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Preparatory School to the Winter College on Optics in Imaging Science

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Optical interference, scalar diffraction and polarization.

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From Maxwell's equations to the Helmholtz Equation

Free space:

$$\left.\begin{array}{ll} \nabla\cdot\vec{E}=0 & (i)\\ \nabla\cdot\vec{B}=0 & (ii)\\ \nabla\times\vec{E}=-\frac{\partial\vec{B}}{\partial t} & (iii)\\ \nabla\times\vec{B}=\mu_0\varepsilon_0\frac{\partial\vec{E}}{\partial t} & (iv)\end{array}\right\}\text{ Maxwell's equations}$$

Take the curl of (iii):

$$\underbrace{\nabla\times(\nabla\times\vec{E})}=\underbrace{-\nabla\times\left(\frac{\partial\vec{B}}{\partial t}\right)}$$

$$\nabla(\underbrace{\nabla\cdot\vec{E}}_{\text{use }(i)})-\nabla^2\vec{E}=-\frac{\partial}{\partial t}(\underbrace{\nabla\times\vec{B}}_{\text{use }(iv)})$$

$$-\nabla^2\vec{E}=-\mu_0\varepsilon_0\frac{\partial^2\vec{E}}{\partial t^2}\Longrightarrow\boxed{\nabla^2\vec{E}-\mu_0\varepsilon_0\frac{\partial^2\vec{E}}{\partial t^2}=\vec{0}}\text{ wave equation}$$

(same for $\vec{B}$)

$\mu_0\varepsilon_0=\frac{1}{c^2}$, c = speed of light in vacuum.

Let $\vec{E}(\vec{r},t)$ be expressed as an inverse FT in time:

$$\underbrace{\vec{E}(\vec{r},t)}_{\text{real}}=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\underbrace{\tilde{\vec{E}}(\vec{r},\omega)}_{\text{complex}}\,e^{-i\omega t}\,d\omega$$

notice that a different sign convention is used for time FT.

Because $\vec{E}$ is real: $\vec{E}=\vec{E}^*$, that is,

$$\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\tilde{\vec{E}}(\vec{r},\omega)e^{-i\omega t}d\omega=\left[\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\tilde{\vec{E}}(\vec{r},\omega)e^{-i\omega t}d\omega\right]^*=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\tilde{\vec{E}}^*(\vec{r},\omega)e^{i\omega t}d\omega$$

Change variables $\omega'=-\omega$, $d\omega'=-d\omega$

$$=-\frac{1}{\sqrt{2\pi}}\int_{\infty}^{-\infty}\tilde{\vec{E}}(\vec{r},-\omega)e^{-i\omega' t}d\omega'=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\tilde{\vec{E}}^*(\vec{r},-\omega)e^{-i\omega' t}d\omega',$$

so $$\boxed{\tilde{\vec{E}}^*(\vec{r},-\omega)=\tilde{\vec{E}}(\vec{r},\omega)}$$

This allows us to simplify:

$$\vec{E}(\vec{r},t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty} \tilde{\vec{E}}(\vec{r},\omega)\, e^{-i\omega t}\, d\omega$$

$$= \frac{1}{\sqrt{2\pi}} \left[\int_{-\infty}^{0} \tilde{\vec{E}}(\vec{r},\omega)\, e^{-i\omega t}\, d\omega + \int_{0}^{\infty} \tilde{\vec{E}}(\vec{r},\omega')\, e^{-i\omega' t}\, d\omega' \right]$$

use $\omega = -\omega'$ use $\omega = \omega'$

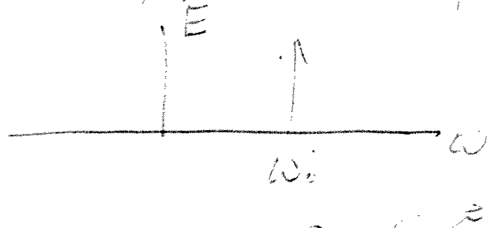
$$= \frac{1}{\sqrt{2\pi}} \left[+ \int_{0}^{\infty} \underbrace{\tilde{\vec{E}}(\vec{r},-\omega)}_{\tilde{\vec{E}}^*(\vec{r},\omega)}\, e^{i\omega t}\, d\omega + \int_{0}^{\infty} \tilde{\vec{E}}(\vec{r},\omega)\, e^{-i\omega t}\, d\omega \right]$$

$$= \frac{1}{\sqrt{2\pi}} \int_{0}^{\infty} \left[\tilde{\vec{E}}^*(\vec{r},\omega)\, e^{i\omega t} + \tilde{\vec{E}}(\vec{r},\omega)\, e^{-i\omega t} \right] d\omega$$

$$= \frac{2}{\sqrt{2\pi}} \int_{0}^{\infty} \mathrm{Re}\left\{ \tilde{\vec{E}}(\vec{r},\omega)\, e^{-i\omega t} \right\} d\omega$$

