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ENTANGLEMENT AT WORK

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- **Lecture I: Mainly bipartite entanglement of Gaussian states**
- **Lecture II: Multipartite entanglement of Gaussian states, teleportation with continuous variables, quantum information with non-Gaussian states**
- **Lecture III: Entanglement at work in quantum spin systems. Long-distance entanglement and teleportation, ground-state factorization, hierarchical entanglement and collective phenomena, role of entanglement in the understanding of frustrated complex quantum many-body systems**

Entanglement in Continuous Variable Systems (mainly Gaussian States of Light)

Introduction to continuous variable systems

A continuous variable (CV) system of N canonical bosonic modes is described by a Hilbert space $\mathcal{H} = \bigotimes_{k=1}^{N} \mathcal{H}_k$, the tensor product of infinite dimensional Fock spaces $\mathcal{H}_k$'s, each of them associated to a single mode. Free Hamiltonian of an arbitrary number N of harmonic oscillators of different frequencies (*modes* of the field):

$$\hat{H} = \sum_{k=1}^{N} \hbar\omega_k \left(\hat{a}_k^\dagger \hat{a}_k + \frac{1}{2} \right) . \tag{1}$$

Bosonic commutation relations:

$$\left[\hat{a}_k, \hat{a}^\dagger_{k'}\right] = \delta_{kk'}\,, \quad [\hat{a}^{,}_k \hat{a}_{k'}] = \left[\hat{a}^\dagger_k, \hat{a}^\dagger_{k'}\right] = 0\,. \tag{2}$$

(natural units with $\hbar = 2$.) The corresponding quadrature phase operators (position and momentum) for each mode are:

$$\hat{q}_k = (\hat{a}_k + \hat{a}^\dagger_k)\,, \tag{3}$$

$$\hat{p}_k = (\hat{a}_k - \hat{a}^\dagger_k)/i\,. \tag{4}$$

Vector form of the canonical operators:

$$\hat{R} = (\hat{q}_1, \hat{p}_1, \ldots, \hat{q}_N, \hat{p}_N)^{\mathsf{T}} , \tag{5}$$

enables the compact-form commutation relations:

$$[\hat{R}_k, \hat{R}_l] = 2i\Omega_{kl} \ . \tag{6}$$

Here Ω is the symplectic form:

$$\Omega = \bigoplus_{k=1}^{N} \omega \, , \quad \omega = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \ . \tag{7}$$

The space $\mathcal{H}_k$ is spanned by the Fock basis $\{|n\rangle_k\}$: Eigenstates of the number operator $\hat{n}_k = \hat{a}_k^\dagger \hat{a}_k$.

For each mode k there exists a different vacuum state $|0\rangle_k \in \mathcal{H}_k$ such that $\hat{a}_k|0\rangle_k = 0$.

The vacuum state of the global Hilbert space will be denoted by $|0\rangle = \bigotimes_k |0\rangle_k$.

In the single-mode Hilbert space $\mathcal{H}_k$, the eigenstates of $\hat{a}_k$ constitute the important set of *coherent states*, which is over-complete in $\mathcal{H}_k$.

Coherent States

Defined by applying the Weyl displacement operator $\hat{D}_k$ to the vacuum $|0\rangle_k$, $|\alpha\rangle_k = \hat{D}_k(\alpha)|0\rangle_k$, where

$$\hat{D}_k(\alpha) = \mathrm{e}^{\alpha\hat{a}_k^\dagger - \alpha^*\hat{a}_k}\,, \tag{8}$$

and for $\alpha \in \mathbb{C}$ it satisfies $\hat{a}_k|\alpha\rangle_k = \alpha|\alpha\rangle_k$. In the Fock basis of mode k it reads

$$|\alpha\rangle_k = \mathrm{e}^{-\frac{1}{2}|\alpha|^2} \sum_{n=1}^{\infty} \frac{\alpha^n}{\sqrt{n!}} |n\rangle_k\,. \tag{9}$$

Tensor products of coherent states for N modes are obtained by applying the N-mode Weyl operators $\hat{D}_\xi$ to the global vacuum $|0\rangle$, where

$$\hat{D}_\xi = \mathrm{e}^{i\hat{R}^\mathsf{T}\Omega\xi}\,, \quad \text{with} \quad \xi \in \mathbb{R}^{2N}\,. \tag{10}$$

One then has $|\xi\rangle = \hat{D}_\xi|0\rangle$.

Quantum phase-space picture - Characteristic functions

Quantum states: positive trace-class operators $\{\varrho\}$ [i.e. compact (bounded and continuous) and with a trace that is always defined and finite] on the Hilbert space $\mathcal{H} = \bigotimes_{k=1}^{N} \mathcal{H}_k$. Alternative and fully equivalent description: s-ordered *characteristic functions*

$$\chi_s(\xi) = \mathrm{Tr}\left[\varrho \hat{D}_\xi\right] \mathrm{e}^{s\|\xi\|^2/2} \,, \tag{11}$$

with $\xi \in \mathbb{R}^{2N}$, $\|\,\cdot\,\|$ standing for the Euclidean norm of $\mathbb{R}^{2N}$. The vector ξ belongs to the real $2N$-dimensional space $\Gamma = (\mathbb{R}^{2N}, \Omega)$, which is called *phase space*, in analogy with classical Hamiltonian dynamics.

Phase space: Quasi-probability distributions

In phase space the tensor product structure is replaced by a direct sum structure, so that the N-mode phase space $\Gamma = \bigoplus_k \Gamma_k$, where $\Gamma_k = (\mathbb{R}^2, \omega)$ is the local phase space associated with mode k.

The family of characteristic functions is in turn related, via complex Fourier transform, to the *quasi-probability distributions* W_s, which constitute another set of complete descriptions of the quantum states

$$W_s(\xi) = \frac{1}{\pi^2} \int_{\mathbb{R}^{2N}} \kappa \chi_s(\kappa)\, \mathrm{e}^{i\kappa^{\mathsf{T}}\Omega\xi}\, \mathrm{d}^{2N}\,. \tag{12}$$

Phase space: Quasi-probability distributions (continued)

The value $s = -1$ corresponds to the Husimi 'Q-function': $Q \equiv W_{-1}(\xi) = \langle\xi|\varrho|\xi\rangle/\pi$, i.e. the regular probability distribution for state ϱ to be found in the coherent state $|\xi\rangle$. The case $s = 0$ corresponds to the Wigner function W: $W \equiv W_0$. Likewise, for the sake of simplicity, χ will stand for the symmetrically ordered characteristic function χ_0. Finally, the case $s = 1$ yields the singular P-representation introduced, independently, by Glauber and Sudarshan.

