\, d\psi\; e^{-Q\Xi(\phi,\psi,\chi)} \\ &= \int \frac{d\chi}{{\rm vol}U(k)}\; e^{-Q\Xi(\phi,d\phi,\chi)} = \int \prod_{i=1}^{k} \frac{d\chi_i}{2\pi {\rm i}} \prod_{i<j}^{k} (\chi_i - \chi_j)^2\, {\rm Sdet}^{-\frac{1}{2}}(Q^2) \\ &= \int \prod_{i=1}^{k} \frac{d\chi_i}{2\pi {\rm i}} \prod_{i<j}^{k} (\chi_i - \chi_j)^2 \prod_{\phi} \prod_{q\in\mathcal{W}^+(\phi)} \Omega_q^{(-)^{F_\phi+1}} \,. \end{aligned} \tag{6.59}$$

The factor $\prod_{i<j}(\chi_i - \chi_j)^2$, known as Vandermonde determinant, comes from the Jacobian resulting from bringing χ into the diagonal form $\chi = {\rm diag}(\chi_1, \chi_2, ...\chi_k)$. In the second line we perform the Grassmanian integrations resulting into the replacements of ψ by $d\phi$ or follows from The second line follows from the localization formula around the fixed point $\phi = 0$ where $Q^2\phi = \Omega_\phi \phi = 0$. In this identification fermions play the role of the differentials $\psi = d\phi$.

The integral over χ_i in the second line above has to be thought of as a multiple contour integral with the pole prescription

$${\rm Im}\,\epsilon_1 >> {\rm Im}\,\epsilon_2 >> {\rm Im}\,\epsilon_3 >> {\rm Im}\,\epsilon_4 > 0 \tag{6.60}$$

Writing

$$Z_k = \int \frac{1}{k!} \prod_{i=1}^{k} \frac{d\chi_i}{2\pi {\rm i}}\; z_k^{\rm gauge} z_k^{\rm matter} \tag{6.61}$$

and using the eigenvalues in Tab. 6.2, one finds

$$\begin{aligned} z_k^{\rm gauge} &= \prod_{i,j}{}' \frac{\chi_{ij}(\chi_{ij}+\epsilon_1+\epsilon_2)}{(\chi_{ij}+\epsilon_1)(\chi_{ij}+\epsilon_2)} \prod_{i,u} \frac{1}{(\chi_i - a_u + \frac{\epsilon_1+\epsilon_2}{2})(-\chi_i + a_u + \frac{\epsilon_1+\epsilon_2}{2})} \\ z_k^{\rm fund} &= \prod_{i,f} (\chi_i - m_f) \\ z_k^{\rm adj} &= \prod_{i,j}{}' \frac{(\chi_{ij}+\epsilon_1+\epsilon_3)(\chi_{ij}+\epsilon_2+\epsilon_3)}{(\chi_{ij}+\epsilon_3)(\chi_{ij}+\epsilon_4)} \prod_{i,u} (\chi_i - a_u + \tfrac{\epsilon_3-\epsilon_4}{2})(-\chi_i + a_u + \tfrac{\epsilon_3-\epsilon_4}{2}) \end{aligned} \tag{6.62}$$

Poles are specified by a N-set of two-dimensional Young Tableaux with k boxes with

$$\chi_i = \chi_{\vec{Y},i} = \chi_{\vec{Y},u,I_u,J_u} = a_u + (I_u - \tfrac{1}{2})\epsilon_1 + (J_u - \tfrac{1}{2})\epsilon_2 \tag{6.63}$$

with I_u, J_u running over the rows and columns of the tableaux. The partition function can then be written as

$$Z_k = \sum_{\vec{Y}} \mathrm{Res}_{\chi_{\vec{Y},i}} z_k^{\mathrm{gauge}} z_k^{\mathrm{matter}} \tag{6.64}$$

The relation $\sum_{I=1}^4 \epsilon_I = 0$ should be imposed only after the integral is performed. The mass of the adjoint matter is parametrized by $\epsilon_{3,4}$ according to $\epsilon_3 = m_{\mathrm{adj}}$, $\epsilon_4 = -m_{\mathrm{adj}} - \epsilon_1 - \epsilon_2$.

6.5 The integral: An Alternative derivation

Here we rederive the instanton partition function using localization in the ADHM moduli space. For simplicity we restrict ourselves to the $\mathcal{N} = 2$ case. The ADHM manifold in this case is given by a $U(k)$ quotient of the hypersurface defined by the ADHM constraints

$$\begin{aligned} D_{\mathbb{C}} &= [B_1, B_2] + IJ = 0 \\ D_{\mathbb{R}} &= [B_1, B_1^\dagger] + [B_2, B_2^\dagger] + II^\dagger - JJ^\dagger = \xi\, \mathbb{1}_{k\times k} \end{aligned} \tag{6.65}$$

on $\mathbb{R}^{4k^2+4kN}$ parametrized by $B_{1,2}, I, J$. This manifold admits a $G = U(k)\times U(N)\times SO(4)$ action. Parametrizing by $\chi_i, a_u, \epsilon_{1,2}$ the $U(1)^{k+N+2}$ Cartan subgroup of G, the infinitesimal variations $\delta_\xi = Q^2$ of the ADHM coordinates read