Monochromatic field

there is only one frequency:



so $$\vec{E}(\vec{r},t) = \frac{2}{\sqrt{2\pi}} \mathrm{Re}\left\{ \tilde{\vec{E}}(\vec{r},\omega_0)\, e^{-i\omega_0 t} \right\}$$

Let $$\vec{U}(\vec{r}) = \frac{\tilde{\vec{E}}(\vec{r},\omega_0)}{\sqrt{2\pi}}$$

then $$\vec{E}(\vec{r},t) = 2\,\mathrm{Re}\left\{ \vec{U}(\vec{r})\, e^{-i\omega_0 t} \right\}$$

In practice, true monochromatic fields do not exist, but lasers can come close

$\Delta\omega \ll \omega_0$

True monochromatic fields would in principle exist for all time!

Time-dependent intensity:

$$I(\vec{r},t) \propto \underbrace{\vec{E}\cdot\vec{E}}_{\to 2 \text{ for convenience}} = 2\,\mathrm{Re}\{\vec{U}e^{-i\omega_0 t}\}\cdot\mathrm{Re}\{\vec{U}e^{-i\omega_0 t}\}$$

$$= 2\left[\left(\mathrm{Re}\{U_x e^{-i\omega_0 t}\}\right)^2 + \left(\mathrm{Re}\{U_y e^{-i\omega_0 t}\}\right)^2 + \left(\mathrm{Re}\{U_z e^{-i\omega_0 t}\}\right)^2\right]$$

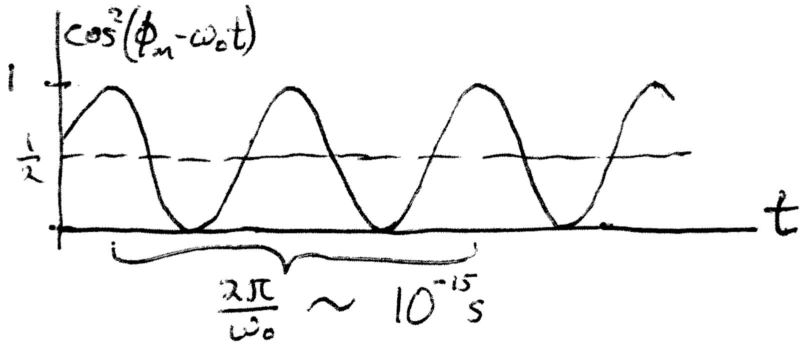
Let $U_m = |U_m|\, e^{i\phi_m}$, $m = x, y, z$. Then

$$I(\vec{r},t) \propto 2\left[|U_x|^2\cos^2(\phi_x-\omega_0 t) + |U_y|^2\cos^2(\phi_y-\omega_0 t) + |U_z|^2\cos^2(\phi_z-\omega_0 t)\right]$$

In practice, the oscillations are so fast that the eye or a detector only sees an average:

$$I(\vec{r}) = \langle I(r,t)\rangle_t$$

Notice: $$\langle \cos^2(\phi_m-\omega_0 t)\rangle_t = \frac{1}{2}$$



Then, $I(\vec{r}) \propto 2\left[|U_x|^2\frac{1}{2} + |U_y|^2\frac{1}{2} + |U_z|^2\frac{1}{2}\right] = \vec{U}^*(\vec{r})\cdot\vec{U}(\vec{r}) = |\vec{U}(\vec{r})|^2$

Substitute now $\vec{E} = 2\,\mathrm{Re}\{\vec{U}(\vec{r})\, e^{-i\omega_0 t}\}$
into wave eq.

$$\nabla^2 \vec{E} = 2\,\mathrm{Re}\left\{\nabla^2 \vec{U}(\vec{r})\, e^{-i\omega_0 t}\right\}$$

$$\frac{\partial^2}{\partial t^2}\vec{E} = 2\,\mathrm{Re}\left\{-\omega_0^2\, \vec{U}(\vec{r})\, e^{-i\omega_0 t}\right\}$$

so

$$\mathrm{Re}\left\{\left[\nabla^2 \vec{U} + \underbrace{\frac{\omega_0^2}{c^2}}_{k_0^2} \vec{U}\right] e^{-i\omega_0 t}\right\} = 0$$

for all t

therefore

$$\boxed{(\nabla^2 + k_0^2)\, \vec{U}(\vec{r}) = 0.}$$

Free-space vector Helmholtz Eq.