Phase-space distributions (continued)

The quasi-probability distributions of integer order W_{-1}, W_0 and W_1 are respectively associated the anti-normally ordered, symmetrically ordered and normally ordered expressions of operators. More precisely, if the operator $\hat{O}$ can be expressed as $\hat{O} = f(\hat{a}_k, \hat{a}_k^\dagger)$ for $k = 1, \ldots, N$, where f is, say, symmetrically ordered function of the field operators, then one has

$$\mathrm{Tr}[\varrho \hat{O}] = \int_{\mathbb{R}^{2N}} W_0(\kappa) \bar{f}(\kappa) \, \mathrm{d}^{2N}\kappa \,,$$

where $\bar{f}(\kappa) = f(\kappa_k + i\kappa_{k+1}, \kappa_k - i\kappa_{k+1})$ and f takes the same form as the operator function previously introduced.

Quantum phase-space vs. Hilbert-space picture - Synopsis

	Hilbert space $\mathcal{H}$	Phase space Γ
dimension	∞	$2N$
structure	$\bigotimes$	$\bigoplus$
description	ϱ	χ_s, W_s

Schematic comparison between Hilbert-space and phase-space pictures for N-mode continuous variable systems.

Mathematical description of Gaussian states.

The set of *Gaussian states* is, by definition, the set of states with Gaussian characteristic functions and quasi-probability distributions on the multimode quantum phase space. Gaussian states include, among others, coherent, squeezed, and thermal states. Therefore they are of central importance in quantum optics and in quantum information and quantum communication with CV systems. Their entanglement properties will thus be one of the main subjects of this set of lectures.

Covariance matrix formalism.

From the definition it follows that a Gaussian state ϱ is completely characterized by the first and second statistical moments of the quadrature field operators, which will be denoted, respectively, by the vector of first moments $\bar{R} = \left(\langle \hat{R}_1 \rangle, \langle \hat{R}_1 \rangle, \ldots, \langle \hat{R}_N \rangle, \langle \hat{R}_n \rangle\right)$ and by the covariance matrix (CM) $\boldsymbol{\sigma}$ of elements

$$\boldsymbol{\sigma}_{ij} = \frac{1}{2}\langle \hat{R}_i \hat{R}_j + \hat{R}_j \hat{R}_i \rangle - \langle \hat{R}_i \rangle \langle \hat{R}_j \rangle \ . \tag{13}$$

First moments can be arbitrarily adjusted by local unitary operations.

Covariance matrix formalism - the fundamental physical constraints.

The CM $\boldsymbol{\sigma}$ contains the complete, locally-invariant, information on a Gaussian state: therefore the requirements that the associated ϱ must be positive and that the canonical commutation relations hold translate on the following constraint on the CM:

$$\boldsymbol{\sigma} + i\Omega \geq 0 \ , \tag{14}$$

Ineq. (14) is the necessary and sufficient relation that the matrix $\boldsymbol{\sigma}$ has to fulfill to be the CM corresponding of a physical Gaussian state. Moreover, Ineq. (14) is the necessary condition for the CM of all CV states (characterized in principle by the moments of any order). The constraint implies $\boldsymbol{\sigma} \geq 0$. Ineq. (14) is also the expression of the uncertainty principle.

The two-mode squeezed state and the EPR state.

The acclaimed representative of two-mode Gaussian states: the *two-mode squeezed state* $|\psi^{sq}\rangle_{i,j} = \hat{U}_{i,j}(r)\,(|0\rangle_i \otimes |0\rangle_j)$ with squeezing factor $r \in \mathbb{R}$, where the (phase-free) two-mode squeezing operator is

$$\hat{U}_{i,j}(r) = \exp\left[-\frac{r}{2}(\hat{a}_i^\dagger \hat{a}_j^\dagger - \hat{a}_i \hat{a}_j)\right] , \tag{15}$$

In the limit of infinite squeezing ($r \longrightarrow \infty$), the state approaches the ideal Einstein-Podolsky-Rosen (EPR) state, simultaneous eigenstate of total momentum and relative position of the two subsystems, which thus share *infinite entanglement*.

Squeezing and teleportation.

Quantum teleportation of an *unknown coherent state* in the Braunstein-Kimble-Vaidman protocol using the EPR state as infinitely entangled resource is realized with absolute, *i.e.* unit fidelity. But: The EPR state is un-normalizable and unphysical. However, in principle, an EPR state can be approximated with an arbitrarily high degree of accuracy by two-mode squeezed states with sufficiently large squeezing.

Therefore, two-mode squeezed states are of key importance as entangled resources for practical implementations of CV quantum information protocols. They play a central role in the study of the entanglement properties of Gaussian states.

Symplectic operations.

Unitary operations which preserve the Gaussian character of the states on which they act. They are all those generated by Hamiltonian terms at most quadratic in the field operators. As a consequence of the Stone-Von Neumann theorem, the so-called *metaplectic* representation entails that any such unitary operation at the Hilbert space level corresponds, in phase space, to a symplectic transformation, *i.e.* to a linear transformation S which preserves the symplectic form Ω:

$$S^{\mathsf{T}}\Omega S = \Omega\,. \tag{16}$$

Important examples in quantum optics and quantum information with CV systems: Ideal beam splitters, phase shifters, and squeezers.

Example: the beam splitter.

A common unitary operation is the ideal (phase-free) *beam splitter*, whose action $\hat{B}_{i,j}$ on a pair of modes i and j is defined as

$$\hat{B}_{i,j}(\theta) : \begin{cases} \hat{a}_i \longrightarrow \hat{a}_i \cos\theta + \hat{a}_j \sin\theta \\ \hat{a}_j \longrightarrow \hat{a}_i \sin\theta - \hat{a}_j \cos\theta \end{cases} , \tag{17}$$

with $\hat{a}_l$ being the annihilation operator of mode k. A beam splitter with transmittivity τ corresponds to a rotation of $\theta = \arccos\sqrt{\tau}$ in phase space ($\theta = \pi/4$ corresponds to a balanced 50:50 beam splitter, $\tau = 1/2$).

