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ENTANGLEMENT AT WORK
Third Lecture

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Entanglement at work: Quantum Spin Systems

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Motivations

Interfacing Quantum Information and Condensed Matter Physics:

1. *Suitably engineered systems of condensed matter for the implementation of quantum computing protocols* (NMR, Quantum dots, Spin chains, Optical lattices, Coupled cavity arrays, etc.)

2. *Scaling and area laws for the entropy of entanglement: mathematical control and design of numerical techniques for the efficient simulation of complex quantum systems* (DMRG (Density Matrix Renormalization Group), PEPS (Projected Entangled-Pair States), MERA (Multi-scale Entanglement Renormalization Ansatz, etc.)

3. **Is it possible to use the tools of entanglement theory to gain novel insight on the physics of complex and quantum many-body systems that are complementary (and/or deeper) to the understanding obtained using the standard tools of quantum statistical mechanics (order parameters, n-point correlation functions, etc.)?**

1. Long-distance and modular entanglement in quantum spin systems: Pairing quantum objects that are distant and non-interacting, thanks to monogamy and modularity of entanglement

2. Occurrence of separable ground states of strongly interacting systems: General theory and possible development of approximation schemes for the study of non-exactly solvable models

3. Hierarchy of bipartite-multipartite entanglement measures applied to the study of quantum spin models: Sequence of novel transitions in the symmetry broken phase. Successive oscillations in the magnetic order of local domains, parity inversion points, and multipartite-to-bipartite transmutations of entanglement

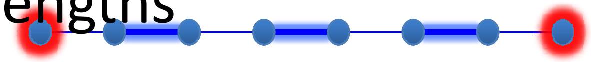
4. General theory of classical and quantum frustration in quantum many-body systems. Understanding frustration in terms of bipartite geometric entanglement and related, properly defined, entanglement monotones

Long Distance Entanglement

- Many body systems as quantum teleportation resource
- End sites act as Alice and Bob
- Ground State Entanglement between end sites determines teleportation fidelity
- Exploit monogamy to achieve entangled end points
- Consider robustness against thermal fluctuations

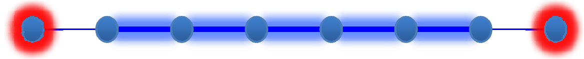
Ground state in XX model

Alternating coupling strengths



- Dimerized ground state
- Monogamy favors dimerization in strong bonds
- Gap closes exponentially
- LDE grows fast with degree of dimerization
- **Not robust** against temp.

Uniform coupling strengths



- GS not dimerized
- Monogamy decouples ends from bulk
- Gap closes algebraically
- LDE grows moderately with degree of weak coupling
- **Relativelly Robust** against temperature

Implementation in cold atoms, cavities, ion traps…
Disorder in coupling strength and role of localization
Alternative models with reasonable engineering and high robustness

Long-range vs. long-distance entanglement

Qubit teleportation:

Existence of a good A-B quantum channel: Highest possible end-to-end entanglement => highest possible end-to-end teleportation fidelity. Difficult without long-range interactions. Long-range interactions involve many different subsystems => strong source of noise and decoherence.

Systems possessing only long-distance entanglement: Open spin chains with alternating couplings, or average bulk interactions with modified, "weak" end bonds. At exactly **T = 0**, these systems can support maximal entanglement and perfect qubit teleportation (i.e. with unit fidelity) between the ends A and B of the chain ("Long-Distance-Entanglement" - **LDE**):

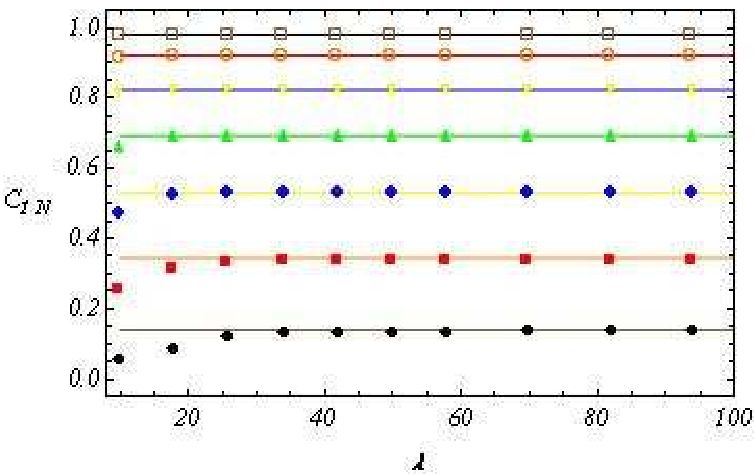
$$H = \sum_{i=1}^{N-1} J_i \left(S_i^x S_{i+1}^x + S_i^y S_{i+1}^y \right)$$

Long-distance entanglement – The ideal case

<u>Infinite-distance entanglement and perfect teleportation (zero temperature)</u>:

($\lambda = J_1 / J_2$; $1 = J_2 / J_2$)

Dimerized 1-D **XX** spin chain with fully alternated couplings (**λ << 1**). **<u>Maximal</u>** Alice-Bob entanglement. **<u>Unit fidelity</u>** for teleporting an unknown input state from Alice to Bob at zero temperature. Arbitrary length, finite size or thermodynamic limit.