For a monochromatic field in a linear, isotropic dielectric: $\vec{D} = \varepsilon(\vec{r}, \omega)\,\vec{E}$,

so

$$\Big(\nabla^2 + \underbrace{\omega_0^2 \mu_0 \varepsilon}_{\underbrace{\omega_0^2 \mu_0 \varepsilon_0}_{k_0^2}\ \underbrace{\frac{\varepsilon}{\varepsilon_0}}_{n^2(\vec{r},\omega_0)}}\Big)\, \vec{U}(\vec{r}) = 0$$

$$\boxed{\left[\nabla^2 + k_0^2 n^2(\vec{r}, \omega_0)\right] \vec{U}(\vec{r}) = 0}$$

Vector Helmholtz Eq. for inhomogeneous media.

This is the basis of most of what follows

Plane waves

One solution of the equation $\nabla^2 U(\vec{r}) + k^2 U(\vec{r}) = 0$ is

$$\vec{U}(\vec{r}) = \vec{A} e^{i\vec{K}\cdot\vec{r}}$$

check: $\nabla^2 \vec{U} = (i\vec{K})\cdot(i\vec{K})\vec{A}e^{i\vec{K}\cdot\vec{r}} = -\vec{K}\cdot\vec{K}\,\vec{U}$

so $\left(k^2 - \vec{K}\cdot\vec{K}\right)\vec{U}(\vec{r}) = 0$

$\vec{K}\cdot\vec{K} = k^2$ or, inother words,

$K_x^2 + K_y^2 + K_z^2 = k^2$, so only two components are independent.

Why do we call it plane wave?

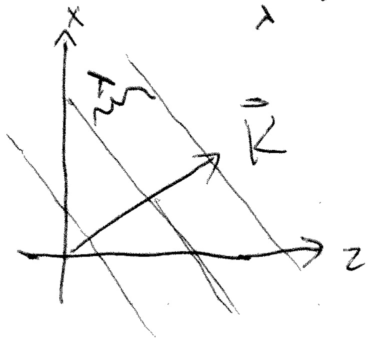
because $\vec{K}\cdot\vec{r}$ is constant over all points $\vec{r}$ within any plane perpendicular to $\vec{K}$.

Let us write $\vec{K} = k\,\hat{u}$, where $\hat{u}$ is a unitvector.

Then $e^{ik(\hat{u}\cdot\vec{r})} = \cos[k(\hat{u}\cdot\vec{r})] + i\sin[k(\hat{u}\cdot\vec{r})]$

oscillates in the direction of $\hat{u}$, with period λ, where

$k = \frac{2\pi}{\lambda}$. λ isthe wavelength.



Additionally, we have

$$\nabla \cdot \vec{E} = 0 \implies \nabla \cdot \vec{U} = 0$$

but $\nabla \cdot \left[\vec{A}\, e^{i\vec{k}\cdot\vec{r}} \right] = i \vec{A} \cdot \vec{k}\, e^{i\vec{k}\cdot\vec{r}} = 0$

so $\vec{A}$ must be perpendicular to $\vec{k}$

Therefore, there are <u>two</u> independent polarizations.

<u>Exercise</u>. Find the intensity of a superposition of two plane waves:

$$\vec{U}(\vec{r}) = \vec{A}_1\, e^{i\vec{k}_1\cdot r} + \vec{A}_2\, e^{i\vec{k}_2\cdot\vec{r}}$$

Paraxial approximation

Let us parametrize K_z in terms of K_x & K_y:

$$K_x^2 + K_y^2 + K_z^2 = k^2 \Rightarrow K_z = \pm\sqrt{k^2 - (K_x^2 + K_y^2)}$$

choose + sign for forward propagating waves.