Symplectics of beam splitters (continued).

Beam splitter operations are described by the symplectic transformation

$$B_{i,j}(\tau) = \begin{pmatrix} \sqrt{\tau} & 0 & \sqrt{1-\tau} & 0 \\ 0 & \sqrt{\tau} & 0 & \sqrt{1-\tau} \\ \sqrt{1-\tau} & 0 & -\sqrt{\tau} & 0 \\ 0 & \sqrt{1-\tau} & 0 & -\sqrt{\tau} \end{pmatrix} . \tag{18}$$

Single-mode symplectic operations are easily introduced as linear combinations of planar (orthogonal) rotations and of single-mode squeezings of the form

$$S_j(r) = \mathrm{diag}\,(\,\mathrm{e}^{r},\,\mathrm{e}^{-r}\,)\,, \tag{19}$$

acting on mode j, for $r > 0$.

Active and passive operations.

The beam splitters $B_{ij}(\tau)$, Eq. (18), preserve the value of $\mathrm{Tr}\,\boldsymbol{\sigma}$. Since $\mathrm{Tr}\,\boldsymbol{\sigma}$ gives the contribution of the second moments to the average of the Hamiltonian $\bigoplus_k \hat{a}_k^\dagger \hat{a}_k$, these transformations are said to be *passive* (they belong to the compact subgroup of $Sp_{(2N,\mathbb{R})}$). Instead, the squeezers $U_{i,j}(r)$, Eq. (15), do not preserve $\mathrm{Tr}\,\boldsymbol{\sigma}$ (they belong to the non compact subgroup of $Sp_{(2N,\mathbb{R})}$). This mathematical difference between squeezers and phase-space rotations accounts, in a quite elegant way, for the difference between *active* (energy non preserving) and passive (energy preserving) optical transformations.

Diagonalization and symplectic eigenvalues.

Diagonalization of a Gaussian state in the basis of normal modes: symplectic transformation. the CM of a N-mode Gaussian state can always be written in the diagonal form

$$\boldsymbol{\sigma} = S^{\mathsf{T}} \boldsymbol{\nu} S \ , \tag{20}$$

where $S \in Sp_{(2N,\mathbb{R})}$ and $\boldsymbol{\nu}$ is the CM

$$\boldsymbol{\nu} = \bigoplus_{k=1}^{N} \begin{pmatrix} \nu_k & 0 \\ 0 & \nu_k \end{pmatrix} \ , \tag{21}$$

corresponding to a tensor product state with a diagonal density matrix

$$\varrho^{\otimes} = \bigotimes_{k} \frac{2}{\nu_k + 1} \sum_{n=0}^{\infty} \left(\frac{\nu_k - 1}{\nu_k + 1} \right) |n\rangle_{kk}\langle n| \ . \tag{22}$$

Symplectic invariants.

The N quantities ν_k's form the *symplectic spectrum* of the CM $\boldsymbol{\sigma}$, and are invariant under the action of global symplectic transformations on the CM $\boldsymbol{\sigma}$. The symplectic eigenvalues can be computed as the orthogonal eigenvalues of the matrix $|i\Omega\boldsymbol{\sigma}|$ and are thus determined by N invariants of the characteristic polynomial of such a matrix.

First global symplectic invariant: The determinant of the CM (whose invariance is a consequence of the fact that $\mathrm{Det}\, S = 1\ \forall S \in Sp_{(2N,\mathbb{R})}$). Computed in the diagonal form, it reads

$$\mathrm{Det}\,\boldsymbol{\sigma} = \prod_{k=1}^{N} \nu_k^2 \,. \tag{23}$$

Symplectic invariants (continued).

Another important invariant under global symplectic operations is the sum of the determinants of all 2×2 sub-matrices of a CM $\boldsymbol{\sigma}$, which can be readily computed in terms of its symplectic eigenvalues as

$$\Delta(\boldsymbol{\sigma}) = \sum_{k=1}^{N} \nu_k^2 \, . \tag{24}$$

The invariance of $\Delta(\boldsymbol{\sigma})$ in the multimode case follows from its invariance in the case of two-mode states and from the fact that any symplectic transformation can be decomposed as the product of two-mode transformations.

Synopsis.

	Hilbert space $\mathcal{H}$	Phase space Γ
dimension	∞	$2N$
structure	$\bigotimes$	$\bigoplus$
description	ϱ	$\boldsymbol{\sigma}$
bona fide	$\varrho \geq 0$	$\boldsymbol{\sigma} + i\Omega \geq 0$
operations	$U : U^\dagger U = I$ $\varrho \mapsto U\varrho U^\dagger$	$S : S^\mathsf{T}\Omega S = \Omega$ $\boldsymbol{\sigma} \mapsto S\boldsymbol{\sigma}S^\mathsf{T}$
spectra	$U\varrho U^\dagger = \mathrm{diag}\{\lambda_k\}$ $0 \leq \lambda_k \leq 1$	$S\boldsymbol{\sigma}S^\mathsf{T} = \mathrm{diag}\{\nu_k\}$ $1 \leq \nu_k < \infty$
pure states	$\lambda_i = 1,\ \lambda_{j\neq i} = 0$	$\nu_j = 1,\ \forall j = 1 \ldots N$
purity	$\mathrm{Tr}\,\varrho^2 = \sum_k \lambda_k^2$	$1/\sqrt{\mathrm{Det}\,\boldsymbol{\sigma}} = \prod_k \nu_k^{-1}$

Comparison between Hilbert-space and phase-space representations for N-mode CV states. The first two rows apply to general states. The remaining ones are special to Gaussian states (CM description and symplectic group).

Degree of information contained in a Gaussian state - Measures of information.

The degree of *information* contained in a quantum state corresponds to the amount of knowledge that we possess *a priori* on predicting the outcome of any test performed on the state.

The simplest measure of such information is the *purity* of a quantum state ϱ:

$$\mu(\varrho) = \mathrm{Tr}\, \varrho^2 \,. \tag{25}$$

For states belonging to a given Hilbert space $\mathcal{H}$ with $\dim \mathcal{H} = D$, the purity varies in the range

$$\frac{1}{D} \leq \mu \leq 1 \,.$$

Measures of information (continued).

