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ENTANGLEMENT AT WORK
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ENTANGLEMENT AT WORK

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- **Lecture I: Mainly bipartite entanglement of Gaussian states**
- **Lecture II: Multipartite entanglement of Gaussian states, quantum information with Gaussian and non-Gaussian states**
- **Lecture III: Entanglement at work in quantum spin systems. Long-distance, modular, and hierarchical entanglement. Factorization, entanglement, and frustration in complex quantum many-body systems**

LECTURE II

Entanglement and quantum information with Gaussian and non-Gaussian states

Multipartite entanglement in Gaussian states:

Distributed entanglement, monogamy, and teleportation

- Structure of multipartite entanglement in CV systems. Monogamy relations for Gaussian states
- Multipartite entanglement at work: The optimal fidelity of CV teleportation networks

Monogamy

- State ϱ_{ABC} of three qubits
- The bipartite entanglement $E(\cdot|\cdot)$ quantified by the linear entropy (also known as the tangle) between, say, qubit A and the remaining two-qubits partition (BC) is never smaller than the sum of the A|B and A|C bipartite entanglements in the reduced states ϱ_{AB}, ϱ_{AC}:

$$E^{A|(BC)} \geq E^{A|B} + E^{A|C} \ . \tag{1}$$

- Monogamy extends to systems with an arbitrary number of qubits N.

Squared negativities as continuous-variable tangles.

- For qubits: Linear entropy (tangle) = concurrence. Analog in CV systems?
- Hint: Concurrence and negativity coincide on pure states (they both reduce to the entropy of entanglement)

Given an arbitrary pure state $|\psi\rangle$ of a N-mode CV system, define the CV tangle as the square of the logarithmic negativity:

$$E_\tau(\psi) \equiv \log^2 \|\tilde{\varrho}\|_1 \,, \quad \varrho = |\psi\rangle\langle\psi| \,. \tag{2}$$

Proper measure of bipartite entanglement, being a convex, increasing function of the logarithmic negativity $E_{\mathcal{N}}$.

Convex roof.

Def. (2) is naturally extended to generic mixed states ρ of N-mode CV systems by convex roof:

$$E_\tau(\rho) \equiv \inf_{\{p_i,\psi_i\}} \sum_i p_i E_\tau(\psi_i) \ . \tag{3}$$

Gaussian tangles.

We now consider the particular case of Gaussian states.

A) Pure States.

Multimode Gaussian state $|\psi\rangle$, with CM $\boldsymbol{\sigma}^p$, of $N+1$ with generic $1|(N-1)$ bipartition:

$$E_\tau(\boldsymbol{\sigma}^p) = \log^2\left(1/\mu_1 - \sqrt{1/\mu_1^2 - 1}\right) \ , \tag{4}$$

where $\mu_1 = 1/\sqrt{\text{Det}\,\boldsymbol{\sigma}_1}$ is the local purity of the reduced state of mode 1 with CM $\boldsymbol{\sigma}_1$.

The Gaussian contangle.

B) Mixed states.

Gaussian convex roof for multimode mixed Gaussian state with CM $\boldsymbol{\sigma}$. Consider only convex decompositions in terms of pure Gaussian states $\boldsymbol{\sigma}^p$. The infimum of the average contangle, taken over all pure Gaussian state decompositions, defines then the Gaussian continuous-variable tangle (*Gaussian contangle*) G_τ:

$$G_\tau(\boldsymbol{\sigma}) \equiv \inf_{\{\pi(d\boldsymbol{\sigma}^p),\boldsymbol{\sigma}^p\}} \int \pi(d\boldsymbol{\sigma}^p) E_\tau(\boldsymbol{\sigma}^p) \ . \tag{5}$$

It follows from the convex roof construction that the Gaussian contangle $G_\tau(\boldsymbol{\sigma})$ is an upper bound to the true contangle $E_\tau(\boldsymbol{\sigma})$ (minimized over *all* CV states),

$$E_\tau(\boldsymbol{\sigma}) \leq G_\tau(\boldsymbol{\sigma}) \ . \tag{6}$$

Computing the Gaussian contangle.