$$\begin{aligned} Q^2 I &= (\chi_i - a_u + \tfrac{\epsilon}{2}) I_{iu} = 0 \\ Q^2 J &= (a_u - \chi_i + \tfrac{\epsilon}{2}) J_{ui} = 0 \\ Q^2 B_\ell &= (\chi_i - \chi_j + \epsilon_\ell) B_{\ell,ij} = 0 \\ Q^2 N_{\mathbb{C}} &= (\chi_i - \chi_j + \epsilon) N_{\mathbb{C},ij} = 0 \\ Q^2 N_{\mathbb{R}} &= (\chi_i - \chi_j) N_{\mathbb{R},ij} = 0 \qquad i \neq j \end{aligned} \tag{6.66}$$

with $\epsilon = \epsilon_1 + \epsilon_2$. Here we introduce the fermionic auxiliary variables $N_{\mathbb{C},\mathbb{R}}$ (the superpartners of $D_{\mathbb{C},\mathbb{R}}$) to account for the subtraction of the degrees of freedom corresponding to the ADHM constraints.

The solutions of (6.66) can be put in one to one correspondence with a set of n Young tableaux $(Y_1, \dots Y_n)$ with $k = \sum_u k_u$ boxes distributed between the Y_u's. The boxes in a Y_u diagram are labelled either by the instanton index $I_u = 1, \dots, k_u$ or by a pair of integers I_u, J_u denoting the vertical and horizontal position respectively in the Young diagram. The explicit solutions to (6.66) can then be written as

$$\chi_i = \chi_{\vec{Y},i} = \chi_{\vec{Y},u,I_u,J_u} = a_u + (I_u - \tfrac{1}{2})\epsilon_1 + (J_u - \tfrac{1}{2})\epsilon_2 \tag{6.67}$$

and all components of I, J, B_ℓ vanishing except for those with zero Q^2-eigenvalues, i.e.

$$B_{1;I,J;I+1,J}; B_{2;I,J;I,J+1}; I_{u,I=J=1} \tag{6.68}$$

These moduli are fixed by solving the ADHM constraint. To compute the SdetQ^2, it is convenient to first compute its trace

$$T = \mathrm{tr}_{\mathfrak{M}}\, e^{iQ^2} \tag{6.69}$$

Introducing

$$\begin{aligned} V &= \sum_i e^{i\chi_i^{\vec{Y}}} = \sum_{(I_u,J_u)\in Y_u} T_{a_u}\, T_1^{-J_u+1} T_2^{-I_u+1} \\ W &= \sum_{u=1}^{n} T_{a_u} \end{aligned} \tag{6.70}$$

with $T_{1,2} = e^{i\epsilon_{1,2}}$ and $T_{a_u} = e^{ia_u}$ one can write

$$\begin{aligned} T &= V^* \times V \times [T_1 + T_2 - T_1T_2 - 1] + W^* \times V + V^* \times W \times T_1T_2 \\ &= \sum_{u,v}^{n} \sum_{s\in Y_j} \left(T_{a_{uv}} T_1^{-h_v(s)} T_2^{v_v(s)+1} + T_{a_{vu}} T_1^{h_v(s)+1} T_2^{-v_u(s)} \right) \end{aligned} \tag{6.71}$$

with $a_{uv} = a_u - a_v$. $h_u(s)$ $(v_u(s))$ is the horizontal(vertical) distance from s till the right (top) end of the $u(v)$ diagram, i.e. the number of black (white) circles in Fig.1.

The exponents in (6.71) are the eigenvalues of the operator Q^2 which enters in the localization formula. Using these eigenvalues, the partition function of $\mathcal{N} = 2$ SYM for winding number k is

$$\mathcal{Z}_k = \sum_{\vec{Y}} \frac{1}{\mathrm{Sdet}Q^2} = \sum_{\vec{Y}} \prod_{u,v=1}^{n} \prod_{s\in Y_u} \frac{1}{E_{uv}(s)(\epsilon - E_{uv}(s))} \tag{6.72}$$

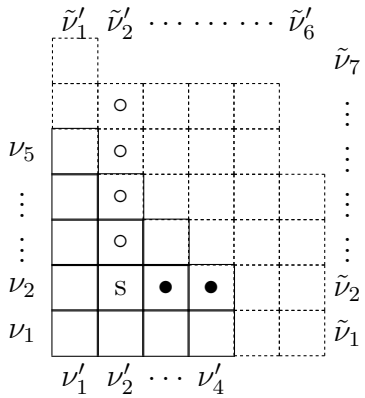

Figure 1: Two generic Young diagrams denoted by the indices $\lambda, \tilde{\lambda}$ in the main text.

and

$$E_{uv}(s) \quad = \quad a_{uv} - \epsilon_1 h_v(s) + \epsilon_2(v_u(s)+1) \tag{6.73}$$

In presence of matter we write

$$\mathcal{Z}_k = \sum_{\vec{Y}} z^{\vec{Y}}_{\text{gauge}} z^{\vec{Y}}_{\text{matter}} \tag{6.74}$$

with

$$\begin{aligned} z^{\vec{Y}}_{\text{gauge}} &= \prod_{u,v=1}^{n} \prod_{s\in Y_u} \frac{1}{E_{uv}(s)(\epsilon - E_{uv}(s))} \\ z^{\vec{Y}}_{\text{adj}} &= \prod_{u,v=1}^{n} \prod_{s\in Y_u} (E_{uv}(s)-m)(E_{uv}(s)-\epsilon+m) \\ z^{\vec{Y}}_{\text{fund}} &= \prod_{u=1}^{n} \prod_{s\in Y_u} (\chi(s)+m) \end{aligned} \tag{6.75}$$

