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School on New Trends in Quantum Dynamics and Quantum Entanglement

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QUANTUM MEASURES FOR NON-MARKOVIANITY

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Quantum measures for non-Markovianity

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1.1. Open quantum systems: Basic concepts

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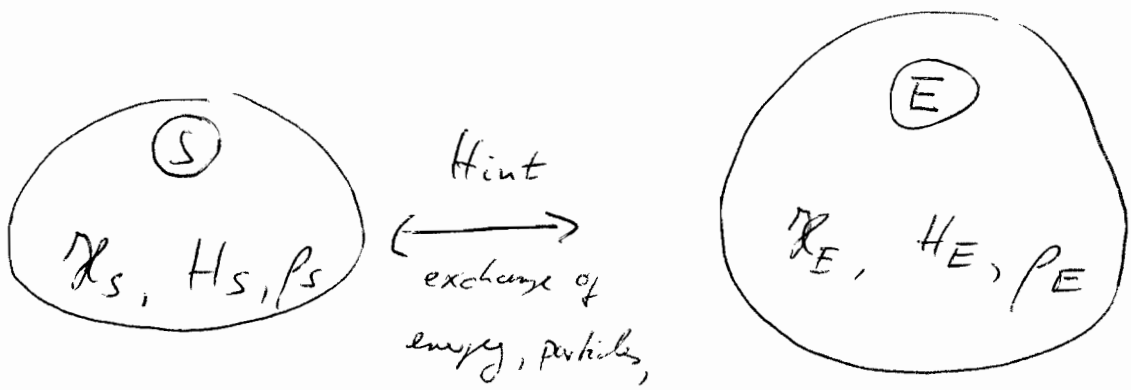
1. Open quantum systems and distinguishability of quantum states

1.1. Open quantum systems: Basic concepts



"open system"
"system"

"environment",
"reservoir, bath"

System S: Hilbert space $\mathcal{H}_S$, Hamiltonian H_S

Environment E: Hilbert space $\mathcal{H}_E$, Hamiltonian H_E

Total system S+E:

Hilbert space $\mathcal{H} = \mathcal{H}_S \otimes \mathcal{H}_E$ tensor product

Hamiltonian: $H = H_S + H_E + H_{int}$

$\equiv H_S \otimes I_S + I_E \otimes H_E + H_{int}$

States of total system:

density matrix: $\rho_{SE} = \rho$

positive operator: $\rho \geq 0 \iff$ ρ selfadjoint and all eigenvalues = prob. positive

normalization: $\mathrm{tr}\,\rho = 1$ total trace in $\mathcal{H}$

Reduced states of system S and environment E:

$\rho_S = \mathrm{tr}_E \rho_{SE}$ partial trace over environment E

$\rho_E = \mathrm{tr}_S \rho_{SE}$ partial trace over system S

Expectation values of system operators:

$A = A_S \otimes I_E$, A_S = observable of S

I_E = unit operator on $\mathcal{H}_E$

$\Rightarrow \langle A \rangle = \mathrm{tr}_S \{ A_S \rho_S \}$

Central goals of the theory

- Methods for "elimination" of degrees of freedom of E
- Construct effective (approximate) description for dynamics of open system's density matrix $\rho_S = \rho_S(t)$
- Derivation of effective equations of motion for $\rho_S(t)$, "master equations", mathematical structure of such equations ($\rightarrow$ Bassano Vacchini's talk)

Physical phenomena

- Dissipation and damping of populations
- relaxation to equilibrium
- quantum transport
- destruction of quantum coherence (decoherence)
- build up of quantum correlations / entanglement
- non-equilibrium stationary states
- ...

1.2. Quantum dynamical maps

Basic assumptions:

- Total system $S+E$ is closed $\Rightarrow$ dynamics of ρ is given by some unitary time-evolution operator:

$$U(t) = e^{-iHt}, \qquad H = \text{total Hamiltonian}$$
$$\hbar = 1$$

- At initial time $t=0$ system and environment are statistically independent

$$\rho(0) = \rho_S(0) \otimes \rho_E(0) \qquad \text{tensor product state}$$

Diagram of dynamical evolution:

$$\rho(0) = \rho_S(0) \otimes \rho_E(0) \xrightarrow{\text{unitary evolution}} \rho(t) = U(t)\left[\rho_S(0) \otimes \rho_E(0)\right] U^\dagger(t)$$

$$\downarrow \text{tr}_E \qquad\qquad\qquad\qquad \downarrow \text{tr}_E$$

$$\rho_S(0) \xrightarrow[\text{map}]{\text{dynamical}} \rho_S(t) = \text{tr}_E\left\{U(t)\left[\rho_S(0) \otimes \rho_E(0)\right] U^\dagger(t)\right\}$$

$$\equiv \Phi_t \rho_S(0)$$

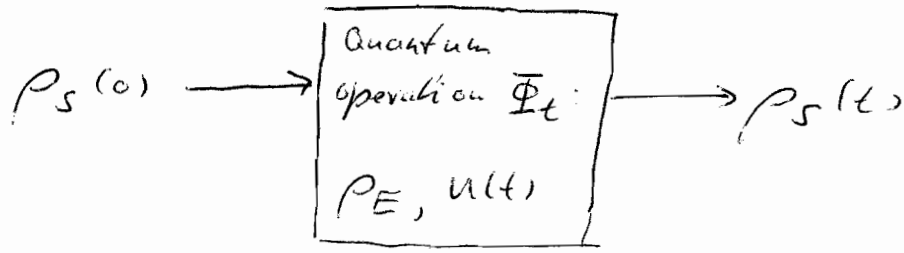
Consider fixed $t \geq 0$ and fixed environmental state $\rho_E = \rho_E(0)$. Then

$$\Phi_t : \rho_S(0) \longmapsto \rho_S(t) = \Phi_t \rho_S(0)$$

defines a linear map, or "superoperator", the so-called quantum dynamical map. Also: "quantum operation" "quantum channel"

→ Hint to Nila Datta's talk!