It can be shown that $G_\tau(\boldsymbol{\sigma})$ is a bipartite entanglement monotone under Gaussian LOCC. Therefore, for Gaussian states, the Gaussian contangle takes the simple form

$$G_\tau(\boldsymbol{\sigma}) = \inf_{\boldsymbol{\sigma}^p \le \boldsymbol{\sigma}} E_\tau(\boldsymbol{\sigma}^p) \,, \tag{7}$$

where the infimum runs over all pure Gaussian states with CM $\boldsymbol{\sigma}^p \le \boldsymbol{\sigma}$.

If $\boldsymbol{\sigma}_{i|j}$ is the CM of a (generally mixed) bipartite Gaussian state where subsystem $\mathcal{S}_i$ comprises one mode only, then the Gaussian contangle G_τ can be computed as

$$G_\tau(\boldsymbol{\sigma}_{i|j}) \equiv G_\tau(\boldsymbol{\sigma}^{opt}_{i|j}) = g[m^2_{i|j}], \quad g[x] = \text{arcsinh}^2[\sqrt{x-1}]. \tag{8}$$

Here $\boldsymbol{\sigma}^{opt}_{i|j}$ corresponds to a pure Gaussian state, and $m_{i|j} \equiv m(\boldsymbol{\sigma}^{opt}_{i|j}) = \sqrt{\text{Det}\, \boldsymbol{\sigma}^{opt}_i} = \sqrt{\text{Det}\, \boldsymbol{\sigma}^{opt}_j}$, with $\boldsymbol{\sigma}^{opt}_{i(j)}$ being the reduced CM of subsystem $\mathcal{S}_i$ ($\mathcal{S}_j$) obtained by tracing over the degrees of freedom of subsystem $\mathcal{S}_j$ ($\mathcal{S}_i$). The CM $\boldsymbol{\sigma}^{opt}_{i|j}$ denotes the pure bipartite Gaussian state which minimizes $m(\boldsymbol{\sigma}^{p}_{i|j})$ among all pure-state CMs $\boldsymbol{\sigma}^{p}_{i|j}$ such that $\boldsymbol{\sigma}^{p}_{i|j} \leq \boldsymbol{\sigma}_{i|j}$. For a separable state $m(\boldsymbol{\sigma}^{opt}_{i|j}) = 1$.

The Gaussian contangle G_τ is completely equivalent to the Gaussian entanglement of formation, which quantifies the cost of creating a given mixed Gaussian state out of an ensemble of pure, entangled Gaussian states.

The Gaussian tangle.

Finally, for a $1 \times (N-1)$ bipartition associated to a pure Gaussian state $\varrho^p_{A|B}$ with $\mathcal{S}_A = \mathcal{S}_1$ (a subsystem of a single mode) and $\mathcal{S}_B = \mathcal{S}_2 \ldots \mathcal{S}_N$, one can define the Gaussian tangle:

$$\tau_G(\varrho^p_{A|B}) = \mathcal{N}^2(\varrho^p_{A|B}). \tag{9}$$

Here, $\mathcal{N}(\varrho)$ is the negativity of the Gaussian state ϱ. The functional τ_G, like the negativity $\mathcal{N}$, vanishes on separable states and does not increase under LOCC, *i.e.* , it is a proper measure of pure-state bipartite entanglement. It can be naturally extended to mixed Gaussian states $\rho_{A|B}$ via the convex roof construction

$$\tau_G(\varrho_{A|B}) = \inf_{\{p_i, \varrho_i^{(p)}\}} \sum_i p_i \tau_G(\varrho^p_i), \tag{10}$$

where the infimum is taken over all convex decompositions of $\varrho_{A|B}$ in terms of pure

Gaussian states ϱ_i^p: $\rho_{A|B} = \sum_i p_i \varrho_i^p$. By virtue of the Gaussian convex roof construction, the Gaussian tangle τ_G Eq. (10) is an entanglement monotone under Gaussian LOCC.

Monogamy inequality for Gaussian states.