6.6 $k = 1, 2$ explicit computations

6.6.1 $\mathcal{N} = 2$ SYM with gauge group $SU(2)$

Here we present some explicit computations using the residue formulas (7.100,6.62)

k=1

The partition function read

$$Z_1 \quad = \quad \frac{\epsilon}{\epsilon_1\epsilon_2} \sum_{\vec{Y}} \text{Res}_{\vec{Y}} \prod_u \frac{1}{(\chi_1 - a_u + \frac{\epsilon_1+\epsilon_2}{2})(-\chi_1 + a_u + \frac{\epsilon_1+\epsilon_2}{2})} \tag{6.76}$$

There are two poles in the upper half plane :
I) $Y_1 = \square$, $Y_2 = \bullet$: $\chi_1 = a_1 + \frac{1}{2}\epsilon$

$$Z_{1,I} = -\frac{1}{\epsilon_1\epsilon_2 a_{12}(a_{12}+\epsilon)} \tag{6.77}$$

II) $Y_1 = \bullet$, $Y_2 = \square$: $\chi_1 = a_2 + \frac{1}{2}\epsilon$

$$Z_{1,II} = -\frac{1}{\epsilon_1\epsilon_2 a_{12}(a_{12}-\epsilon)} \tag{6.78}$$

Altogether one finds

$$Z_1 = -\frac{2}{\epsilon_1\epsilon_2(a_{12}^2-\epsilon^2)} \tag{6.79}$$

k=2

The partition function read

$$Z_k = \left(\frac{\epsilon}{\epsilon_1\epsilon_2}\right)^2 \sum_{\vec{Y}} \mathrm{Res}_{\vec{Y}} \frac{\chi_{12}^2(\chi_{12}^2-\epsilon^2)}{(\chi_{12}^2-\epsilon_1^2)(\chi_{12}^2-\epsilon_2^2)} \prod_{i,u} \frac{1}{(\chi_i^{\vec{Y}} - a_u + \frac{\epsilon_1+\epsilon_2}{2})(-\chi_i^{\vec{Y}} + a_u + \frac{\epsilon_1+\epsilon_2}{2})}$$

There are five poles in the upper half plane :
I) $Y_1 = \square\square$, $Y_2 = \bullet$: $\chi_1 = a_1 + \frac{1}{2}\epsilon$, $\chi_2 = a_1 + \frac{1}{2}\epsilon + \epsilon_1$

$$Z_{2,I} = -\frac{1}{2\epsilon_1^2\epsilon_2(\epsilon_1-\epsilon_2)a_{12}(a_{12}+\epsilon_1)(a_{12}+\epsilon_1+\epsilon_2)(a_{12}+2\epsilon_1+\epsilon_2)} \tag{6.80}$$

II) $Y_1 = \begin{smallmatrix}\square\\\square\end{smallmatrix}$, $Y_2 = \bullet$: $\chi_1 = a_1 + \frac{1}{2}\epsilon$, $\chi_2 = a_1 + \frac{1}{2}\epsilon + \epsilon_2$

$$Z_{2,II} = \frac{1}{2\epsilon_2^2\epsilon_1(\epsilon_1-\epsilon_2)a_{12}(a_{12}+\epsilon_2)(a_{12}+\epsilon_1+\epsilon_2)(a_{12}+\epsilon_1+2\epsilon_2)} \tag{6.81}$$

III) $Y_1 = \bullet$, $Y_2 = \square\square$: $\chi_1 = a_2 + \frac{1}{2}\epsilon$, $\chi_2 = a_2 + \frac{1}{2}\epsilon + \epsilon_1$

$$Z_{2,III} = -\frac{1}{2\epsilon_1^2\epsilon_2(\epsilon_1-\epsilon_2)a_{12}(a_{12}-\epsilon_1)(a_{12}-\epsilon_1-\epsilon_2)(a_{12}-2\epsilon_1-\epsilon_2)} \tag{6.82}$$

IV) $Y_1 = \bullet$, $Y_2 = \begin{smallmatrix}\square\\\square\end{smallmatrix}$: $\chi_1 = a_2 + \frac{1}{2}\epsilon$, $\chi_2 = a_2 + \frac{1}{2}\epsilon + \epsilon_2$

$$Z_{2,V} = \frac{1}{2\epsilon_2^2\epsilon_1(\epsilon_1-\epsilon_2)a_{12}(a_{12}-\epsilon_2)(a_{12}-\epsilon_1-\epsilon_2)(a_{12}-\epsilon_1-2\epsilon_2)} \tag{6.83}$$

V) $Y_1 = \square$, $Y_2 = \square$: $\chi_1 = a_1 + \frac{1}{2}\epsilon$, $\chi_2 = a_2 + \frac{1}{2}\epsilon$

$$Z_{2,V} = \frac{1}{\epsilon_1^2\epsilon_2^2(a_{12}^2-\epsilon_1^2)(a_{12}^2-\epsilon_2^2)} \tag{6.84}$$