Schematically:



Properties of the dynamical map:

- Φ_t is trace-preserving:

$$tr_S\{\Phi_t A\} = tr_S\{A\}$$

- Φ_t maps positive operators to positive operators, i.e. Φ_t is a positive map:

$$A \geq 0 \Rightarrow A' = \Phi_t A \geq 0.$$

$\Rightarrow$ Φ_t can be regarded as a convex-linear map on the state space of S:

$$S(\mathcal{H}_S) = \{ \rho_S \text{ on } \mathcal{H}_S \mid \rho_S \geq 0,\ \mathrm{tr}_S \rho_S = 1 \}$$

$$= \text{state space of reduced system } S$$

$$\Phi_t : S(\mathcal{H}) \longrightarrow S(\mathcal{H})$$

- Φ_t is not only positive, but also <u>completely positive</u>.

<u>Definition of complete positivity</u>

<u>Hilbert space:</u> $\mathcal{H}_S \otimes \mathbb{C}^n$, $n = 1, 2, 3, \ldots$

$\mathbb{C}^n$: Hilbert space of auxiliary n-level system

<u>Tensor extension of Φ</u> defined by:

$$(\Phi \otimes I_n)(A \otimes B) = (\Phi A) \otimes B$$

+ linear extension

identity on second factor!

Definition: Φ is called <u>completely positive</u> if $\Phi \otimes I_n$ is <u>positive for all n</u>:

$$C \geq 0 \Rightarrow (\Phi \otimes I_n) C \geq 0$$

Applications to <u>entanglement theory</u>:

If Φ is positive, but not completely positive:

$(\Phi \otimes I)\rho \geq 0$ necessary condition for separability of ρ

$(\Phi \otimes I)\rho \not\geq 0 \Rightarrow \rho$ must be entangled

$W = (\Phi \otimes I)|\Psi\rangle\langle\Psi| =$ "entanglement witness"

$|\Psi\rangle$ = maximally entangled state

(Jamiołkowski isomorphism, Choi-Theorem)

<u>Notation</u>: Φ = trace-preserving quantum operation

= quantum channel

= CPT map

Theorem:

The following statements are equivalent:

1. Φ is completely positive

2. Φ has a <u>Kraus representation</u>

$$\Phi A = \sum_i \Omega_i A \Omega_i^+$$

Ω_i = "Kraus operators" (arbitrary operators)

(→ starting point for proof of Lindblad theorem)

3. Φ can be represented as (Stinespring dilation):

$$\Phi A = tr_E \{ U (A \otimes \rho_E) U^+ \}$$

U unitary, ρ_E density matrix

(→ any CPT map can be regarded as dynamical map of some open quantum system)

1.3 The trace distance between quantum states

Trace norm of a trace class operator A:

$$\|A\| = \mathrm{tr}\left\{\sqrt{A^+A}\right\} = \mathrm{tr}\,|A| < \infty$$

A selfadjoint with eigenvalues a_i:

$$\|A\| = \sum_i |a_i| = \text{sum of absolute eigenvalues (counting multiplicity)}$$

$$A = \sum_i a_i\, |i\rangle\langle i|$$

Consider two quantum states ρ_1 and ρ_2, represented by positive trace class operators with unit trace.

Definition of trace distance:

$$D(\rho_1,\rho_2) = \frac{1}{2}\|\rho_1-\rho_2\| = \frac{1}{2}\,\mathrm{tr}\,|\rho_1-\rho_2|$$

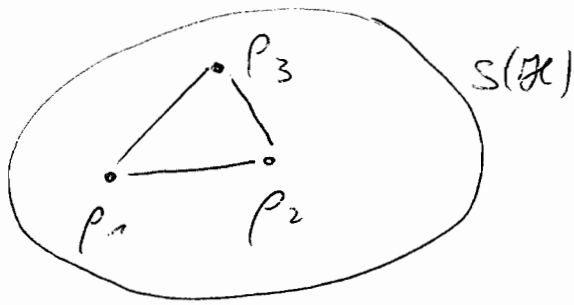
This is a natural metric on the state space $S(\mathcal{H})$ with the following properties:

1. Bounds

$$0 \leq D(\rho_1, \rho_2) \leq 1$$

$D(\rho_1, \rho_2) = 0 \iff \rho_1 = \rho_2$ ('norm')

$D(\rho_1, \rho_2) = 1 \iff \rho_1$ and ρ_2 have orthogonal supports ($\rho_1 \perp \rho_2$)



2. Triangular inequality

$$D(\rho_1, \rho_2) \leq D(\rho_1, \rho_3) + D(\rho_3, \rho_2)$$

3. Invariance under unitary transformations

$$D(U\rho_1 U^+, U\rho_2 U^+) = D(\rho_1, \rho_2)$$

4. All positive and trace preserving maps are contractions

$$D(\Phi\rho_1, \Phi\rho_2) \leq D(\rho_1, \rho_2)$$

Note: Φ needs not be completely positive!

[Ruskai, 1994]

5. Subadditivity with respect to tensor products

$$D(\rho_1 \otimes \sigma_1, \rho_2 \otimes \sigma_2) \leq D(\rho_1, \rho_2) + D(\sigma_1, \sigma_2)$$

In particular: $D(\rho_1 \otimes \sigma, \rho_2 \otimes \sigma) = D(\rho_1, \rho_2)$

6. Representation as maximum over all projections Π

$$D(\rho_1, \rho_2) = \max_{\Pi} \operatorname{tr}\{\Pi(\rho_1 - \rho_2)\}$$

Physical interpretation of the trace distance

Alice:

preparation

ρ_1 with $p_1 = \frac{1}{2}$

ρ_2 with $p_2 = \frac{1}{2}$

$\Longrightarrow$

Bob:

measurement:

ρ_1 or ρ_2 ?