The minimum is reached by the totally random mixture; the upper bound is saturated by pure states. In the limit of CV systems ($D \to \infty$), the minimum purity tends asymptotically to zero. Accordingly, the "impurity" or degree of *mixedness* of a quantum state ϱ, which characterizes our ignorance before performing any quantum test on ϱ, can be quantified by the functional

$$S_L(\varrho) = \frac{D}{D-1}(1-\mu) = \frac{D}{D-1}\left(1 - \mathrm{Tr}\,\varrho^2\right) . \tag{26}$$

The *linear entropy* S_L (ranging between 0 and 1) defined by Eq. (26) is a very useful measure of mixedness in quantum mechanics and quantum information theory due to its direct connection with the purity and its computational simplicity.

Purity and von Neumann entropy.

The *von Neumann entropy*

$$S_V(\varrho) = -\mathrm{Tr}\, \varrho \log \varrho \tag{27}$$

is *subadditive*. Bipartite system $\mathcal{S}$ (Hilbert space $\mathcal{H} = \mathcal{H}_1 \otimes \mathcal{H}_2$) in the state ϱ. Then

$$S_V(\varrho) \leq S_V(\varrho_1) + S_V(\varrho_2)\,, \tag{28}$$

where $\varrho_{1,2}$ are the reduced density matrices $\varrho_{1,2} = \mathrm{Tr}_{2,1}\, \varrho$ associated to subsystems $\mathcal{S}_{1,2}$. The von Neumann entropy is *additive* on tensor product states:

$$S_V(\varrho_1 \otimes \varrho_2) = S_V(\varrho_1) + S_V(\varrho_2)\,. \tag{29}$$

The purity, Eq. (25), is instead *multiplicative* on product states, as the trace of a product equates the product of the traces:

$$\mu(\varrho_1 \otimes \varrho_2) = \mu(\varrho_1) \cdot \mu(\varrho_2)\,. \tag{30}$$

Classical and quantum information.

Considering a joint classical probability distribution over the variables X and Y, one has for the Shannon entropy,

$$S(X,Y) \geq S(X),\, S(Y)\,. \tag{31}$$

The Shannon entropy of a joint probability distribution is always greater than the Shannon entropy of each marginal probability distribution, meaning that there is less information in a global classical system than in any of its parts.

Classical and quantum information (continued).

Bipartite quantum system in a pure state: $\varrho = |\psi\rangle\langle\psi|$. We have then for the von Neumann entropies: $S_V(\varrho) = 0$, $S_V(\varrho_1) = S_V(\varrho_2) \geq 0$. It is impossible to reconstruct the complete information about how the global system was prepared in the state ϱ (apart from the trivial instance of ϱ being a product state $\varrho = \varrho_1 \otimes \varrho_2$), by looking separately at the two subsystems. Information is rather encoded in nonlocal and non-factorizable quantum correlations – **entanglement** – between the two subsystems. This clearly highlights the fundamental difference between classical and quantum distributions of information.

Entropic measures for Gaussian states.

Purity of a Gaussian state:

$$\mu(\varrho) = \frac{1}{\prod_i \nu_i} = \frac{1}{\sqrt{\mathrm{Det}\,\boldsymbol{\sigma}}}\,. \tag{32}$$

It depends only on the global symplectic invariant Det $\boldsymbol{\sigma}$ Eq. (23). The purity is related to the linear entropy S_L via Eq. (26), which in CV systems simply becomes $S_L = 1-\mu$.

The von Neumann entropy of a Gaussian state:

$$S_V(\varrho) = \sum_{i=1}^{N} f(\nu_i)\ , \tag{33}$$

where

$$f(x) \equiv \frac{x+1}{2}\log\left(\frac{x+1}{2}\right) - \frac{x-1}{2}\log\left(\frac{x-1}{2}\right)\ . \tag{34}$$

Notice that S_V diverges on maximally mixed CV states, while S_L is normalized to 1.

Entanglement of pure states.

It is by now well understood that entanglement in a pure bipartite quantum state $\varrho = |\psi\rangle\langle\psi|$ is equivalent to the degree of mixedness of each subsystem. Accordingly, it is properly quantified by the *entropy of entanglement* $E_V(|\psi\rangle)$, defined as the von Neumann entropy, Eq. (27), of the reduced density matrices,

$$E_V(|\psi\rangle) = S_V(\varrho_1) = S_V(\varrho_2) = -\sum_{k=1}^{d} \lambda_k^2 \log \lambda_k^2 \,. \tag{35}$$

The entropy of entanglement is by definition invariant under local unitary operations

$$E_V\Big((\hat{U}_1 \otimes \hat{U}_2)|\psi\rangle\Big) = E_V\Big(|\psi\rangle\Big) \,. \tag{36}$$

Local Operations and Classical Communication (LOCC).

It can be shown that $E_V(|\psi\rangle)$ cannot increase under LOCC (Local operations and classical communication) performed on the state $|\psi\rangle$: Suppose one starts with a state $|\psi\rangle$ of the global system $\mathcal{S}$, to perform local measurements on $\mathcal{S}_1$ and $\mathcal{S}_2$, and to obtain, after the measurement, the state $|\varphi_1\rangle$ with probability p_1, the state $|\varphi_2\rangle$ with probability p_2, and so on. Then

$$E_V(|\psi\rangle) \geq \sum_k p_k E_V(|\varphi_k\rangle)\,. \tag{37}$$

Note that entanglement cannot increase *on average*, that is nothing prevents, for a given k, that $E_V(|\varphi_k\rangle) > E_V(|\psi\rangle)$. The concept of *entanglement distillation* is based on this fact: with a probability p_k, it is possible to increase entanglement via LOCC manipulations.

Entanglement of mixed states

A mixed state ϱ can be decomposed as a convex combination of pure states,

$$\varrho = \sum_k p_k |\psi_k\rangle\langle\psi_k| \,. \tag{38}$$

Eq. (38) tells us how to create the state described by the density matrix ϱ: we have to prepare the state $|\psi_1\rangle$ with probability p_1, the state $|\psi_2\rangle$ with probability p_2, etc. For instance, we could collect N copies ($N \gg 1$) of the system, prepare $n_k \simeq N p_k$ of them in the state $|\psi_k\rangle$, and pick a copy at random.