Altogether one finds

$$Z_2 = \frac{2a_{12}^2 - 8\epsilon_1^2 - 8\epsilon_2^2 - 17\epsilon_1\epsilon_2}{\epsilon_1^2\epsilon_2^2(a_{12}^2 - (\epsilon_1+\epsilon_2)^2)(a_{12}^2 - (\epsilon_1+2\epsilon_2)^2)(a_{12}^2 - (2\epsilon_1+\epsilon_2)^2)} \tag{6.85}$$

The prepotential

For the prepotential one finds

$$\begin{aligned} F &= -\epsilon_1\epsilon_2 \ln Z(q) = -\epsilon_1\epsilon_2 Z_1 q + -\epsilon_1\epsilon_2(Z_2 - \tfrac{1}{2}Z_1^2)q + \dots && (6.86) \\ &= \frac{2q}{a_{12}^2 - \epsilon^2} + \frac{q^2(5a_{12}^2 + 7\epsilon_1^2 + 7\epsilon_2^2 + 16\epsilon_1\epsilon_2)}{(a_{12}^2 - (\epsilon_1+\epsilon_2)^2)(a_{12}^2 - (\epsilon_1+2\epsilon_2)^2)(a_{12}^2 - (2\epsilon_1+\epsilon_2)^2)} + \dots \\ &= \frac{2q}{a_{12}^2} + \frac{5q^2}{a_{12}^4} + O(\epsilon^2, q^3) && (6.87) \end{aligned}$$

6.6.2 $\mathcal{N} = 2^*$ SYM with gauge group $SU(N)$

Here we present some explicit calculations using the localization formulas (6.72,6.75).

It is useful to introduce the following definitions:

$$f(x) = \frac{(x-m)(x+m-\epsilon)}{x(x-\epsilon)} \qquad T_u(x) = \prod_{v\neq u} f(a_{uv} + x) \tag{6.88}$$

In terms of these definitions we can rewrite:

$$\mathcal{Z}_k = \sum_{\vec{Y}} \prod_{w,w'=1}^{N} \prod_{s\in Y_w} f(E_{ww'}(s)) \tag{6.89}$$

k=1

$Y_u = \square$, $Y_{v\neq u} = \bullet$. From the above definitions we have $v(s) = h(s) = 0$ for $w = w' = u$ while $v(s) = -1, h(s) = 0$ for $w' \neq w = u$. Summing up over diagrams of this kind one finds

$$Z_1 = \sum_u f(\epsilon_2) T_u(0) \tag{6.90}$$

k=2

We have three diagrams:

I) $Y_u = \square$, $Y_v = \square$, $Y_{w\neq u,v} = \bullet$:

$$Z_2^I = \frac{1}{2}\sum_{u\neq v} f(\epsilon_2)^2 f(a_{uv} + \epsilon_2) f(a_{vu} + \epsilon_2) \frac{T_u(0)T_v(0)}{f(a_{uv})f(a_{vu})} \tag{6.91}$$

The contribution $\frac{T_u(0)}{f(a_{uv})}$ comes from the product in (6.89) with $w, w' \neq u, v$ for which $h(s) = 0, v(s) = -1$. The term $w, w' = u, v$, i.e. $h(s) = v(s) = 0$ gives $f(\epsilon_2)$ or $f(a_{\alpha\beta} + \epsilon_2)$ in the case of $w = w' = u$ and $w = u, w' = v$ respectively. Similar contributions come from terms with $u \leftrightarrow v$ exchanged.

II) $Y_u = \square\square$, $Y_{v\neq u} = \bullet$:

$$Z^{II} = \sum_u f(\epsilon_2) f(\epsilon_2 - \epsilon_1) T_u(0) T_u(-\epsilon_1) \tag{6.92}$$

Now $f(\epsilon_2)f(\epsilon_2 - \epsilon_1)$ comes from the terms in (6.89) with $w = w' = u$ i.e. $v(s) = 0, h(s) = 0, 1$, while the product over $w' \neq w = u$, $v(s) = -1, h(s) = 0, 1$ brings the T_u contributions.

Finally the third diagram is the transposition of the one above and its contribution can be read from (6.92) by exchanging $\epsilon_1 \leftrightarrow \epsilon_2$.

The prepotential

Setting $\epsilon_1 = -\epsilon_2 = \hbar$ one finds:

$$\begin{aligned}
\mathcal{F}_1 &= -\lim_{\hbar\to 0} \hbar^2 Z_1 = m^2 \sum_u T_u \\
\mathcal{F}_2 &= -\lim_{\hbar\to 0} \hbar^2 \left(Z_2 - \frac{1}{2} Z_1^2 \right) = \sum_u \left(\frac{1}{4} m^4 T_u T_u'' - \frac{3}{2} m^2 T_u^2 \right) \\
&\quad + m^4 \sum_{u\neq v} T_u T_v \left(\frac{1}{a_{uv}^2} - \frac{1}{2(a_{uv} - m)^2} - \frac{1}{2(a_{uv} + m)^2} \right)
\end{aligned} \tag{6.93}$$

with $T_u = T_u(0)$. The pure $\mathcal{N} = 2$ analog of formulae (6.90,6.92) can be simply obtained by omitting m-dependent factors.