Define $\underline{K} = (K_x, K_y)$. Then

$$K_z(\underline{K}) = \sqrt{k^2 - |\underline{K}|^2}.$$

Paraxial approximation: $\vec{K}$ is at small angles from the z axis, i.e.

$$K_z \approx k >> |\underline{K}|$$

Then $$K_z(\underline{K}) = k\sqrt{1 - \frac{|\underline{K}|^2}{k^2}} \approx k\left(1 - \frac{|\underline{K}|^2}{2k^2}\right)$$
$$\underline{= k - \frac{|\underline{K}|^2}{2k}}$$

since $\vec{A} \cdot \vec{K} = 0$, $A_z \approx 0$, $\vec{A} \approx A_x \hat{x} + A_y \hat{y}$.

so $$\vec{U}(\vec{r}) = (A_x \hat{x} + A_y \hat{y})\, e^{i\vec{K}\cdot\vec{r}}$$
$$= (A_x \hat{x} + A_y \hat{y})\, e^{i\underline{K}\cdot\underline{x}}\, e^{ikz}\, e^{-i\frac{|\underline{K}|^2}{2k}z}$$

where $\underline{x} = (x, y)$

Consider now a continuous superposition of plane waves in different directions $\underline{K}$. Each might have a different complex amplitude $\vec{A}$:

$$\vec{U}(\vec{r}) = e^{ikz} \iint \vec{A}(\underline{K})\, e^{i\underline{K}\cdot\underline{x}}\, e^{-i\frac{|\underline{K}|^2 z}{2k}}\, dK_x\, dK_y.$$

Change variables to $\underline{K} = 2\pi\underline{\nu}$, $k = 2\pi/\lambda$

$$\vec{U}(\vec{r}) = e^{ikz} \iint \vec{A}(2\pi\underline{\nu})\, e^{i2\pi\underline{\nu}\cdot\underline{x}}\, e^{-i\frac{\cancel{2}\pi\lambda|\underline{\nu}|^2 z}{\cancel{2}}}\, d\nu_x\, d\nu_y\, (2\pi)^2$$

$$= e^{ikz} \iint \left[4\pi^2 \vec{A}(2\pi\underline{\nu})\right] e^{i2\pi\underline{\nu}\cdot\underline{x}}\, e^{-i\pi\lambda z|\underline{\nu}|^2}\, d\nu_x\, d\nu_y.$$

Note that for $z=0$

$$\vec{U}(\underline{x},0) = \vec{U}_0(\underline{x}) = \iint \left[4\pi^2 \vec{A}(2\pi\underline{\nu})\right] e^{i2\pi\underline{\nu}\cdot\underline{x}}\, d\nu_x\, d\nu_y$$

so $\quad \tilde{\vec{U}}_0(\underline{\nu}) = \hat{\mathfrak{F}}_{\underline{x}\to\underline{\nu}}\, \vec{U}_0(\underline{x}) = 4\pi^2 \vec{A}(2\pi\underline{\nu})$

so

$$\vec{U}(\underline{x},z) = e^{ikz} \iint \tilde{\vec{U}}_0(\underline{\nu})\, e^{-i\pi\lambda z|\underline{\nu}|^2}\, e^{i2\pi\underline{\nu}\cdot\underline{x}}\, d\nu_x\, d\nu_y$$

$$= e^{ikz}\, \hat{\mathfrak{F}}^{-1}_{\underline{\nu}\to\underline{x}}\left[e^{-i\pi\lambda z|\underline{\nu}|^2}\, \hat{\mathfrak{F}}_{\underline{x}\to\underline{\nu}}\left[\vec{U}_0(\underline{x})\right]\right]$$

Let $\tilde{G}(\underline{\nu}) = e^{-i\pi\lambda z|\underline{\nu}|^2}$

Then $\vec{U}(\underline{x},z) = e^{ikz}\, \hat{\mathfrak{F}}^{-1}_{\underline{\nu}\to\underline{x}}\left[\tilde{\vec{U}}_0(\underline{\nu})\, \tilde{G}(\underline{\nu})\right]$

$$= e^{ikz}\, \vec{U}_0(\underline{x}) * G(\underline{x})$$

Where

$$G(\underline{x}) = \hat{\mathcal{F}}^{-1}_{\underline{\nu}\to\underline{x}}\ \tilde{G}(\underline{\nu}) = \iint_{-\infty}^{\infty} e^{-i\pi\lambda z|\underline{\nu}|^2} e^{i2\pi\underline{x}\cdot\underline{\nu}}\, d\nu_x d\nu_y$$