A mixed state is separable (unentangled) if there exists at least one decomposition that is a convex combination of product states:

$$\varrho = \sum_k p_k \varrho_k^{(1)} \otimes \varrho_k^{(2)} \,. \tag{39}$$

Separability criteria

The difficulty in checking whether a mixed state is entangled or separable lies in the fact that the decomposition of Eq. (38) is not unique: There exist infinitely many decompositions of a generic ϱ, meaning that mixed states can be prepared in infinitely many different ways. Deciding separability according to the above definition would imply checking all the infinitely many decompositions of a state ϱ and looking for the existence of at least one, expressed as a convex combination of product states, Eq. (39), to conclude that the state is not entangled. This is clearly impractical. For this reason, several *operational* criteria have been developed in order to detect entanglement in mixed quantum states. Some of them, of special relevance to Gaussian states of CV systems, are discussed in the following.

Separability and distillability: The PPT criterion.

It can be shown that if a state ϱ_s is separable, then its partial transpose $\varrho_s^{\mathsf{T}_1}$ (with respect *e.g.* to subsystem $\mathcal{S}_1$) is a valid density matrix, in particular positive semidefinite, $\varrho_s^{\mathsf{T}_1} \geq 0$. Obviously, the same holds for $\varrho_s^{\mathsf{T}_2}$. Positivity of the partial transpose (PPT) is therefore a necessary condition for separability. The converse (*i.e.* $\varrho^{\mathsf{T}_1} \geq 0 \Rightarrow \varrho$ separable) is in general false: PPT entangled states (with $\varrho^{\mathsf{T}_1} \geq 0$) have been shown to exist. These states are known as *bound entangled*, because their entanglement cannot be distilled to obtain maximally entangled states. The existence of bound entangled (undistillable) states with negative partial transposition has been conjectured, but at present there is not yet evidence of this property.

The PPT criterion for Gaussian states.

The PPT criterion is necessary and sufficient for the separability of all $(1 \times N)$-mode Gaussian states and all $M \times N$ bisymmetric Gaussian states.

The PPT criterion yields that a Gaussian state $\boldsymbol{\sigma}_{A|B}$ (with $N_A = 1$ and N_B arbitrary) is *separable* if and only if the partially transposed $\tilde{\boldsymbol{\sigma}}_{A|B}$ is a *bona fide* CM, that is it satisfies the uncertainty principle Eq. (14),

$$\tilde{\boldsymbol{\sigma}}_{A|B} + i\Omega \geq 0\,. \tag{40}$$

This property in turn reflects the positivity of the partially transposed density matrix ϱ^{T_A} associated to the state ϱ.

The PPT criterion: Symplectic representation.

The PPT criterion has a powerful symplectic representation. The partially transposed matrix $\tilde{\boldsymbol{\sigma}}$ of any N-mode Gaussian CM is still a positive and symmetric matrix. As such, it admits a diagonal normal-mode decomposition, Eq. (20), of the form

$$\tilde{\boldsymbol{\sigma}} = S^{\mathsf{T}} \tilde{\boldsymbol{\nu}} S \ , \tag{41}$$

where $S \in Sp_{(2N,\mathbb{R})}$ and $\tilde{\boldsymbol{\nu}}$ is the CM

$$\tilde{\boldsymbol{\nu}} = \bigoplus_{k=1}^{N} \begin{pmatrix} \tilde{\nu}_k & 0 \\ 0 & \tilde{\nu}_k \end{pmatrix} \ , \tag{42}$$

The $\tilde{\nu}_k$'s are the N symplectic eigenvalues of the partially transposed CM $\tilde{\boldsymbol{\sigma}}$.

Symplectic spectra and entanglement.

The symplectic spectrum $\{\nu_k\}$ of $\boldsymbol{\sigma}$ encodes the structural and informational properties of a Gaussian state. The partially transposed spectrum $\{\tilde{\nu}_k\}$ encodes the characterization of entanglement in the state. In terms of the latter, the PPT condition (40) can be equivalently recast in the form

$$\tilde{\nu}_k \geq 1 \, . \tag{43}$$

We can rearrange the modes of a N-mode state such that the corresponding symplectic eigenvalues of the partial transpose $\tilde{\boldsymbol{\sigma}}$ are sorted in ascending order

$$\tilde{\nu}_- \equiv \tilde{\nu}_1 \leq \tilde{\nu}_2 \leq \ldots \leq \tilde{\nu}_{N-1} \leq \tilde{\nu}_N \equiv \tilde{\nu}_+ \, .$$

Then the PPT criterion across an arbitrary bipartition becomes $\tilde{\nu}_1 \geq 1$ for all separable Gaussian states. If $\tilde{\nu}_1 < 1$, the corresponding Gaussian state $\boldsymbol{\sigma}$ is entangled.

Synopsis.

	Physical	Separable
density matrix	$\varrho \geq 0$	$\varrho^{\mathsf{T}_A} \geq 0$
covariance matrix	$\boldsymbol{\sigma} + i\Omega \geq 0$	$\tilde{\boldsymbol{\sigma}} + i\Omega \geq 0$
symplectic spectrum	$\nu_k \geq 1$	$\tilde{\nu}_k \geq 1$

Conditions of existence and conditions of separability for Gaussian states in different representations. The second column qualifies the PPT condition.

The distillability problem for Gaussian states: the entanglement of any non-PPT bipartite Gaussian state is distillable by LOCC. However, this entanglement can be distilled only by non Gaussian LOCC: Distilling Gaussian states with Gaussian operations is impossible.

Quantification of bipartite entanglement in Gaussian states.

For pure states the "unique" measure of bipartite entanglement is the entropy of entanglement (von Neumann entropy of the reduced state). Other entanglement monotones, such as the geometric entanglement, are however very important, conceptually and operationally. We will come back on this when discussing spin systems.

For general, mixed, states there is no "unique" measure, as each acceptable measure, *i.e.* satisfying a minimal set of conditions, has a different operational motivation and meaning, accounting for different, and in some cases inequivalent, operational characterizations and orderings of entangled states.

The situation is even more intricate and interesting in the case of multipartite entanglement, both for pure and for mixed states.

Entanglement of formation.