$\Downarrow$ optimal strategy

correct state identification with success probability:

*) $$p_{Bob} = \frac{1}{2}\left(1 + D(\rho_1, \rho_2)\right)$$

$D(\rho_1, \rho_2)$ = bias in favour of correct state identification

$D(\rho_1, \rho_2)$ = measure for distinguishability of states

Example

pure states $\rho_1 = |\Psi_1\rangle\langle\Psi_1|$, $\rho_2 = |\Psi_2\rangle\langle\Psi_2|$

$$\Rightarrow \quad D(\rho_1, \rho_2) = \sqrt{1 - |\langle\Psi_1|\Psi_2\rangle|^2}$$

$$\Rightarrow \quad P_{Bob} = 1 \quad \text{for} \quad |\Psi_1\rangle \perp |\Psi_2\rangle$$

Bob measures projection $\Pi = |\Psi_1\rangle\langle\Psi_1|$

(or $\Pi = |\Psi_2\rangle\langle\Psi_2|$)

More generally: $P_{Bob} = 1$ for $\rho_1 \perp \rho_2$

Φ = trace-preserving quantum operation (CPT map):

$$D(\Phi\rho_1, \Phi\rho_2) \leq D(\rho_1, \rho_2)$$

$\Rightarrow$ a quantum operation can <u>never increase</u> <u>the distinguishability</u> between two states!

(Note: Assumption of trace-preservation is important $\rightarrow$ non-selective measurements)

Proof of $P_{Bob} = \frac{1}{2}(1 + D(\rho_1, \rho_2))$:

$$D(\rho_1, \rho_2) = \max_{\Pi} \operatorname{tr}\{\Pi(\rho_1 - \rho_2)\}$$

Bob measures projection Π:

$$\Pi = 1 \Rightarrow \text{state was } \rho_1$$
$$\Pi = 0 \Rightarrow \text{state was } \rho_2$$

Success probability:

$$P_{Bob} = \frac{1}{2} \cdot P(\Pi = 1 \mid \rho_1) + \frac{1}{2} \cdot P(\Pi = 0 \mid \rho_2)$$

$$= \frac{1}{2}\left(\operatorname{tr}\{\Pi \rho_1\} + \operatorname{tr}\{(I - \Pi)\rho_2\} \right)$$

$$= \frac{1}{2}\left(1 + \operatorname{tr}\{\Pi(\rho_1 - \rho_2)\} \right)$$

$$= \frac{1}{2}\left(1 + D(\rho_1, \rho_2)\right)$$

2. Information flow and measures for quantum non-Markovianity

2.1. Quantum Markovian semigroups

Goals

- What is the characteristic feature of non-Markovian quantum dynamics?
- Memory effects and flow of information between system and environment?
- Definition of non-Markovianity without reference to specific representation (e.g. in terms of approximate master equation etc.)?
- Quantitative measure for degree of non-Markovianity!
- Experimentally measurable quantity?

→ Experiments: Guang-Can Guo and Chuang-Feng Li, Key Laboratory of Science and Technology of China, CAS, Hefei, China

→ Jyrki Piilo's talk next week!

Consider first quantum dynamical semigroup:

Dynamical maps: $\Phi_t = e^{\mathcal{L}t}$, $t \geq 0$

$\boxed{\Phi_{t+\tau} = \Phi_t \circ \Phi_\tau}$ $\Rightarrow$ $\mathcal{L}$ = generator of the semigroup

Semigroup $\{ \Phi_t \mid t \geq 0, \Phi_0 = id \}$

Theorem (Gorini, Kossakowski, Sudarshan, Lindblad, 1976):

$\mathcal{L}$ is the generator of a quantum dynamical semigroup (semigroup of CPT maps) if and only if it has the structure ("Lindblad generator")

$$\mathcal{L}\rho_S = -i[H, \rho_S] + \sum_i \gamma_i \left[A_i \rho_S A_i^+ - \frac{1}{2} \{ A_i^+ A_i, \rho_S \} \right]$$

with: $H^+ = H$, $\gamma_i \geq 0$, A_i arbitrary system operators

$\Rightarrow$ quantum Markovian master equation:

$$\frac{d}{dt} \rho_S(t) = \mathcal{L} \rho_S(t)$$

Consider pair of initial states:

$$\rho_{1,2}(0) \longrightarrow \rho_{1,2}(t) = e^{\mathcal{L}t}\rho_{1,2}(0) = \Phi_t \rho_{1,2}($$

Dynamical behavior of the trace distance

$$D(\rho_1(t+\tau), \rho_2(t+\tau)) = D(\Phi_\tau \rho_1(t), \Phi_\tau \rho_2(t))$$

(Semigroup property)

$$\leq D(\rho_1(t), \rho_2(t))$$

(contraction property of CPT maps)

$\forall\, t, \tau \geq 0$

Implications

- monotonic decrease of the trace distance
- as time increases, any two initial states $\rho_{1,2}(0)$ become less and less distinguishable
- Interpretation: Continuous flow of information from system to environment, loss of relative information on the pair of states $\rho_{1,2}$

$\Rightarrow$ characteristic feature of Markovian dynamics!

2.2. Time-local master equations and divisibility

Natural generalization of Lindblad master equation:

$$\frac{d}{dt}\rho_S(t) = K(t)\,\rho_S(t)$$

$K(t)$ = time-dependent generator (TCL generator)

Most general form of Hermiticity-preserving generator:
(and trace-preserving)

$$K(t)\rho_S = -i\,[H(t), \rho_S] + \sum_i \gamma_i(t)\left[A_i(t)\,\rho_S\,A_i^\dagger(t) - \frac{1}{2}\{A_i^\dagger(t)A_i(t), \rho\}\right]$$

By contrast to Lindblad structure:

Hamiltonian $H(t)$, "Lindblad"-operators $A_i(t)$

and decay rates in general time-dependent!

$\Rightarrow$ Dynamical maps Φ_t do no longer form a semigroup!

Define instead two-parameter family of maps:

$$\Phi_{t_2,t_1} = \mathcal{T}_{\leftarrow}\exp\left[\int_{t_1}^{t_2} dt\, K(t)\right], \quad t_2 \geq t_1$$

$\mathcal{T}_{\leftarrow}$: chronological time-ordering operator

Instead of the group property we have:

$$\Phi_{t_3,t_1} = \Phi_{t_3,t_2}\,\Phi_{t_2,t_1}, \quad t_3 \geq t_2 \geq t_1$$

dynamical map $\Phi_t = \Phi_{t,0}$

Assumption: $\gamma_i(t) \geq 0 \quad \forall i \; \forall t \geq 0$

decay rates of all channels positive

$\Rightarrow$ Conclusions:

- The generator $K(t)$ is in Lindblad form for each fixed $t \geq 0$
- The maps Φ_{t_2,t_1} are CPT for all $t_2 \geq t_1 \geq 0$
- For this type of processes we have a monotonic decrease of the trace distance and, hence, of the distinguishability.