- Gaussian state distributed among N parties (each owning a single mode) $\mathcal{S}_k$ ($k = 1, \ldots, N$), and E is a proper measure of bipartite entanglement. Corresponding monogamy constraint:

$$E^{\mathcal{S}_i|(\mathcal{S}_1 \ldots \mathcal{S}_{i-1}\mathcal{S}_{i+1} \ldots \mathcal{S}_N)} \geq \sum_{j \neq i}^{N} E^{\mathcal{S}_i|\mathcal{S}_j} \tag{11}$$

The left-hand side of inequality (11) quantifies the bipartite entanglement between a probe subsystem $\mathcal{S}_i$ and the remaining subsystems taken as a whole. The right-hand side quantifies the total bipartite entanglement between $\mathcal{S}_i$ and each one of the other subsystems $\mathcal{S}_{j \neq i}$ in the respective reduced states.

Residual multipartite entanglement.

The non negative difference between these two entanglements, minimized over all choices of the probe subsystem, is referred to as the *residual multipartite entanglement*:

$$\tau_{res} \equiv E^{\mathcal{S}_i|(\mathcal{S}_1 \ldots \mathcal{S}_{i-1} \mathcal{S}_{i+1} \ldots \mathcal{S}_N)} - \sum_{j \neq i}^{N} E^{\mathcal{S}_i|\mathcal{S}_j} \,. \tag{12}$$

It quantifies the purely quantum correlations that are not encoded in pairwise form, so it includes all manifestations of genuine K-partite entanglement, involving K subsystems (modes) at a time, with $2 < K \leq N$.

Monogamy of the Gaussian contangle and Gaussian tangle.

It is known that:

- The Gaussian contangle (and the Gaussian tangle, as an implication) is monogamous in all 3-mode and in all fully symmetric N-mode Gaussian states.
- The Gaussian tangle satisfies inequality (11) in *all* Gaussian states.
- A full analytical proof of the monogamy inequality for the contangle in all Gaussian states beyond the symmetry, is currently lacking. Strong numerical evidence obtained for randomly generated non symmetric Gaussian states.

Some consequences.

- Monogamy helps in characterizing and quantifying multipartite entanglement.
- The monogamy constraints on entanglement sharing are essential for the security of CV quantum cryptographic schemes because they limit the information that might be extracted from the secret key by a malicious eavesdropper.
- Monogamy is useful as well in investigating the range of correlations in Gaussian matrix-product states of harmonic rings and in understanding the entanglement frustration occurring in ground states of many-body harmonic lattice systems, which may be now extended to arbitrary states beyond symmetry constraints.

Multipartite Gaussian entanglement: the simplest nontrivial case.

Simplest non-trivial case: Three-mode Gaussian states of CV systems with a 6×6 CM $\boldsymbol{\sigma} \equiv \boldsymbol{\sigma}_{123}$, and three 4×4 CMs $\boldsymbol{\sigma}_{ij}$ of the reduced two-mode Gaussian states of modes i and j. The fundamental symplectic objects are the local (two-mode) invariants Δ_{ij} and the three-mode (global) seralian $\Delta \equiv \Delta_{123}$, with the uncertainty relation for the reduced two-mode Gaussian states that reads

$$\Delta_{ij} - \mathrm{Det}\, \boldsymbol{\sigma}_{ij} \leq 1 \, . \tag{13}$$

Residual contangle as genuine tripartite entanglement.

The monogamy constraint leads naturally to the definition of the *residual contangle* as a quantifier of genuine tripartite entanglement in three-mode Gaussian states, much in the same way as in systems of three qubits. However, at variance with the three-qubit case (where the residual tangle of pure states is invariant under qubit permutations), here the residual contangle is partition-dependent according to the choice of the probe mode, with the obvious exception of the fully symmetric states. A *bona fide* quantification of tripartite entanglement is then provided by the *minimum* residual contangle

$$E_\tau^{i|j|k} \equiv \min_{(i,j,k)} \left[E_\tau^{i|(jk)} - E_\tau^{i|j} - E_\tau^{i|k} \right] \, , \tag{14}$$

where the symbol (i, j, k) denotes all the permutations of the three mode indexes. This definition ensures that $E_\tau^{i|j|k}$ is invariant under all permutations of the modes and is thus a genuine three-way property of any three-mode Gaussian state. We can adopt an analogous definition for the minimum residual Gaussian contangle G_τ^{res}, sometimes referred to as *arravogliament*:

$$G_\tau^{res} \equiv G_\tau^{i|j|k} \equiv \min_{(i,j,k)} \left[G_\tau^{i|(jk)} - G_\tau^{i|j} - G_\tau^{i|k} \right] . \tag{15}$$