An immediate generalization to mixed states is certainly the *entanglement of formation* $E_F(\varrho)$, defined as the convex-roof extension of the entropy of entanglement Eq. (35), *i.e.* the weighted average of pure-state entanglement,

$$E_F(\varrho) = \min_{\{p_k, |\psi_k\rangle\}} \sum_k p_k \, E_V(|\psi_k\rangle) \,, \tag{44}$$

minimized over all decompositions of the mixed state $\varrho = \sum_k p_k |\psi_k\rangle\langle\psi_k|$. This is clearly an optimization problem of formidable difficulty, and an explicit solution is known only for the mixed states of two qubits, and for highly symmetric states like Werner states and isotropic states in arbitrary dimension. In CV systems, an explicit expression for the entanglement of formation is available only for symmetric, two-mode Gaussian states. The entanglement of formation is not additive (very recent result).

PPT-derived measures of entanglement for Gaussian states.

An important class of entanglement monotones, and very useful in the study of Gaussian states, is defined by the negativities, which quantify the violation of the PPT criterion for separability, *i.e.* how much the partial transposition of ϱ fails to be positive. The *negativity* $\mathcal{N}(\varrho)$ is defined as

$$\mathcal{N}(\varrho) = \frac{\left\|\varrho^{\mathsf{T}_i}\right\|_1 - 1}{2}\,, \tag{45}$$

where

$$\|\hat{O}\|_1 = \mathrm{Tr}\,\sqrt{\hat{O}^\dagger \hat{O}} \tag{46}$$

is the trace norm of the operator $\hat{O}$.

Negativities.

The negativity has the advantage of being a computable measure of entanglement:

$$\mathcal{N}(\varrho) = \max\left\{0, -\sum_k \lambda_k^-\right\} , \tag{47}$$

where the $\left\{\lambda_k^-\right\}$'s are the negative eigenvalues of the partial transpose.

The negativity can be defined for CV systems as well even though a related measure is more often used, the *logarithmic negativity* $E_{\mathcal{N}}(\varrho)$,

$$E_{\mathcal{N}}(\varrho) = \log \|\varrho^{\mathsf{T}_i}\|_1 = \log\left[1 + 2\mathcal{N}(\varrho)\right] . \tag{48}$$

The logarithmic negativity is additive and, despite not being convex, is an entanglement monotone under LOCC; it is an upper bound for the distillable entanglement, $E_{\mathcal{N}}(\varrho) \geq E_D(\varrho)$, and coincides with the entanglement cost under PPT-preserving operations.

Negativities (continued).

Both the negativity and the logarithmic negativity fail to be continuous in trace norm on infinite-dimensional Hilbert spaces; however, this problem can be circumvented by restricting to physical states of finite mean energy.

The great advantage of the negativities is that they are easily *computable* for all Gaussian states and provide a proper quantification of entanglement at least for arbitrary $1 \times N$ and bisymmetric $M \times N$ Gaussian states, directly quantifying the degree of violation of the necessary and sufficient PPT criterion for separability, Eq. (43).

Negativities: explicit symplectic expressions.

The negativity of a Gaussian state with CM $\boldsymbol{\sigma}$ is

$$\mathcal{N}(\boldsymbol{\sigma}) = \begin{cases} \frac{1}{2}\left(\prod_k \tilde{\nu}_k^{-1} - 1\right), & \text{for } k : \tilde{\nu}_k < 1 \ . \\ \\ 0 \quad \text{if } \tilde{\nu}_i \geq 1 \ \forall \, i \ . \end{cases} \tag{49}$$

The set $\{\tilde{\nu}_k\}$ is constituted by the symplectic eigenvalues of the partially transposed CM $\tilde{\boldsymbol{\sigma}}$. Accordingly, the logarithmic negativity reads

$$E_{\mathcal{N}}(\boldsymbol{\sigma}) = \begin{cases} -\sum_k \log \tilde{\nu}_k, & \text{for } k : \tilde{\nu}_k < 1 \ . \\ \\ 0 \quad \text{if } \tilde{\nu}_i \geq 1 \ \forall \, i \ . \end{cases} \tag{50}$$

The largest role of the smallest symplectic eigenvalue.

It can be shown that in a $(N_A + N_B)$-mode Gaussian state with CM $\boldsymbol{\sigma}_{A|B}$, at most

$$N_{\min} \equiv \min\{N_A,\, N_B\} \tag{51}$$

symplectic eigenvalues $\tilde{\nu}_k$ of the partial transpose $\tilde{\boldsymbol{\sigma}}_{A|B}$ can violate the PPT inequality (43) with respect to a $N_A \times N_B$ bipartition. Therefore, in all $1 \times N$ and all bisymmetric $M \times N$ Gaussian states the negativities are quantified only in terms of the smallest symplectic eigenvalue $\tilde{\nu}_-$ of the partially transposed CM. For $\tilde{\nu}_- \geq 1$ the state is separable, otherwise it is entangled; the smaller $\tilde{\nu}_-$, the more entangled is the state. In the limit of a vanishing partially transposed symplectic eigenvalue, $\tilde{\nu}_- \longrightarrow 0$, the negativities grow unboundedly.

Symplectic invariants, purities, and the bipartite entanglement of Gaussian states.

To study entanglement and informational properties (like global and marginal entropies) of two-mode Gaussian states, we can consider without loss of generality states whose CM $\boldsymbol{\sigma}$ is in the $Sp_{(2,\mathbb{R})} \oplus Sp_{(2,\mathbb{R})}$-invariant form (standard form in the technical jargon):

$$\boldsymbol{\sigma} = \begin{pmatrix} \boldsymbol{\alpha} & \boldsymbol{\gamma} \\ \boldsymbol{\gamma}^{\mathsf{T}} & \boldsymbol{\beta} \end{pmatrix} = \begin{pmatrix} a & 0 & c_+ & 0 \\ 0 & a & 0 & c_- \\ c_+ & 0 & b & 0 \\ 0 & c_- & 0 & b \end{pmatrix} . \tag{52}$$

Characterizing the bipartite entanglement of Gaussian states via local and global entropies.