  $\Rightarrow$ continuous flow of information from system to environment.

"time-dependent Markovian dynamics"

Proof:

Pair of states: $\rho_{1,2}(t) = \Phi_{t,0}\,\rho_{1,2}(0)$

$$\rho_{1,2}(t+\tau) = \underbrace{\Phi_{t+\tau,t}}_{\text{CPT}}\,\rho_{1,2}(t)$$

$\Rightarrow$

$$D(\rho_1(t+\tau), \rho_2(t+\tau)) = D(\Phi_{t+\tau,t}\,\rho_1(t), \Phi_{t+\tau,t}\,\rho_2(t))$$

$$\leq D(\rho_1(t), \rho_2(t))$$

(contraction property of CPT maps)

□

Divisibility

The dynamical map $\Phi_t = \Phi_{t,0}$ is defined to be divisible if for all $t, \tau \geq 0$ the CPT map $\Phi_{t+\tau,0}$ can be written

$$\Phi_{t+\tau,0} = \underbrace{\Phi_{t+\tau,t}}_{\stackrel{!}{=}\text{CPT}}\,\Phi_{t,0}$$

CPT by construction / assumption!

Possible violation of divisibility:

- If $\Phi_{t,0}$ is not invertible, a linear map $\Phi_{t+\tau,t}$ may not exist!
- Even if $\Phi_{t+\tau,t}$ exists, it may be not completely positive and not even positive!

From above considerations:

Time-dependent Markov processes, TCL generator with positive rates $\Rightarrow$ Dynamical maps divisible?

Theorem: Under certain conditions, the converse is also true: The classes of time-dependent Markovian process and divisible process are essentially the same.

Proof: Assume $\Phi_{t,0}$ is divisible with a unique map $\Phi_{t+\tau,t}$ depending smoothly on τ.

Then we have:

$$\rho(t+\tau) = \underbrace{\Phi_{t+\tau,\,t}}_{=CPT}\,\rho(t)$$

$$\Rightarrow \quad \frac{d}{dt}\rho(t) = \underbrace{\left[\left.\frac{d}{d\tau}\right|_{\tau=0} \Phi_{t+\tau,\,t}\right]}_{\equiv K(t)\ \text{TCL generator!}}\rho(t)$$

Since $\Phi_{t+\tau,\,t} = CPT$ and $\Phi_{t,\,t} = \mathbb{1}$

the generator $K(t)$ must be in Lindblad form for each fixed $t \geq 0$, and, hence, the rates $\gamma_i(t)$ must be positive. □

2.3. Definition and measure for quantum non-Markovianity

(HPB, E.M. Laine, J. Piilo, Phys. Rev. Lett. 103, 210401 (2009); Phys. Rev. A 81, 062115)

A quantum process given by a dynamical map Φ_t is said to be non-Markovian if there is a pair of initial states $\rho_{1,2}(0)$ and a certain time $t \geq 0$ such that

$$\sigma(t, \rho_{1,2}(0)) \equiv \frac{d}{dt} D(\rho_1(t), \rho_2(t)) > 0$$

$$\rho_{1,2}(t) = \Phi_t \rho_{1,2}(0)$$

non-monotonic behavior of trace distance

Physical interpretation:

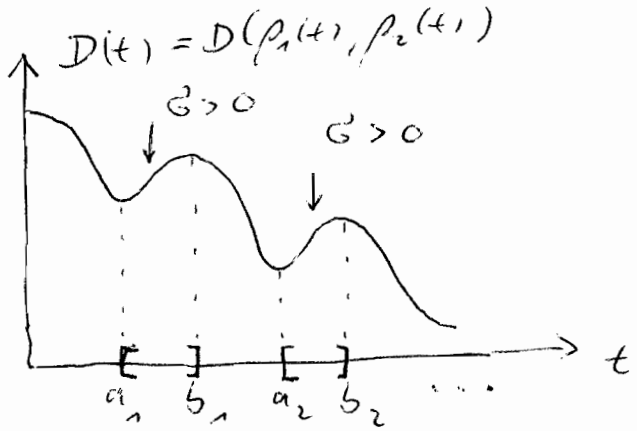
Non-Markovian processes feature an increase of the distinguishability over certain time intervals, i.e. an associated backflow of information from the environment E to the system S

$\Rightarrow$ memory effects. Through this backflow of information the earlier system states may have an influence on the later system states

Measure for non-Markovianity

$$\mathcal{N}(\Phi) = \max_{\rho_{1,2}(0)} \int_{\dot{G}>0} dt\ \dot{G}(t, \rho_{1,2}(0))$$

$$= \max_{\rho_{1,2}(0)} \sum_i \left[D(\rho_1(b_i), \rho_2(b_i)) - D(\rho_1(a_i), \rho_2(a_i)) \right]$$



- measures total increase of trace distance

  = total backflow of information from E to S

  → Summation over all intervalls $[a_i, b_i]$ over which $\dot{G} > 0$

- maximization over all pairs of initial states

  $\rho_{1,2}(0) \in S(\mathcal{H}_S)$ [e.g. performed through Monte Carlo sampling]

Some basic properties

- $\mathcal{N}(\Phi)$ = functional of the whole family $\{\Phi_t \mid t \geq 0\}$ of dynamical maps

- $\mathcal{N}(\Phi) \geq 0$ and $\mathcal{N}(\Phi) = 0$ if and only if proces Φ_t is Markovian

- $\mathcal{N}(\Phi) = +\infty$ possible $\Rightarrow$ process Φ_t can never be approximated by Markovian dynamics

- Semigroups and time-dependent Markov processes (= divisible process): $\mathcal{N}(\Phi) = 0$

- Proces given by time-local master equation $\frac{d}{dt}\rho_S(t) = K(t)\rho_S(t)$:

  $\mathcal{N}(\Phi) > 0$ $\Rightarrow$ at least one of the rates $\gamma_i(t)$ must be negative

  $\Rightarrow$ process must be non-divisible

Hence:

$\gamma_i(t) < 0$ for some $t > 0$

and some decay mode i

is necessary (but not sufficient) for non-Markovianity.