6.7 $\mathcal{N} = 4$

The $\mathcal{N} = 4$ case follows from $\mathcal{N} = 2$ plus an adjoint matter sending the mass of the adjoint to zero. This corresponds to take $\epsilon_3 = 0$, $\epsilon_4 = -\epsilon$. Plugging into the instanton partition function one finds that z_k^{gauge} cancels against z^{adj} and one finds

$$Z(q) = \sum_{\vec{Y}} q^{|\vec{Y}|} = \frac{1}{\prod_{n=1}^{\infty} (1 - q^n)^N} \tag{6.94}$$

In particular there are no correction to the prepotential as expected.

6.8 Black hole counting

In this section we derive a microscopic formula for the partition function of a black hole made out of D4-D2-D0 bound states wrapping a four cycle inside a CY. We will restrict ourselves to the case where both the cycle and the CY are compact. The lift of this brane system to M-theory is well known and a microscopic derivation of the corresponding black hole entropy based on a two-dimensional $(4,0)$ SCFT has been derived by Maldacena-Strominger-Witten. The aim of this section is to test our instanton partition function formula against supergravity. We consider a single D4-brane wrapping a very ample divisor P inside a CY. The conjugacy class $[P] \in H^2(CY,\mathbb{Z})$ can be expanded as $[P] = p^A \alpha_A$ with α_A a basis in $H^2(CY,\mathbb{Z})$.

According to [**?**] the black hole partition function is defined as

$$Z_{BH} = \sum_{Q_0,Q_A} \Omega(Q_0,Q_A,p^A)\, e^{-Q_0\varphi_0 - Q_A\varphi^A} \tag{6.95}$$

with $\Omega(Q_0,Q_A,p^A)$ the multiplicity of a bound state of Q_0 D0-branes, Q_A D2-branes and a D4 brane wrapping $P = p^A\Sigma_A$. φ_0, φ^A are the D0,D2 chemical potentials. D0,D2 branes can be thought of as instantons and fluxes respectively in the world-volume theory of the D4-brane

$$Q_0 = k = \frac{1}{8\pi^2}\int_M \mathrm{tr}\, F\wedge F \qquad Q_A = \frac{1}{2\pi}\int \mathrm{tr}\, F\wedge \alpha_A \tag{6.96}$$

Self-duality implies that $Q_{A>b_2^+} = 0$.

The black hole partition function can be then read from the instanton partition function formula

$$\begin{aligned} Z &= \frac{1}{\hat{\eta}(e^{-\varphi_0})^{\chi(P)}} \sum_{Q_A\in\mathbb{Z}^{b_2^+(P)}} e^{-\frac{1}{12}D^{AB}\,Q_A\,Q_B\,\varphi_0 - \varphi^A\,Q_A} \\ &= \sum_{Q_0,Q_A} \Omega(Q_0,Q_A,p^A)\, q^{Q_0}\, e^{-\varphi^A Q_A} \end{aligned} \tag{6.97}$$

with

$$\begin{aligned} \chi(P) &= \int_P c_2(P) = 6D_{ABC}p^Ap^Bp^C + c_{2A}p^A \\ b_2^+(P) &= 2\,D_{ABC}p^Ap^Bp^C + \tfrac{1}{6}c_{2A}p^A \\ D_{ABC} &\equiv \tfrac{1}{6}\int_{CY}\alpha_A\wedge\alpha_B\wedge\alpha_C \qquad c_{2A} \equiv \int_{CY}\alpha_A\wedge c_2(CY) \\ C_{AB} &= -\int_P \alpha_A\wedge\alpha_B = -6\,D_{AB} \qquad D_{AB} \equiv D_{ABC}\,p^C \\ q &= e^{-\varphi_0} \qquad e^{-z_A} = e^{-C_{AB}\varphi^B} \end{aligned} \tag{6.98}$$

and C^{AB}, D^{AB} the inverse of C_{AB}, D_{AB} respectively.

Notice that (6.97) is the partition function of $\chi(P)$ free bosons (b_2^+ of them living in the lattice $H^2(P,\mathbb{Z})$) in two-dimensions. The black hole entropy follows from the Cardy formula

$$\begin{aligned} S_{BH} &\approx \ln\Omega(Q_0, Q_A, p^A) \approx 2\pi\sqrt{\tfrac{1}{6}\chi(P)Q_{0,\mathrm{reg}}} \\ &= 2\pi\sqrt{(D_{ABC}p^Ap^Bp^C + \tfrac{1}{6}\,c_{2A}p^A)(Q_0 + \tfrac{1}{12}D^{AB}Q_AQ_B)} \end{aligned} \tag{6.99}$$

with $Q_{0,\mathrm{reg}} = Q_0 + \frac{1}{12}D^{AB}Q_AQ_B$ the number of regular instantons coming from the expansion of $\hat{\eta}^{-\chi}$ in (6.97). (6.99) agrees with the micro/macroscopic M5-brane/supergravity results.