For two-mode states, the uncertainty principle Ineq. (14) can be recast as a constraint on the $Sp_{(4,\mathbb{R})}$ invariants (invariants under global, two-mode symplectic operations) $\mathrm{Det}\boldsymbol{\sigma}$ and $\Delta(\boldsymbol{\sigma}) = \mathrm{Det}\boldsymbol{\alpha} + \mathrm{Det}\boldsymbol{\beta} + 2\,\mathrm{Det}\boldsymbol{\gamma}$,

$$\Delta(\boldsymbol{\sigma}) \leq 1 + \mathrm{Det}\boldsymbol{\sigma} \; . \tag{53}$$

The symplectic eigenvalues of a two-mode Gaussian state will be denoted as ν_- and ν_+, with $\nu_- \leq \nu_+$, with the uncertainty principle reducing to

$$\nu_- \geq 1 \; . \tag{54}$$

A simple expression for the $\nu_\mp$ can be found in terms of the two $Sp_{(4,\mathbb{R})}$ invariants

$$2\nu_\mp^2 = \Delta(\boldsymbol{\sigma}) \mp \sqrt{\Delta^2(\boldsymbol{\sigma}) - 4\,\mathrm{Det}\,\boldsymbol{\sigma}} \; . \tag{55}$$

Local and global entropies, and the bipartite entanglement of Gaussian states (continued).

According to Eq. (52), two-mode Gaussian states can be classified in terms of their four standard form covariances a, b, c_+, and c_-. These standard covariances of the CM can be recast in terms of symplectic invariants which admit a direct physical interpretation for generic Gaussian states. This is extremely useful and illuminating.

Covariances, symplectic invariants, and purities.

Namely, the parameters of Eq. (52) can be determined in terms of the two local symplectic invariants

$$\mu_1 = (\mathrm{Det}\,\boldsymbol{\alpha})^{-1/2} = 1/a\,, \quad \mu_2 = (\mathrm{Det}\,\boldsymbol{\beta})^{-1/2} = 1/b\,, \tag{56}$$

i.e. the purities of the reduced single-mode states, and of the two global symplectic invariants

$$\mu = (\mathrm{Det}\,\boldsymbol{\sigma})^{-1/2} = [(ab - c_+^2)(ab - c_-^2)]^{-1/2}\,, \quad \Delta = a^2 + b^2 + 2c_+c_-\,, \tag{57}$$

which are, respectively, the global purity Eq. (32) and the seralian Eq. (24). Eqs. (56–57) can be inverted to give a physical parametrization of two-mode states in terms of the four independent parameters μ_1, μ_2, μ, and Δ. This parametrization is particularly useful for the evaluation of entanglement.

Partial transposition, symplectic invariants, and purity.

The PPT condition for separability, Eq. (40) takes a very simple form for two-mode Gaussian states. In terms of symplectic invariants, partial transposition corresponds to flipping the sign of $\mathrm{Det}\,\boldsymbol{\gamma}$,

$$\boldsymbol{\sigma} = \begin{pmatrix} \boldsymbol{\alpha} & \boldsymbol{\gamma} \\ \boldsymbol{\gamma}^{\mathsf{T}} & \boldsymbol{\beta} \end{pmatrix} \quad \xrightarrow{\varrho \to \varrho^{\mathsf{T}_i}} \quad \tilde{\boldsymbol{\sigma}} = \begin{pmatrix} \boldsymbol{\alpha} & \tilde{\boldsymbol{\gamma}} \\ \tilde{\boldsymbol{\gamma}}^{\mathsf{T}} & \boldsymbol{\beta} \end{pmatrix} , \tag{58}$$

with $\mathrm{Det}\,\tilde{\boldsymbol{\gamma}} = -\mathrm{Det}\,\boldsymbol{\gamma}$. For a standard form CM Eq. (52), this simply means $c_+ \to c_+$, $c_- \to -c_-$. Accordingly, the seralian $\Delta = \mathrm{Det}\,\boldsymbol{\alpha} + \mathrm{Det}\,\boldsymbol{\beta} + 2\,\mathrm{Det}\,\boldsymbol{\gamma}$, Eq. (24), is mapped, under partial transposition, into

$$\begin{aligned} \tilde{\Delta} &= \mathrm{Det}\,\boldsymbol{\alpha} + \mathrm{Det}\,\boldsymbol{\beta} + 2\,\mathrm{Det}\,\tilde{\boldsymbol{\gamma}} = \mathrm{Det}\,\boldsymbol{\alpha} + \mathrm{Det}\,\boldsymbol{\beta} - 2\,\mathrm{Det}\,\boldsymbol{\gamma} \\ &= \Delta - 4\,\mathrm{Det}\,\boldsymbol{\gamma} = -\Delta + 2/\mu_1^2 + 2/\mu_2^2 \,. \end{aligned} \tag{59}$$

Partial transposition, symplectic invariants, and purity - II.

From Eq. (55), the symplectic eigenvalues of the partial transpose $\tilde{\boldsymbol{\sigma}}$ of a two-mode CM $\boldsymbol{\sigma}$ are promptly determined in terms of symplectic invariants,

$$2\tilde{\nu}_{\mp}^{2} = \tilde{\Delta} \mp \sqrt{\tilde{\Delta}^{2} - \frac{4}{\mu^{2}}}\,. \tag{60}$$

The PPT criterion is then reexpressed by the following inequality

$$\tilde{\Delta} \leq 1 + 1/\mu^{2}\,, \tag{61}$$

equivalent to separability.

Partial transposition, symplectic invariants, and purity - III.

The state $\boldsymbol{\sigma}$ is separable if and only if $\tilde{\nu}_- \geq 1$. Accordingly, the logarithmic negativity Eq. (50) is a decreasing function of $\tilde{\nu}_-$ alone,

$$E_{\mathcal{N}} = \max\{0, -\log \tilde{\nu}_-\}\,, \tag{62}$$

as for the largest symplectic eigenvalue of the partial transpose one has $\tilde{\nu}_+ > 1$ for all two-mode Gaussian states.

Note that from Eqs. (52,53,59,61) the following necessary condition for two-mode entanglement follows,

$$\boldsymbol{\sigma} \text{ entangled} \quad \Rightarrow \quad \mathrm{Det}\,\boldsymbol{\gamma} < 0\,. \tag{63}$$

Entanglement versus information.

As discussed in the previous sections, the concepts of entanglement and information encoded in a quantum state are closely related. Specifically, for pure states bipartite entanglement is equivalent to the lack of information (mixedness) of the reduced state of each subsystem. For mixed states, each subsystem has its own level of impurity, and moreover the global state is itself characterized by a nonzero mixedness. Each of these properties can be interpreted as information on the preparation of the respective (marginal and global) states. Therefore, by properly accessing these degrees of information it should be possible to deduce the status of the quantum correlations between the subsystems.

Entanglement versus information (continued).