For master equation with several decay modes, the process can be Markovian, $\mathcal{N} = 0$, although some of the rates are negative: In flow can the backflow of information from E to S due to the negative modes is overcompensated by the flow of information from S to E due to the positive modes.

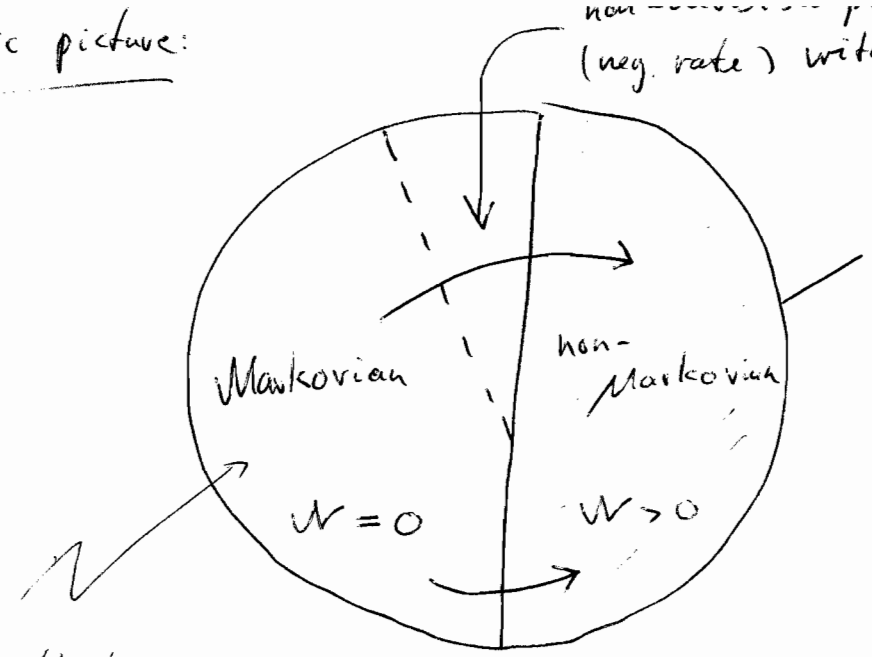
Conclusion

$$\boxed{\mathcal{N}(\Phi) > 0} \Rightarrow \boxed{\text{non-Divisibility}}$$

converse is not true!? (possible for several decay modes)

Counterexample: L. Mazzola, E.M Laine, HPB, S. Maniscalco, J. Piilo, Phys. Rev. A 81, 062120 (2010)

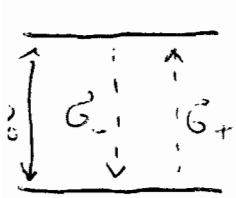
Schematic picture:



Example (E.M. Laine, J. Piilo, HPB, Phys. Rev. A 81, 062115 (2010))

Microscopic model: Two-state system interacting with a reservoir of field modes b_k (simple case of spin-boson model)

Hamiltonian: $H = H_S + H_E + H_{int}$



$$H_S = \omega_0 \sigma_+ \sigma_- , \quad H_E = \sum_k \omega_k b_k^+ b_k$$

$$H_I = \sum_k \left(g_k \sigma_+ \otimes b_k + g_k^* \sigma_- \otimes b_k^+ \right)$$

$$\equiv \sigma_+ B + \sigma_- B^+$$

$\sigma_+ = |1\rangle\langle 0|$, $\sigma_- = |0\rangle\langle 1|$, $[b_k, b_{k'}^+] = \delta_{kk'}$

Two-point reservoir correlation function

Initial state of total system: $\rho(0) = \rho_S(0) \otimes |0\rangle\langle 0|$

$|0\rangle$ = vacuum (ground) state of reservoir

two-point correlation:

$$f(t-t_1) = \langle 0 | B(t)\, B^\dagger(t_1) | 0 \rangle\, e^{i\omega_0 (t-t_1)}$$

$$= \sum_k |g_k|^2\, e^{i(\omega_0 - \omega_k)(t-t_1)}$$

$$B(t) = \sum_k g_k\, b_k\, e^{-i\omega_k t}$$

interaction picture reservoir operator

Function $G(t)$ defined by:

$$\frac{d}{dt} G(t) = -\int_0^t dt_1\, f(t-t_1)\, G(t_1)$$

$$G(0) = 1.$$

$\Rightarrow$ $G(t)$ completely defines/determines the exact dynamical map of the problem.

$$\Phi_t : \rho_S(0) = \begin{pmatrix} \rho_{11}(0) & \rho_{10}(0) \\ \rho_{01}(t) & \rho_{00}(0) \end{pmatrix} \longmapsto \rho_S(t) = \begin{pmatrix} \rho_{11}(t) & \rho_{10}(t) \\ \rho_{01}(t) & \rho_{00}(t) \end{pmatrix}$$

$$\rho_{11}(t) = |G(t)|^2 \rho_{11}(0)$$

$$\rho_{00}(t) = \rho_{00}(0) + \left(1 - |G(t)|^2\right) \rho_{11}(0)$$

$$\rho_{10}(t) = G(t)\, \rho_{10}(0)$$

$$\rho_{01}(t) = G^*(t)\, \rho_{01}(0)$$

Exact time-local (TCL) master equation:

$$\frac{d}{dt}\rho_S(t) = K(t)\rho_S(t)$$

$$= -\frac{i}{2} S(t) \left[\sigma_+\sigma_-, \rho_S(t)\right]$$

$$+ \gamma(t) \left[\sigma_- \rho_S(t) \sigma_+ - \frac{1}{2}\left\{\sigma_+\sigma_-, \rho_S(t)\right\}\right]$$