One can verify that

$$(G_\tau^{i|(jk)} - G_\tau^{i|k}) - (G_\tau^{j|(ik)} - G_\tau^{j|k}) \geq 0 \tag{16}$$

if and only if $\mu_i \leq \mu_j$, and therefore the absolute minimum in Eq. (14) is attained by the decomposition realized with respect to the reference mode l of largest local purity μ_l, i.e. for the single-mode reduced state with CM of smallest determinant.

The residual (Gaussian) contangle, Eq. (15), is a proper measure of tripartite entanglement as it is non-increasing under (Gaussian) LOCC. In the CV setting we have proven that for pure three-mode Gaussian states G_τ^{res} is an entanglement monotone under tripartite Gaussian LOCC, and that it is non-increasing even under probabilistic operations, which is a stronger property than being only monotone on average. Therefore, the residual Gaussian contangle G_τ^{res} is a proper and computable measure of genuine multipartite (specifically, tripartite) entanglement in three-mode Gaussian states.

Equivalence between network teleportation fidelity and multipartite entanglement

Quantum teleportation using quadrature entanglement in continuous variable systems is in principle imperfect, due to the impossibility of achieving infinite squeezing.

Without using entanglement, by purely classical communication, an average fidelity of $\mathcal{F}_{cl} = 1/2$ is the best that can be achieved if the alphabet of input states includes all coherent states with even weight. The *fidelity*, which quantifies the success of a teleportation experiment, is defined as

$$\mathcal{F} \equiv \langle \psi^{in} | \varrho^{out} | \psi^{in} \rangle \;, \tag{17}$$

where "in" and "out" denote the input and the output state. $\mathcal{F}$ reaches unity only for a perfect state transfer, $\varrho^{out} = |\psi^{in}\rangle\langle\psi^{in}|$.

The sufficient entanglement criterion

To accomplish teleportation with high fidelity, the sender (Alice) and the receiver (Bob) must share an entangled state (resource). The sufficient fidelity criterion states that, if teleportation is performed with $\mathcal{F} > \mathcal{F}_{cl}$, then the two parties exploited an entangled state. The converse is generally false, i.e. some entangled resources may yield lower-than-classical fidelities.

The two-user CV teleportation protocol would require, to achieve unit fidelity, the sharing of an ideal (unnormalizable) Einstein-Podolski-Rosen (EPR) resource state, i.e. the eigenstate of relative position and total momentum of a two-mode radiation field. An arbitrarily good approximation of the EPR state is represented by two-mode squeezed Gaussian states with squeezing parameter $r \to \infty$.

Squeezed state resources for teleportation

A two-mode squeezed state can be produced by mixing a momentum-squeezed state and a position-squeezed state, with squeezing parameters r_1 and r_2 respectively, through a 50:50 ideal (lossless) beam splitter. In general, due to experimental imperfections and unavoidable thermal noise the two initial single-mode squeezed states will be mixed. One must then consider two thermal squeezed single-mode states, described by the following quadrature operators in Heisenberg picture:

$$\hat{x}_1^{sq} = \sqrt{n_1} e^{r_1} \hat{x}_1^0 \,, \quad \hat{p}_1^{sq} = \sqrt{n_1} e^{-r_1} \hat{p}_1^0 \,, \tag{18}$$

$$\hat{x}_2^{sq} = \sqrt{n_2} e^{-r_2} \hat{x}_2^0 \,, \quad \hat{p}_2^{sq} = \sqrt{n_2} e^{r_2} \hat{p}_2^0 \,, \tag{19}$$

where the suffix “0” refers to the vacuum.