7 Saddle point analysis

For simplicity we take pure $\mathcal{N}=2$ SYM with gauge group $SU(N)$ and $\tau = i\tau_2$. We write

$$Z(q) = \sum_k q^k Z_k = \sum_k \frac{q^k}{k!}\int\prod_{i=1}^k \frac{d\chi_i}{2\pi\mathrm{i}}\; e^{\ln z_k^{\mathrm{gauge}}} \approx \sum_k \int\prod_{i=1}^k \frac{d\chi_i}{2\pi\mathrm{i}}\; e^{\frac{1}{\epsilon_1\epsilon_2}\mathcal{H}_k(\chi_i)} \tag{7.100}$$

In the limit $\epsilon_\ell \to 0$ one finds

$$\begin{aligned} \mathcal{H}_k(\chi_i) &= \epsilon_1\epsilon_2\left[\sum_{ij}\ln\left(\frac{\chi_{ij}(\chi_{ij}+\epsilon)}{(\chi_{ij}+\epsilon_1)(\chi_{ij}+\epsilon_2)}\right) - \sum_i \ln P(\chi_i + \tfrac{\epsilon}{2})P(\chi_i - \tfrac{\epsilon}{2}) + k\ln q\right] \\ &\approx \left[-\epsilon_1^2\epsilon_2^2\sum_{ij}\frac{1}{\chi_{ij}^2} - 2\epsilon_1\epsilon_2\sum_i \ln P(\chi_i) + k\epsilon_1\epsilon_2\ln q\right] \end{aligned} \tag{7.101}$$

Introducing the density function

$$\rho(x) = \epsilon_1\epsilon_2\sum_i \delta(x-\chi_i) \tag{7.102}$$

one can rewrite (7.101) as

$$\mathcal{H}_k(\rho) = -\int dxdy\frac{\rho(x)\rho(y)}{(x-y)^2} - 2\int dy\rho(y)\ln P(y) + \ln q\int dy\rho(y) \tag{7.103}$$

The Saddle point equation becomes

$$\frac{d\mathcal{H}_k(\rho)}{d\rho(x)} = -2\int_{\mathbb{R}} dy\frac{\rho(y)}{(x-y)^2} - 2\ln P(x) + \ln q = 0 \tag{7.104}$$

It is convenient to rewrite this equation as

$$\begin{aligned}\frac{d\mathcal{H}_k(\rho)}{d\rho(x)} &= 2\int_{\mathbb{R}} dy\rho''(y)\ln|x-y| - 2\ln P(x) + \ln q \\ &= -\int_{\mathbb{R}} dy f''(y)\ln|x-y| + \ln q = 0 \end{aligned} \tag{7.105}$$

with

$$f''(x) = -2\rho''(x) + 2\sum_{u=1}^{N}\delta(x-a_u) \tag{7.106}$$

i.e. [7]

$$f(x) = -2\rho(x) + \sum_{u=1}^{N}|x-a_u| \tag{7.107}$$

the profile function. Summarizing the leading profile function can be found by solving the integral equation

$$\boxed{\int_{\mathbb{R}} dy f''(y)\ln(x-y) = \ln q \qquad \text{with} \qquad a_u = \tfrac{1}{2}\int_{\Sigma_u} x f''(x)} \tag{7.108}$$

and $\Sigma_u = [\alpha_u^-, \alpha_u^+]$ an interval around $x = a_u$.

To find a solution of the saddle point equation is convenient to introduce the holomorphic function (defined in the upper half plane)

$$y(z) = e^{\frac{1}{2}\int_{\mathbb{R}} dy f''(y)\ln(z-y)} \tag{7.109}$$

that encodes all momenta of the profile function. More precisely, expanding around $z \approx \infty$ one can write

$$\Phi'(z) = \partial_z \ln y(z) = \tfrac{1}{2}\sum_{J=0}^{\infty}\frac{1}{z^{J+1}}\int_{\mathbb{R}} dy\, y^J f''(y) \tag{7.110}$$

The saddle point equation can be written in terms of the y-function as

$$|y(z)|^2 = q \qquad z \in [\alpha_u^-, \alpha_u^+] \tag{7.111}$$

This can be solved by taking

$$y(z) = y_-(z) = \frac{P}{2} - \sqrt{\frac{P^2}{4} - q} \qquad P^2 - 4q = \prod_{u=1}^{N}(z-\alpha_u^-)(z-\alpha_u^+) \tag{7.112}$$

[7] Here we use $\frac{d^2|x|}{dx^2} = 2\delta(x)$.

with $y_\pm(z)$ the two roots of the equation

$$y(z)^2 - P(z)y(z) + q = 0 \tag{7.113}$$

Indeed for $z \in [\alpha_u^-, \alpha_u^+]$ the argument of the square root is negative and therefore $|y|^2 = y_+ y_- = q$.

The prepotential:

$$\begin{aligned} q\partial_q F &= -\epsilon_1\epsilon_2 q\partial_q \ln Z = -q\partial_q \mathcal{H}_k(f) = -q\partial_q f\, \mathcal{H}_k'(f) + \int_{\mathbb{R}} f(y)dy \\ &= \int_{\mathbb{R}} f(y)dy = \tfrac{1}{2}\int_{\mathbb{R}} y^2 f''(y)dy = \sum_{u=1}^{N} e_u^2 \end{aligned} \tag{7.114}$$

The right hand expression follows by plugging (7.112) into (7.110) and noticing that q start to contribute to $\Phi'(z)$ at order z^{-2N-1}. This implies in particular that

$$\Phi'(z) = \sum_{J=0}^{2N} \frac{1}{z^{J+1}} \sum_{u=1}^{N} e_u^J + O(z^{-2N-1}) \tag{7.115}$$

i.e.