Hamiltonian term: time-dependent "Lamb shift":

$$S(t) = -2\, \mathrm{Im}\left(\frac{\dot{G}(t)}{G(t)}\right)$$

A single decay channel with rate

$$\gamma(t) = -2\, \mathrm{Re}\left(\frac{\dot{G}(t)}{G(t)}\right) = -\frac{2}{|G(t)|} \frac{d}{dt} |G(t)|$$

Dynamics of trace distance.

$$D(t, \rho_{1,2}(0)) = \underbrace{\frac{2|G(t)|^2 a^2 + |b|^2}{\sqrt{|G(t)|^2 a^2 + |b|^2}}}_{>0} \frac{d}{dt}|G(t)|$$

$$a \equiv \rho_{11}^{1}(0) - \rho_{11}^{2}(0) \quad , \quad b \equiv \rho_{10}^{1}(0) - \rho_{10}^{2}(0)$$

Conclusion

Non-Markovian behavior : $N(\Phi) > 0$

$$\Leftrightarrow \quad \frac{d}{dt}|G(t)| > 0$$

$$\Leftrightarrow \quad \gamma(t) < 0$$

temporarily negative decay rate !

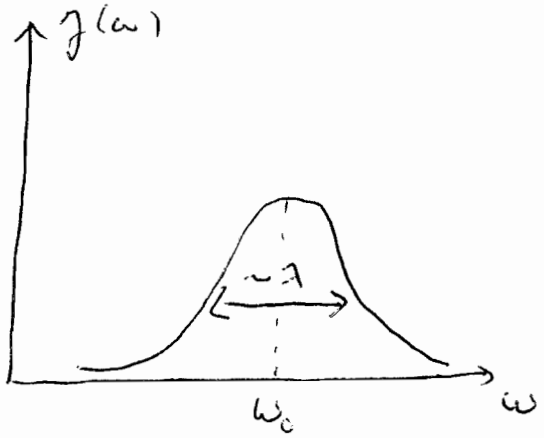
$$\Leftrightarrow$$ process non-divisible !

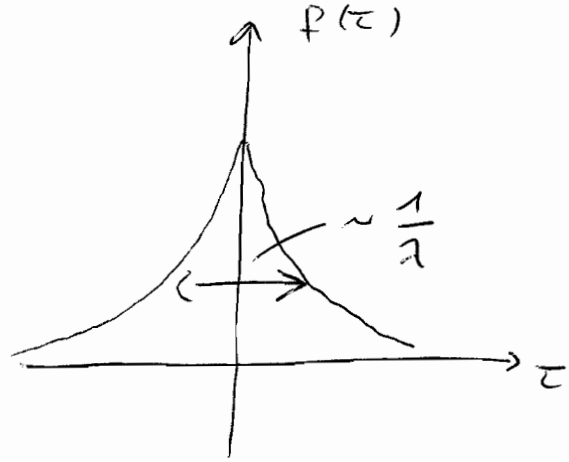
Note: This is the only possible situation here, because we have a single decay channel !

Specific case

Lorentzian spectral density : $J(\omega) = \frac{\gamma_0 \lambda^2}{2\pi} \frac{1}{(\omega_c - \omega)^2 + \lambda^2}$

$\Rightarrow$ exponential two-point correlation : $f(\tau) = \frac{\gamma_0 \lambda}{2} e^{-\lambda|\tau|}$





transition frequency resonant with maximal density

$$\Rightarrow G(t) = e^{-\lambda t/2}\left[\cosh\left(\frac{dt}{2}\right) + \frac{\lambda}{d}\sinh\left(\frac{dt}{2}\right)\right]$$

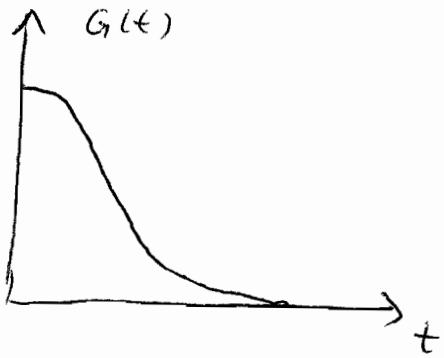
$$d \equiv \sqrt{\lambda^2 - 2\gamma_0\lambda} = \sqrt{\lambda(\lambda - 2\gamma_0)}$$

Weak coupling regime :

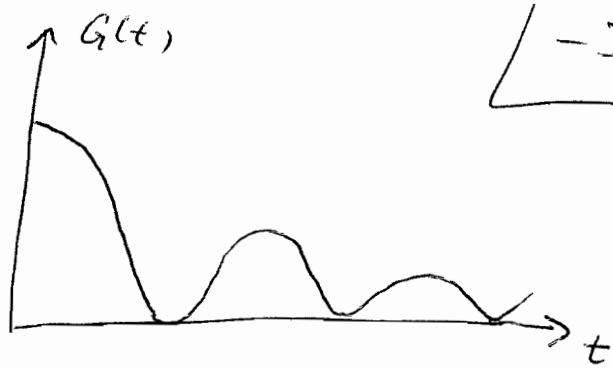
$\gamma_0 < \frac{\lambda}{2} \Rightarrow G(t)$ monotonic $\Rightarrow \mathcal{N} = 0$

Strong coupling regime :

$\gamma_0 > \frac{\lambda}{2} \Rightarrow G(t)$ non-monotonic $\Rightarrow \mathcal{N} > 0$