Squeezing and entanglement

When applied to the two modes of Eqs. (18,19), the beam splitter entangling operation (with phase $\theta = \pi/4$) produces a symmetric mixed state, depending on the individual squeezings $r_{1,2}$ and on the thermal noises $n_{1,2}$. The noise can be difficult to control and reduce in the lab, but it is quantifiable. Keeping n_1 and n_2 fixed, all states produced starting with different r_1 and r_2, but with equal average $\bar{r} \equiv (r_1 + r_2)/2$, are completely equivalent up to local unitary operations and possess, by definition, the same entanglement. Let us recall that a two-mode Gaussian state is entangled if and only if it violates the positivity of partial transpose (PPT) condition $\eta \geq 1$ where η is the smallest symplectic eigenvalue of the partially transposed CM.

Smallest symplectic eigenvalue, entanglement, and fidelity

Review: The CM $\boldsymbol{\sigma}$ of a generic two-mode Gaussian state can be written in the block form $\boldsymbol{\sigma} = \begin{pmatrix} \boldsymbol{\alpha} & \boldsymbol{\gamma} \\ \boldsymbol{\gamma}^{\mathsf{T}} & \boldsymbol{\beta} \end{pmatrix}$, where $\boldsymbol{\alpha}$ and $\boldsymbol{\beta}$ are the CM's of the individual modes, while the matrix $\boldsymbol{\gamma}$ describes intermodal correlations. One then has $2\eta^2 = \Sigma(\boldsymbol{\sigma}) - \sqrt{\Sigma^2(\boldsymbol{\sigma}) - 4\text{Det}\,\boldsymbol{\sigma}}$, where $\Sigma(\boldsymbol{\sigma}) \equiv \text{Det}\,\boldsymbol{\alpha} + \text{Det}\,\boldsymbol{\beta} - 2\text{Det}\,\boldsymbol{\gamma}$. The parameter η provides the quantitative characterization of entanglement, because the logarithmic negativity and, equivalently for symmetric states ($\text{Det}\,\boldsymbol{\alpha} = \text{Det}\,\boldsymbol{\beta}$), the entanglement of formation E_F, are both monotonically decreasing functions of η.

Entanglement of formation

For symmetric Gaussian states the bipartite entanglement of formation E_F reads

$$E_F(\boldsymbol{\sigma}) = \max\{0,\ f(\eta)\}, \tag{20}$$

with

$$f(x) \equiv \frac{(1+x)^2}{4x} \log \frac{(1+x)^2}{4x} - \frac{(1-x)^2}{4x} \log \frac{(1-x)^2}{4x} \,. \tag{21}$$

For the mixed two-mode states considered here, we have

$$\eta = \sqrt{n_1 n_2} e^{-(r_1+r_2)} \,. \tag{22}$$

The entanglement thus depends both on the arithmetic mean of the individual squeezings, and on the geometric mean of the individual noises, which is related to the purity of the state $\mu = (n_1 n_2)^{-1}$. The teleportation success, instead, depends separately on each of the four single-mode parameters.

The necessary and sufficient entanglement criterion

The fidelity (averaged over the complex plane) for teleporting an unknown single-mode coherent state can be computed by writing the quadrature operators in Heisenberg picture:

$$\mathcal{F} \equiv \phi^{-1/2}, \;\; \phi = \left\{ \left[\langle (\hat{x}_{tel})^2 \rangle + 1 \right] \left[\langle (\hat{p}_{tel})^2 \rangle + 1 \right] \right\} /4 \,, \tag{23}$$

where $\langle (\hat{x}_{tel})^2 \rangle$ and $\langle (\hat{p}_{tel})^2 \rangle$ are the variances of the canonical operators $\hat{x}_{tel}$ and $\hat{p}_{tel}$ which describe the teleported mode. Parameterizing the single-mode squeezings r_1 and r_2 by $\bar{r}$ and the relative squeezing $d \equiv (r_1 - r_2)/2$, one finds:

$$\phi(\bar{r}, d, n_{1,2}) = e^{-4\bar{r}} (e^{2(\bar{r}+d)} + n_1)(e^{2(\bar{r}-d)} + n_2) \,. \tag{24}$$

Optimal fidelity of teleportation $\Longleftrightarrow$ entanglement (bipartite)