$$\tfrac{1}{2}\int_{\mathbb{R}} y^J f''(y)dy = \sum_{u=1}^{N} e_u^J \qquad J \leq 2N \tag{7.116}$$

for

$$P(z) = \prod_{u=1} N(z - e_u) \tag{7.117}$$

8 $\mathcal{N} = 1$ Superpotentials

8.1 SQCD with $N_f = N - 1$ flavors

In this section we consider a $\mathcal{N} = 1$ $U(N)$ gauge theory with $N_f = N - 1$ quark-antiquark pairs of chiral fileds in the fundamental and antifundamental representations respectively. In the background of the instanton the effective action is given by the moduli space integral

$$S_{\text{eff}} = \sum_k q^k \Lambda^{k(3N-N_f)} \int d\mathfrak{M}\, e^{-S_{\text{mod}}} \tag{8.118}$$

where the factor Λ^{2N+1} compensates for the dimension of the instanton moduli space

$$\mathfrak{M} = \{a_\mu, w_{ui\dot{\alpha}}, D_c, \mathcal{M}_\alpha, \lambda_{\dot{\alpha}}, \mu_{ui}, \mu'_{if}\} \tag{8.119}$$

with length dimension

$$[\mathfrak{M}] = 4k^2 + 4kN - 6k^2 - k^2 + 3k^2 - kN - kN_f = k(3N - N_f) \tag{8.120}$$

We denote by $Q_{uf}, \tilde{Q}^{uf}$, $u = 1, ...N$, $f = 1, ...N_f$, the quark-antiquark superfields. The classical moduli space of the gauge theory is defined by the D-flatness conditions

$$Q_{uf}\tilde{Q}^{\dagger vf} - \tilde{Q}^{\dagger}_{uf}\tilde{Q}^{vf} = 0 \tag{8.121}$$

For simplicity we take the quark-antiquark superfields in the almost diagonal form

$$Q_{uf} = Q_u\delta_{uf} \qquad \tilde{Q}^{vf} = \tilde{Q}^v\delta_{vf} \qquad Q_N = \tilde{Q}^N = 0 \tag{8.122}$$

Now let us consider the $k = 1$ instanton action in the quark-antiquark background given by (8.122). The action in the instanton moduli space can be written as

$$\begin{aligned} S_{\rm mod} &= \bar{w}^{\dot{\alpha}u} w_{\beta v}\left[(Q_{uf}Q^{\dagger fv} + \tilde{Q}^{\dagger}_{uf}\tilde{Q}^{fv} + i\epsilon)\delta^{\dot{\beta}}_{\dot{\alpha}} + D^c\,(\tau^c)^{\dot{\beta}}_{\dot{\alpha}}\,\delta^v_u\right] \\ &\quad +\lambda^{\dot{\alpha}}(\bar{\mu}^u w_{\dot{\alpha}u} + \bar{w}^u_{\dot{\alpha}}\mu_u) - \frac{i}{2}\bar{\mu}^u\,\tilde{\phi}^{\dagger}_{uf}\,\mu'_f - \frac{i}{2}\bar{\mu}'_f\,\phi^{uf\dagger}\,\mu_u \\ &= \bar{w}^{\dot{\alpha}u} w_{\beta u}\left[(2Q_uQ^{u\dagger} + i\epsilon)\delta^{\dot{\beta}}_{\dot{\alpha}} + D^c\,(\tau^c)^{\dot{\beta}}_{\dot{\alpha}}\right] \\ &\quad +\lambda^{\dot{\alpha}}(\bar{\mu}^u w_{\dot{\alpha}u} + \bar{w}^u_{\dot{\alpha}}\mu_u) - \frac{i}{2}\bar{\mu}^f\,\tilde{Q}^{\dagger}_f\,\mu'_f - \frac{i}{2}\bar{\mu}'_f\,Q^{f\dagger}\,\mu_f \end{aligned} \tag{8.123}$$

The $i\epsilon$ term is introduced for regularizing the Gaussian integral. The third line made use of the D-flatness conditions and of (8.122). We notice that $\mu_f, \bar{\mu}_f$ moduli are soaked by the last two terms while λ-depend terms accounts for $\mu_N, \bar{\mu}_N$ components. After the Grassmanian integrals one finds

$$\int d^2\lambda\, d^{2N}\mu\, d^{2N_f}\mu'\, e^{-S_{\rm mod}} = detM^{\dagger}\;(\bar{w}^{\dot{\alpha}}_N w_{N\dot{\alpha}})\, e^{-\bar{w}^{\dot{\alpha}u}w_{\dot{\beta}u}\left[(2Q_uQ^{u\dagger}+i\epsilon)\delta^{\dot{\beta}}_{\dot{\alpha}}+D^c\,(\tau^c)^{\dot{\beta}}_{\dot{\alpha}}\right]} \tag{8.124}$$

with $M^{f'}_f = Q_{uf}, \tilde{Q}^{uf'}$ the Meson field. The w-integrals leads to

$$\int d^4w_N\,(\bar{w}^{\dot{\alpha}}_N w_{N\dot{\alpha}})\, e^{-\bar{w}^{\dot{\alpha}0}w_{\dot{\beta}0}(i\epsilon\delta^{\dot{\beta}}_{\dot{\alpha}}+D^c\,(\tau^c)^{\dot{\beta}}_{\dot{\alpha}})} = \frac{2i\epsilon}{(\vec{D}^2+\epsilon^2)^2} \tag{8.125}$$

$$\int d^{4N_f}w_f e^{-\bar{w}^{\dot{\alpha}f}w_{\dot{\beta}f}\left[(2Q_fQ^{f\dagger}+i\epsilon)\delta^{\dot{\beta}}_{\dot{\alpha}}+D^c\,(\tau^c)^{\dot{\beta}}_{\dot{\alpha}}\right]} = \frac{1}{\prod_f(\vec{D}^2 - (2Q_fQ^{f\dagger}+i\epsilon)^2)}$$