Indeed, the negativities of arbitrary (mixed) two-mode Gaussian states are analytically constrained by rigorous upper and lower bounds. This follows by re-parameterizing, as already anticipated, the standard form CM Eq. (52) in terms of the invariants $\mu_1,\ \mu_2,\ \mu,\ \Delta$ and by observing that, at fixed purities, the negativities are monotonically decreasing function of Δ. Further constraints imposed on Δ by the uncertainty principle and by the existence condition of the radicals involved in the reparametrization,

$$\frac{2}{\mu}+\frac{(\mu_1-\mu_2)^2}{\mu_1^2\mu_2^2}\leq\Delta\ \leq\ \min\left\{\frac{(\mu_1+\mu_2)^2}{\mu_1^2\mu_2^2}-\frac{2}{\mu}\ ,\ 1+\frac{1}{\mu^2}\right\}\ , \tag{64}$$

immediately lead to the definition of *extremally* – maximally and minimally – entangled Gaussian states at fixed global and local purities.

Extremally entangled Gaussian states.

These maximally and minimally entangled Gaussian states are known, respectively, as "GMEMS" (saturating the leftmost inequality in Eq. (64)), alias nonsymmetric thermal squeezed states, and "GLEMS" (saturating the rightmost inequality in Eq. (64)), alias mixed states of partial minimum uncertainty. Nonsymmetric thermal squeezed states have also been proven to be maximally entangled Gaussian mixed states at fixed global purity and mean energy.

Summarizing, *the entanglement, quantified by the negativities, of two-mode (mixed) Gaussian states is strictly bound from above and from below by the amounts of global and marginal purities, with only one remaining degree of freedom related to the symplectic invariant* Δ.

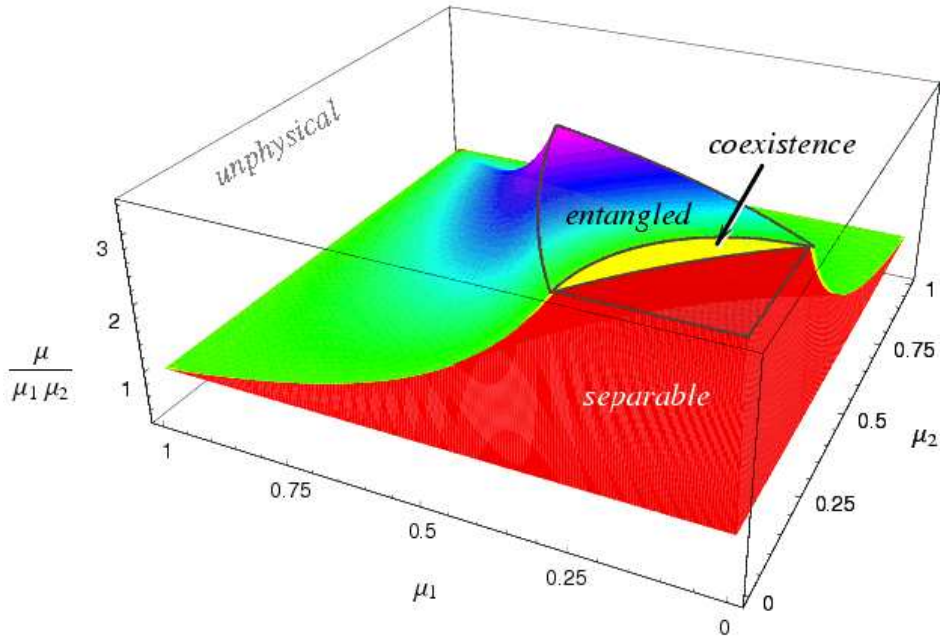
Extremally entangled Gaussian states (continued).

The existence of GMEMS and GLEMS has two consequences. First, it allows for a classification of the properties of separability of all two-mode Gaussian states according to their degree of global and marginal purities (See following Table). Namely, from the separability properties of the extremally entangled states, necessary and/or sufficient conditions for entanglement — which constitute the strongest entropic criteria for separability to date in the case of Gaussian states — are straightforwardly derived, which allow one to decide whether a two-mode Gaussian state is entangled by looking at its degree of global and local purity. There is only a narrow region where, for given purities, both separable and entangled states can coexist (see Figure).

Entanglement on the average.

The second consequence is of a quantitative nature. The difference between the entanglement of maximally and minimally entangled Gaussian states narrows exponentially with increasing entanglement. One can then define the *average logarithmic negativity* (mean value of the entanglements of the GMEMS and the GLEMS corresponding to a given triplet of purities) as a reliable estimator of bipartite entanglement in two-mode Gaussian states and its accurate quantification by knowledge of the global and marginal purities alone. The average logarithmic negativity is exact in the two important instances of nonsymmetric thermal squeezed states and mixed states of partial minimum uncertainty, whose logarithmic negativity is completely determined as a function of the three purities alone.

The landscape.



Entanglement of two-mode Gaussian states in the space of marginal purities $\mu_{1,2}$ and global purity μ. Separable states (red zone) and entangled states (green to magenta zone, according to the average entanglement) are well separated except for a narrow region of coexistence (yellow zone). The mathematical relations defining the boundaries between the three regions are collected in the synopsis.

Synopsis.

Degrees of purity	Entanglement properties
$\mu < \mu_1\mu_2$	unphysical region
$\mu_1\mu_2 \leq \mu \leq \frac{\mu_1\mu_2}{\mu_1+\mu_2-\mu_1\mu_2}$	*separable* states
$\frac{\mu_1\mu_2}{\mu_1+\mu_2-\mu_1\mu_2} < \mu \leq \frac{\mu_1\mu_2}{\sqrt{\mu_1^2+\mu_2^2-\mu_1^2\mu_2^2}}$	*coexistence* region
$\frac{\mu_1\mu_2}{\sqrt{\mu_1^2+\mu_2^2-\mu_1^2\mu_2^2}} < \mu \leq \frac{\mu_1\mu_2}{\mu_1\mu_2+\lvert\mu_1-\mu_2\rvert}$	*entangled* states
$\mu > \frac{\mu_1\mu_2}{\mu_1\mu_2+\lvert\mu_1-\mu_2\rvert}$	unphysical region

Classification of two-mode Gaussian states and of their properties of separability according to their degrees of global purity μ and of marginal purities μ_1 and μ_2.