The fidelity is maximized, for given entanglement and noises of the Gaussian resource state (i.e. for fixed $n_{1,2}, \bar{r}$), by determining d^{opt} and inserting it in Eq. (24):

$$\mathcal{F}^{opt} = 1/(1+\eta)\,. \tag{25}$$

The optimal teleportation fidelity depends only on the entanglement of the resource state, and vice versa. The fidelity criterion becomes *necessary and sufficient* for the presence of the entanglement, if $\mathcal{F}^{opt}$ is considered: the optimal fidelity is classical for $\eta \geq 1$ (separable state) and larger than the classical threshold for any entangled state. Moreover, $\mathcal{F}^{opt}$ provides a quantitative measure of entanglement completely equivalent to the two-mode entanglement of formation: $E_F = \max\{0,\ f(1/\mathcal{F}^{opt} - 1)\}$. In the limit of infinite squeezing ($\bar{r} \to \infty$), $\mathcal{F}^{opt} \to 1$.

The N-user teleportation network

We now extend our analysis to a quantum teleportation-network protocol, involving N users who share a genuine N-partite entangled Gaussian resource, completely symmetric under permutations of the modes. Two parties are randomly chosen as sender (Alice) and receiver (Bob), but this time, in order to accomplish teleportation of an unknown coherent state, Bob needs the results of $N-2$ momentum detections performed by the other cooperating parties. A nonclassical teleportation fidelity (i.e. $\mathcal{F} > \mathcal{F}^{cl} = 1/2$) between *any* pair of parties is sufficient for the presence of genuine N-partite entanglement in the shared resource, while in general the converse is false.

The optimal network fidelity

We begin by considering a mixed momentum-squeezed state described by r_1, n_1 as in Eq. (18), and $N - 1$ position-squeezed states of the form Eq. (19). We then combine the N beams into an N-splitter. The resulting state is a completely symmetric mixed Gaussian state of a N-mode CV system, parameterized by $n_{1,2}$, $\bar{r}$ and d. One then chooses randomly two modes, denoted by the indices k and l, to be respectively the sender and the receiver. the teleported mode is described by the following quadrature operators: $\hat{x}_{tel} = \hat{x}_{in} - \hat{x}_{rel}$, $\hat{p}_{tel} = \hat{p}_{in} + \hat{p}_{tot}$, with $\hat{x}_{rel} = \hat{x}_k - \hat{x}_l$ and $\hat{p}_{tot} = \hat{p}_k + \hat{p}_l + g_N \sum_{j \neq k,l} \hat{p}_j$, where g_N is an experimentally adjustable gain.

The optimal network fidelity (continued)

To compute the teleportation fidelity from Eq. (23), we need the variances of $\hat{x}_{rel}$ and $\hat{p}_{tot}$. From the action of the N-splitter, one finds:

$$\begin{aligned} \langle(\hat{x}_{rel})^2\rangle &= 2n_2 e^{-2(\bar{r}-d)}\,, \\ \langle(\hat{p}_{tot})^2\rangle &= \big\{[2+(N-2)g_N]^2 n_1 e^{-2(\bar{r}+d)} \\ &\quad + 2[g_N-1]^2(N-2)n_2 e^{2(\bar{r}-d)}\big\}/4\,. \end{aligned} \tag{26}$$

Maximizing with respect to g_N (i.e. finding the optimal gain g_N^{opt}) and with respect to d (i.e. finding the optimal d_N^{opt}), one obtains the optimal teleportation-network fidelity, which can be put in the following general form for N modes:

$$\mathcal{F}_N^{opt} = \frac{1}{1+\eta_N}\,, \qquad \eta_N \equiv \sqrt{\frac{N n_1 n_2}{2e^{4\bar{r}} + (N-2)n_1/n_2}}\,. \tag{27}$$

Optimal network fidelity $\Longleftrightarrow$ multipartite entanglement

For $N = 2$, $\eta_2 = \eta$ from Eq. (22), showing that the general multipartite protocol comprises the standard bipartite case. By comparison with Eq. (25), we observe that, for any $N > 2$, the quantity η_N plays the role of a generalized symplectic eigenvalue. If the shared N-mode resources are prepared (or locally transformed) in the optimal form, the teleportation fidelity is guaranteed to be nonclassical as soon as $\bar{r} > 0$ for any N, in which case the considered class of pure states is genuinely multiparty entangled. Therefore *a nonclassical optimal fidelity is necessary and sufficient for the presence of multipartite entanglement in any multimode symmetric Gaussian state*, shared as a resource for CV teleportation.