$$= \underbrace{\iint_{-\infty}^{\infty} e^{-i\pi\lambda z\,|\underbrace{\underline{\nu}-\underline{x}/\lambda z}_{\underline{\nu}',\ d\nu_x d\nu_y = d\nu_x' d\nu_y'}|^2} d\nu_x d\nu_y}_{\frac{1}{i\lambda z}}\ e^{i\pi|\underline{x}|^2/\lambda z}$$

$$= \frac{e^{ik\frac{|\underline{x}|^2}{2z}}}{i\lambda z}$$

So

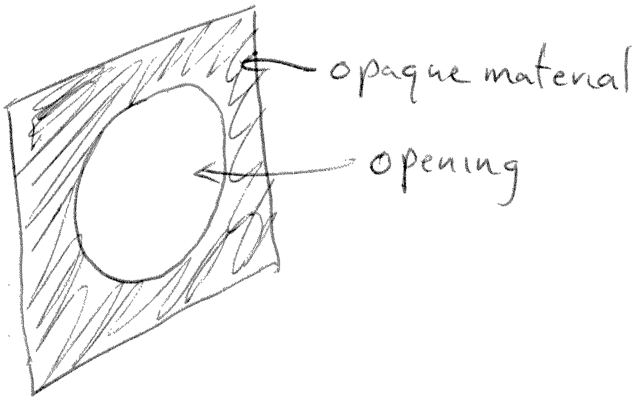
$$\boxed{\begin{aligned}\vec{U}(\underline{x},z) &= e^{ikz}\,\vec{U}_0(\underline{x}) * G(\underline{x})\\ &= \iint_{-\infty}^{\infty} \vec{U}_0(\underline{x}')\ \underbrace{\frac{e^{ik\left[z+\frac{|\underline{x}'-\underline{x}|^2}{2z}\right]}}{i\lambda z}}_{\text{Paraxial approximation to a spherical wave.}}\, dx'dy'\end{aligned}}$$

<u>Fresnel diffraction formula</u>

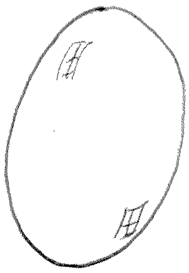
"Planar" obstacles

These are optical elements at planes normal to the optical axis, e.g.

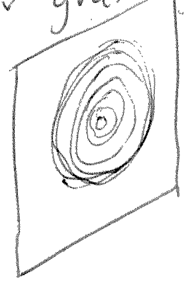
- planar apertures

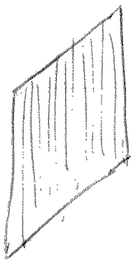


- thin lenses

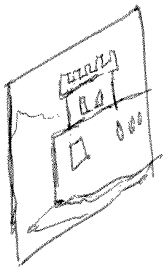


- diffractive lenses or gratings





- Transparencies



Kirchhoff approximation:

The field right after the planar object equals the field right before the planar object times a transmission function $t(x,y)$.

Consider a "scalar field" (e.g. only one polarization)

$$U(\vec{r})$$

and place an element described by $t(x,y)$ at z.

then:

$$\underbrace{U(\underline{x}, z_+)}_{\text{just after}} = t(\underline{x})\, \underbrace{U(\underline{x}, z_-)}_{\text{just before.}}$$

The intensity right after is

$$I(\underline{x}, z_+) = |U(\underline{x}, z_+)|^2 = |t(\underline{x})|^2\, |U(\underline{x}, z_-)|^2 = |t(\underline{x})|^2\, I(\underline{x}, z_-)$$

So $|t(\underline{x})| \leq 1$

If the element is transparent, $|t(\underline{x})| = 1$

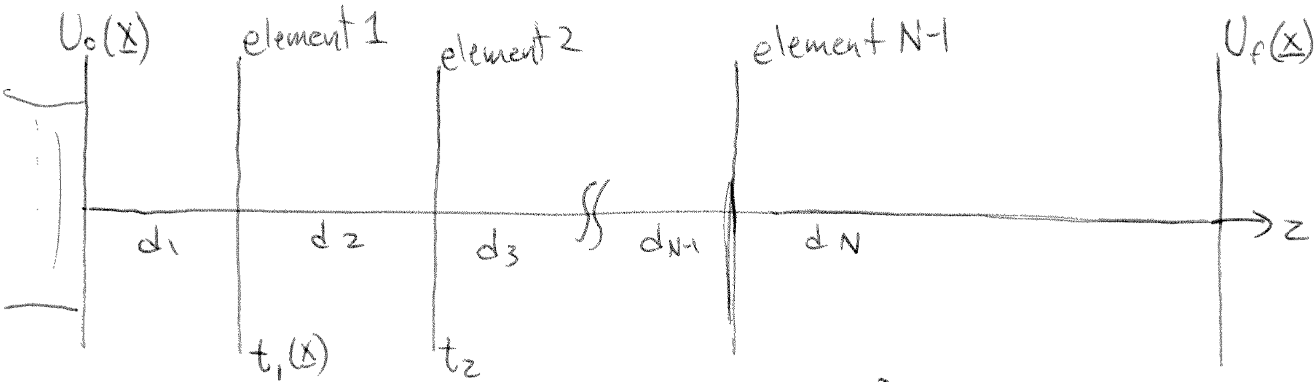
$$t(\underline{x}) = e^{i\phi(\underline{x})},$$