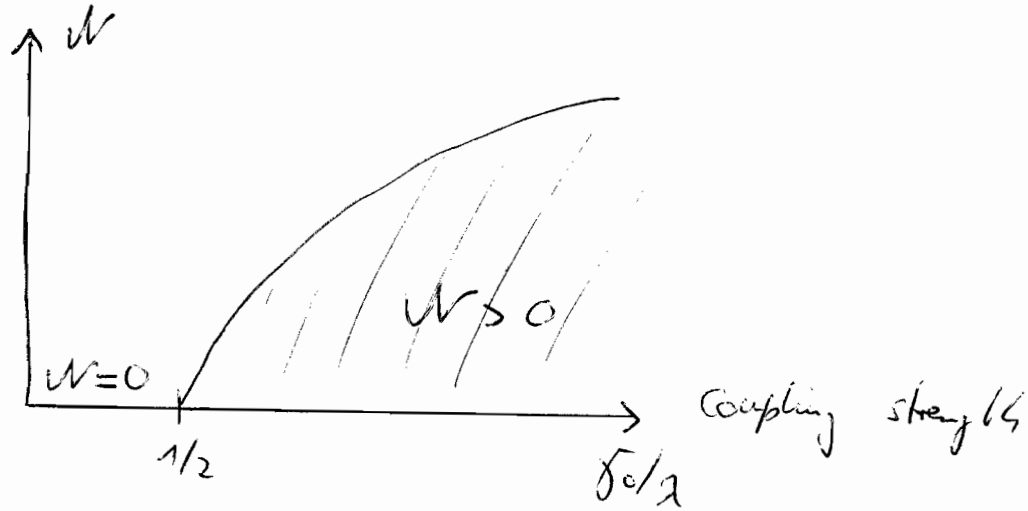
$\gamma_0 < \frac{\lambda}{2}$



$\gamma_0 > \frac{\lambda}{2}$

Physical interpretation: Emission of "virtual photon" which is reabsorbed at a later time ⇒ memory effect

Behavior of measure for non-Markovianity



- Threshold for emergence of non-Markovianity?
- optimal initial pair involves coherences:

$\rho_1(0) = |0\rangle\langle 0| = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$, $\rho_2(0) = \frac{1}{2}(|1\rangle + |0\rangle)(\langle 1| + \langle 0|) = \frac{1}{2}\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$

⇒ Genuine quantum non-Markovianity?

⇒ The quantum non-Markovianity is larger than the "classical" non-Markovianity in which only diagonal states are allowed: $\boxed{\mathcal{N}_{qm}(\Phi) > \mathcal{N}_{cl}(\Phi)}$

Alternative measures for the distance between quantum states

- Relative entropy:

$$S(\rho_1|\rho_2) = \mathrm{tr}\{\rho_1 \ln \rho_1 - \rho_1 \ln \rho_2\}$$

CPT maps are contractions:

$$S(\Phi\rho_1|\Phi\rho_2) \leq S(\rho_1|\rho_2)$$

But: Problems with singularities of relative entropy!

- Hilbert-Schmidt distance:

$$D_{HS}(\rho_1,\rho_2) = \sqrt{\mathrm{tr}\{(\rho_1-\rho_2)^2\}} = \|\rho_1-\rho_2\|_{HS}$$

However: CPT maps are in general no contractions!

3. The role of initial correlations

3.1. General inequalities describing correlated initial states

Dynamical map: $\Phi_t : \rho_S(0) \longmapsto \rho_S(t) = \Phi_t \rho_S(0)$

$$= \mathrm{tr}_E \left\{ U(t) \underbrace{\left[\rho_S(0) \otimes \rho_E \right]}_{\text{uncorrelated initial state with fixed } \rho_E} U^+(t) \right\}$$

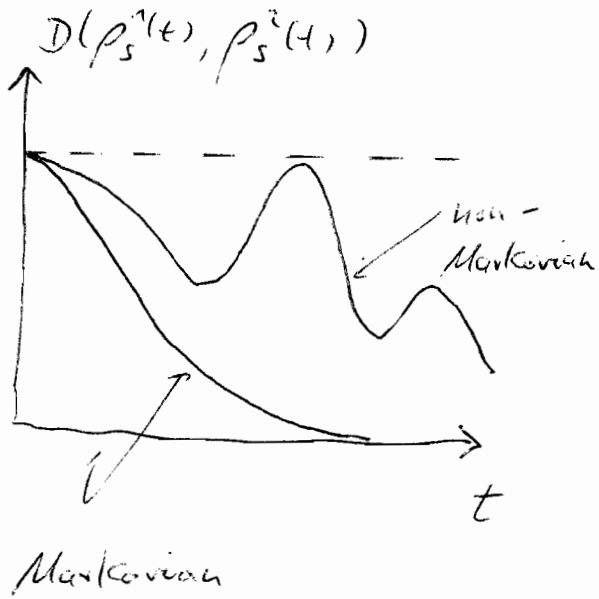
E. M. Laine, J. Piilo, HPB, EPL 92, 60010 (2010)

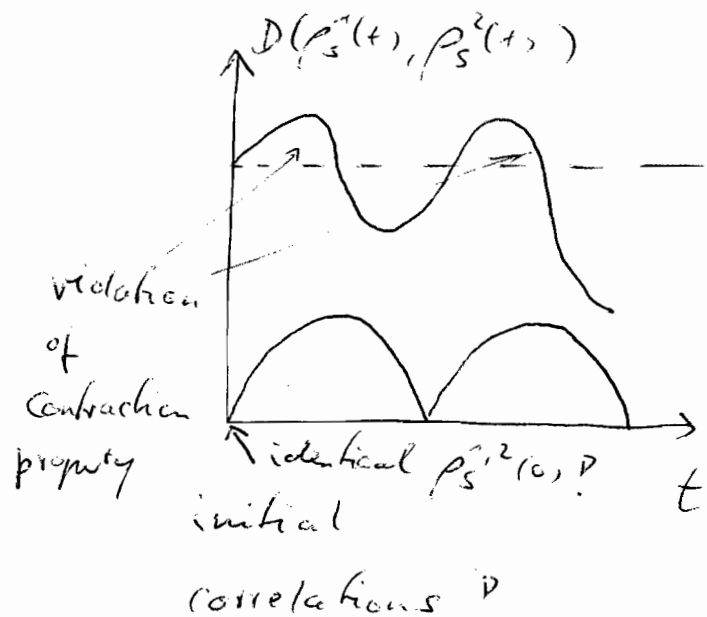
A. Smirne, HPB, J. Piilo, B. Vacchini, PRA 82, 062114 (2010)

$$\Phi_t : S(\mathcal{H}_S) \longrightarrow S(\mathcal{H}_S)$$

Trace distance evolution for pair of initial states:

$$\rho_S^{1,2}(t) = \Phi_t \rho_S^{1,2}(0)$$





- Relation to initial correlations
- Derivation of general upper bounds for increase of trace distance
- Interpretation in terms of information flow between system S and environment E