The role of localizable entanglement

The teleportation network is realized in two steps: first, the $N-2$ cooperating parties perform local measurements on their modes, then Alice and Bob exploit their resulting highly entangled two-mode state to accomplish teleportation. Stopping at the first stage, the protocol describes a concentration, or *localization* of the original N-partite entanglement, into a bipartite two-mode entanglement. The maximum entanglement that can be concentrated on a pair of parties by locally measuring the others, is known as the *localizable entanglement* (LE) of a multiparty system. Here, the LE is the maximal entanglement that can be concentrated onto two modes, by unitary operations and non-unitary momentum detections performed locally on the other $N-2$ modes.

Localized multipartite entanglement: The symplectic eigenvalue η_N

The two-mode entanglement of the resulting state (described by a CM $\boldsymbol{\sigma}_{loc}$) is quantified in terms of the smallest symplectic eigenvalue η_{loc} of its partial transpose. The smallest symplectic eigenvalue η_{loc} is related to the EPR correlations by the expression $4\eta_{loc} = \langle(\hat{x}_{rel})^2\rangle + \langle(\hat{p}_{tot})^2\rangle$. Minimizing η_{loc} with respect to d means finding the optimal set of local unitary operations (not affecting multipartite entanglement) to be applied to the original multimode mixed resource described by $\{n_{1,2}, \bar{r}, d\}$; minimizing then η_{loc} with respect to g_N means finding the optimal set of momentum detections to be performed on the transformed state in order to localize the highest entanglement on a pair of modes. The resulting two-mode state contains a localized entanglement *exactly* quantified by $\eta_{loc}^{opt} = \eta_N$. The quantity η_N in Eq. (27) is thus the smallest symplectic eigenvalue of the partial transpose of the optimal two-mode state that can be extracted from a N-party entangled resource by local measurements on the remaining modes. Eq. (27) thus provides a bright connection between two *operative* aspects of multipartite entanglement in CV systems: the maximal fidelity achievable in a multi-user teleportation network, and the LE.

Entanglement of teleportation

These results yield quite naturally a direct operative way to quantify multipartite entanglement in N-mode (mixed) symmetric Gaussian states, in terms of the so-called *Entanglement of Teleportation*, defined as the normalized optimal fidelity

$$E_T \equiv \max\left\{0, \frac{\mathcal{F}_N^{opt} - \mathcal{F}_{cl}}{1 - \mathcal{F}_{cl}}\right\} = \max\left\{0, \frac{1-\eta_N}{1+\eta_N}\right\} . \tag{28}$$

It ranges from 0 (separable states) to 1 (CV GHZ state). The localizable entanglement of formation E_F^{loc} of N-mode symmetric Gaussian states is a monotonically increasing function of E_T, namely: $E_F^{loc} = f[(1-E_T)/(1+E_T)]$, with $f(x)$ defined after Eq. (20). For $N = 2$ the state is already localized and $E_F^{loc} = E_F$.

Remarkably for three-mode pure (symmetric) Gaussian states, the residual contangle E_τ, the tripartite entanglement monotone under Gaussian LOCC that quantifies

CV entanglement sharing, is also a monotonically increasing function of E_T, thus providing another *equivalent* quantitative characterization of genuine tripartite CV entanglement:

$$E_\tau = \log^2 \frac{2\sqrt{2}E_T - (E_T+1)\sqrt{E_T^2+1}}{(E_T-1)\sqrt{E_T(E_T+4)+1}} - \frac{1}{2}\log^2 \frac{E_T^2+1}{E_T(E_T+4)+1}\,. \tag{29}$$

This finding lends itself immediately to experimental verification, in terms of optimal fidelities in teleportation networks, to verify the sharing properties of tripartite CV entanglement in multi-party Gaussian states.