Finally the D-integral leads at leading order in ϵ to

$$\int dD\frac{i\epsilon\,D^2}{(D^2+\epsilon^2)^2\prod_f(D^2-4(Q_fQ^{f\dagger})^2)} = -\frac{1}{4\prod_f(Q_fQ^{f\dagger})^2} = -\frac{1}{4{\rm det}MM^{\dagger}} \tag{8.126}$$

Collecting all pieces one finds

$$S_{\text{eff}} = \Lambda^{2N+1} \int d\mathfrak{M}\, e^{-S_{\text{mod}}} = c \int d^4x d^2\theta \, \frac{\Lambda^{2N+1}}{\det M(x,\theta)} \tag{8.127}$$

with c a constant.

8.2 Exotic prepotentials

Now we consider the effect of exotic instantons. Let us consider again the case of $\mathcal{N} = 1$ SQCD with gauge group $Sp(N)$ and N_f quark-antiquark fundamentals. This can be realized on D3-branes at a $\mathbb{Z}_2 \times \mathbb{Z}_2$ orientifold singularity with $N_0 = N$, $N_1 = N_f$, $N_2 = N_3 = 0$. Now let us consider the effect of a $k = 1$ instanton at the node 1 of the quiver. Such an instanton carry a gauge group $O(1)$ and therefore. no λ-moduli survives the orientifold projection. The only moduli are then $(a_\mu, M_\alpha)_{i_1 j_1}$ and $\mu'_{u_0 i_1}$. For $k = 1$ the action in the moduli space is simply

$$S_{\text{mod}} = \bar{\mu}^{i_1 u_0} Q_{u_0 v_0} \mu^{v_0 i_1} + \bar{\mu}_{i_1 u_0} \tilde{Q}^{u_0 v_0} \mu_{v_0 i_1} \tag{8.128}$$

We notice that the integral over μ's is different from zero only for a square matrix μ, i.e. $N_f = N_c$. In this case, identifying as before $(a_\mu, M_\alpha)_{i_1 j_1}$ with the spacetime coordinates (x_μ, θ_α) and integrating over μ's one finds

$$S_{\text{eff}} = \Lambda^{3-2N} \int d\mathfrak{M}_1\, e^{-S_{\text{mod}}} = c\, \Lambda^{3-2N} \int d^4x d^2\theta \, \det M(x,\theta) \tag{8.129}$$

with

$$\mathfrak{M}_1 = \{a_\mu, D_c, M_\alpha, \mu_{u_0}) \qquad [\mathfrak{M}] = 4 - 6 - 1 + 2N = 2N - 3 \tag{8.130}$$

A Gamma matrices

In four dimensional Euclidean space we take

$$\gamma_n = \begin{pmatrix} 0 & -i\sigma_n \\ i\bar{\sigma}_n & 0 \end{pmatrix} \tag{A.131}$$

with

$$\sigma_m = (i\vec{\tau}, 1) \qquad \bar{\sigma}_m = (-i\vec{\tau}, 1) \tag{A.132}$$

In terms of these matrices one can write self and antiself-dual tensors

$$\sigma_{mn} = \tfrac{1}{4}(\sigma_m\bar{\sigma}_n - \sigma_n\bar{\sigma}_m) \qquad \bar{\sigma}_{mn} = \tfrac{1}{4}(\bar{\sigma}_m\sigma_n - \bar{\sigma}_n\sigma_m) \tag{A.133}$$

satisfying

$$\sigma_{mn} = \tfrac{1}{2}\epsilon_{mnpq}\sigma_{pq} \qquad \bar{\sigma}_{mn} = -\tfrac{1}{2}\epsilon_{mnpq}\bar{\sigma}_{pq} \tag{A.134}$$

The self and antiself-dual tensors can be expressed in terms of the t'Hooft symbols as

$$\sigma_{mn} = \tfrac{i}{2}\,\eta^c_{mn}\tau^c \qquad \bar{\sigma}_{mn} = \tfrac{i}{2}\,\bar{\eta}^c_{mn}\tau^c \tag{A.135}$$

with

$$\begin{aligned} \eta^c_{AB} &= -\eta^c_{BA} \qquad \bar{\eta}^c_{AB} = -\bar{\eta}^c_{BA} \\ \eta^c_{ab} &= \bar{\eta}^c_{ab} = \epsilon_{cab} \qquad \eta^c_{m4} = -\bar{\eta}^c_{m4} = \delta_{mc} \end{aligned} \tag{A.136}$$

In six-dimensional Euclidean space we take

$$\Gamma_a = \begin{pmatrix} 0 & \Sigma_a \\ \bar{\Sigma}_a & 0 \end{pmatrix} \tag{A.137}$$

with

$$\bar{\Sigma}^a_{AB} = (\eta^c_{AB}, i\bar{\eta}^c_{AB}) \qquad \bar{\Sigma}^a_{AB} = (-\eta^c_{AB}, i\bar{\eta}^c_{AB}) \qquad c = 1,2,3 \quad a = 1,..6 \tag{A.138}$$