for a lens, $\phi(\underline{x}) = k\left[n\Delta_0 - \frac{|\underline{x}|^2}{2f}\right]$, where f = focal length.

For an aperture: $t(\underline{x}) = \begin{cases} 1 & \text{inside aperture} \\ 0 & \text{outside aperture.} \end{cases}$

In general $t(\underline{x}) = |t(\underline{x})|\, e^{i\phi(\underline{x})}$.

Propagation across a system



Let $\tilde{h}(\underline{\nu}, d) = e^{ikd} e^{-i\pi\lambda d|\underline{\nu}|^2}$

Then

$$U_f(\underline{x}) = \hat{\mathcal{F}}^{-1} \tilde{h}(\underline{\nu}, d_N) \hat{\mathcal{F}} t_{N-1}(\underline{x}) \hat{\mathcal{F}}^{-1} \tilde{h}(\underline{\nu}, d_{N-1}) \hat{\mathcal{F}} t_{N-2}(\underline{x}) \cdots \hat{\mathcal{F}}^{-1} \tilde{h}(\underline{\nu}, d_1) \hat{\mathcal{F}} U_0(\underline{x})$$

Resolution

The image of a point object is not a point but a "blur" called the Airy pattern:

$$U_o(\underline{x}) = \delta(\underline{x}-\underline{x}_o) \implies U_i(\underline{x}) \propto \frac{J_1(k\,NA_i\,|\underline{x}-M\underline{x}_o|)}{k\,NA_i\,|\underline{x}-M\underline{x}_o|}e^{i\phi}$$

where it was assumed that the system is paraxial and there are no aberrations, and where

NA_i = numerical aperture in image space
ϕ = some phase
M = transverse magnification.

Consider two object points $\underline{x}_1$ & $\underline{x}_2$ with ideal image locations $\bar{\underline{x}}_1 = M\underline{x}_1$, $\bar{\underline{x}}_2 = M\underline{x}_2$.

- If the illumination is <u>coherent</u> (a collimated laser), then

$$U_i(\underline{x}) \propto \frac{J_1(kNA_i|\underline{x}-\bar{\underline{x}}_1|)}{kNA_i|\underline{x}-\bar{\underline{x}}_1|}e^{i\phi_1} + \frac{J_2(kNA_i|\underline{x}-\bar{\underline{x}}_2|)}{kNA_i|\underline{x}-\bar{\underline{x}}_2|}e^{i\phi_2}$$

$$I_i(\underline{x}) = \left|\frac{J_1(kNA_i|\underline{x}-\bar{\underline{x}}_1|)}{kNA_i|\underline{x}-\bar{\underline{x}}_1|} + e^{i(\phi_2-\phi_1)}\frac{J_2(kNA_i|\underline{x}-\bar{\underline{x}}_2|)}{kNA_i|\underline{x}-\bar{\underline{x}}_2|}\right|^2$$

<u>this depends strongly on $\phi_2-\phi_1$</u>

· If the illumination is spatially incoherent, then the interference terms average to zero, and

$$I_i \propto \left| \frac{J_1(k NA_i |\underline{x} - \bar{\underline{x}}_1|)}{k NA_i |\underline{x} - \bar{\underline{x}}_1|} \right|^2 + \left| \frac{J_2(k NA_i |\underline{x} - \bar{\underline{x}}_2|)}{k NA_i |\underline{x} - \bar{\underline{x}}_2|} \right|^2$$

Rayleigh Resolution criterion

We can distinguish the images of two points if their separation is more than the distance to the first zero:

$$\underline{|\bar{\underline{x}}_2 - \bar{\underline{x}}_1| \geq 0.609 \frac{\lambda}{NA_i}}$$