Consider dynamical map:

$$\Lambda_t : S(\mathcal{H}_S \otimes \mathcal{H}_E) \longrightarrow S(\mathcal{H}_S) \qquad \text{CPT map!}$$

$$\Lambda_t \rho_{SE}(0) = \mathrm{tr}_E \left\{ U(t) \rho_{SE}(0) U^+(t) \right\}$$

$$= \rho_S(t)$$

Initial pair of total system states:

$$\rho_S^{1,2}(t) = \Lambda_t \rho_{SE}^{1,2}(0)$$

From contraction property of CPT maps:

$$D(\rho_S^1(t), \rho_S^2(t)) \leq D(\rho_{SE}^1(0), \rho_{SE}^2(0))$$

→ The distinguishability of the open system states can never be larger than the distinguishability of the total system initial states.

Special case: uncorrelated states with identical environmental state:

$$\rho_{SE}^{1,2}(0) = \rho_S^{1,2}(0) \otimes \rho_E$$

$$\Rightarrow$$

$$D(\rho_S^1(t), \rho_S^2(t)) \leq D(\rho_S^1(0) \otimes \rho_E, \rho_S^2(0) \otimes \rho_E)$$

$$= D(\rho_S^1(0), \rho_S^2(0))$$

$\Rightarrow$ Contraction property of dyn. map Φ_t !

General case:

$$\underbrace{D(\rho_S^1(t), \rho_S^2(t)) - D(\rho_S^1(0), \rho_S^2(0))}_{\text{change of trace distance}} \leq \underbrace{D(\rho_{SE}^1(0), \rho_{SE}^2(0)) - D(\rho_S^1(0), \rho_S^2(0)}_{\equiv I(\rho_{SE}^1(0), \rho_{SE}^2(0))}$$

→ increase of trace distance of system states is bounded by the quantity I.

I = distinguishability of total system states minus distinguishability of open system states

= relative information on total system state which is "outside the open system", i.e., which is "inaccessible" for local measurements on open system

(also information loss, lack of information)

Note: If the bound is actually reached for some time t: initial distinguishability of total system states = distinguishability of open system states at time t

⇒ information on total initial states dynamically transferred completely to open system

How is the upper bound related to correlations in the initial state?

all states at time zero !

2 times triangular inequality !

$$D(\rho_{SE}^{\hat{1}}, \rho_{SE}^{2}) \leq D(\rho_{SE}^{\hat{1}}, \rho_S^{\hat{1}} \otimes \rho_E^{\hat{1}}) + D(\rho_{SE}^{2}, \rho_S^{\hat{1}} \otimes \rho_E^{\hat{1}})$$

$$\leq D(\rho_{SE}^{\hat{1}}, \rho_S^{\hat{1}} \otimes \rho_E^{\hat{1}}) + D(\rho_{SE}^{2}, \rho_S^{2} \otimes \rho_E^{2})$$

$$+ \underbrace{D(\rho_S^{\hat{1}} \otimes \rho_E^{\hat{1}}, \rho_S^{2} \otimes \rho_E^{2})}$$

$$\leq D(\rho_S^{\hat{1}}, \rho_S^{2}) + D(\rho_E^{\hat{1}}, \rho_E^{2})$$

Subadditivity w.r.t. tensor product

$\Rightarrow$

$$D(\rho_S^{\hat{1}}(t), \rho_S^{2}(t)) - D(\rho_S^{\hat{1}}(0), \rho_S^{2}(0))$$

$$\leq D(\rho_{SE}^{\hat{1}}(0), \rho_S^{\hat{1}}(0) \otimes \rho_E^{\hat{1}}(0)) + D(\rho_{SE}^{2}(0), \rho_S^{2}(0) \otimes \rho_E^{2}(0))$$

$$+ D(\rho_E^{\hat{1}}(0), \rho_E^{2}(0))$$

Increase of trace distance between open system states

$\Rightarrow$ correlations in initial state or different environmental state

Measure for correlations:

$D(\rho_{SE}, \rho_S \otimes \rho_E)$ = trace distance / distinguishability of ρ_{SE} and the corresponding product of its marginals

$\rho_S = tr_E \rho_{SE}$, $\rho_E = tr_S \rho_{SE}$

Special case: Initial pair:

$\rho_{SE}^1(0)$ and $\rho_{SE}^2(0) = \rho_S^1(0) \otimes \rho_E^1(0)$

$\Rightarrow$ $\Rightarrow$ $\rho_E^1(0) = \rho_E^2(0)$, $\rho_S^1(0) = \rho_S^2(0)$

$D(\rho_S^1(t), \rho_S^2(t)) \leq D(\rho_{SE}^1(0), \rho_S^1(0) \otimes \rho_E^1(0))$

$\rightsquigarrow$ increase of trace distance (initially zero) is bounded by the correlations in the initial state!

Possible <u>experimental strategy</u> to detect correlations in an unknown (!) initial state $\rho_{SE}(0)$:

- Take initial pair: $\rho_{SE}^{1}(0) = \rho_{SE}(0)$

$$\rho_{SE}^{2}(0) = (\Phi \otimes I)\rho_{SE}(0)$$

Φ local quantum operation

- Any dynamical increase of the trace distance of the reduced system state is then a <u>witness for initial correlations</u> in $\rho_{SE}(0)$ (unknown!):

$$D(\rho_S^1(t), \rho_S^2(t)) - D(\rho_S^1(0), \rho_S^2(0))$$

$$\leq D(\rho_{SE}^1(0), \rho_S^1(0) \otimes \rho_E^1(0)) + D(\rho_{SE}^2(0), \rho_S^2(0) \otimes \rho_E^2(0))$$

Since $\rho_E^1(0) = \rho_E^2(0)$.