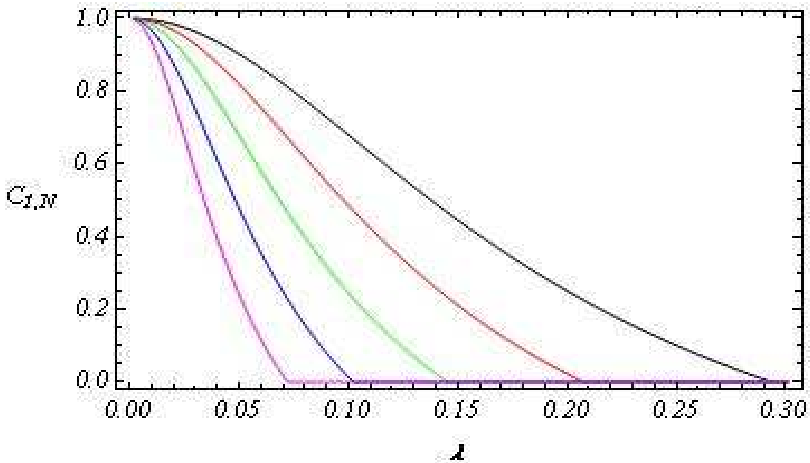


Energy Gap between ground and excited state falls exponentially with N

Long-distance entanglement – The weak end-bond case

($\lambda = J_1 / J_b = J_{N-1} / J_b$; $1 = J_b / J_b$)

"Weak end-bond" **XX** spin chain with O(1) **Bulk** interactions and weak couplings at the end points of the chain (**$\lambda << 1$**). **Large** Alice-Bob entanglement, and large teleportation fidelity at zero temperature. Slowly decreasing with the length of the chain (distance A-B).

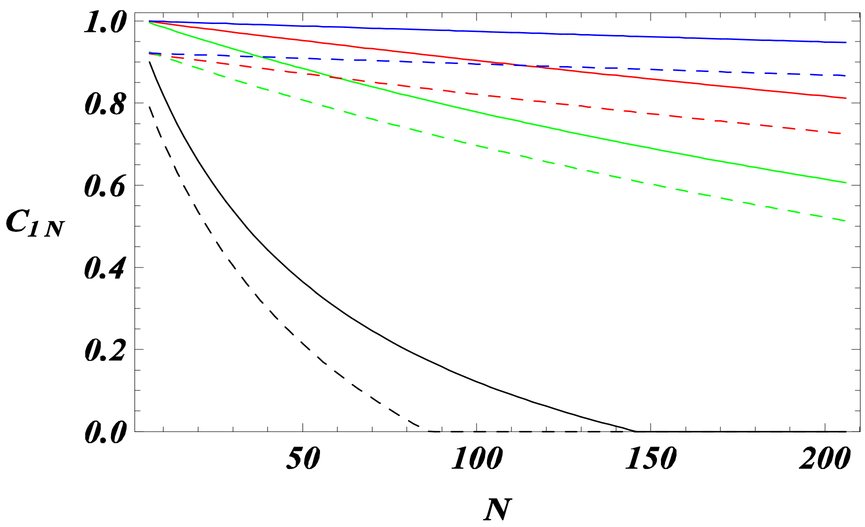


Robust against temperature but entanglement very small except for vanishingly small λ

Long-distance entanglement – Optimizations

($\lambda = J_1 / J_b = J_{N-1} / J_b$; $\mu = J_2 / J_b = J_{N-2} / J_b$; $1 = J_b / J_b$)

Augmented **XX** spin chain with O(1) **Bulk** interactions and **alternating strong/weak** couplings at the **ends** of the chain ($\lambda << 1 < <\mu$). **Large** Alice-Bob entanglement and teleportation fidelity at zero temperature. Slowly decreasing with the length of the chain.



Long-distance entanglement and high-fidelity teleportation at zero and finite T

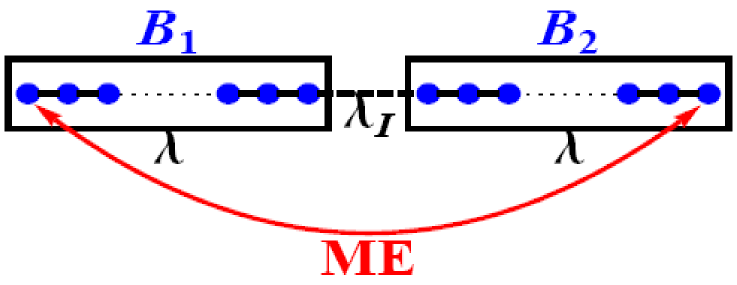
Generalization: Modular Entanglement.

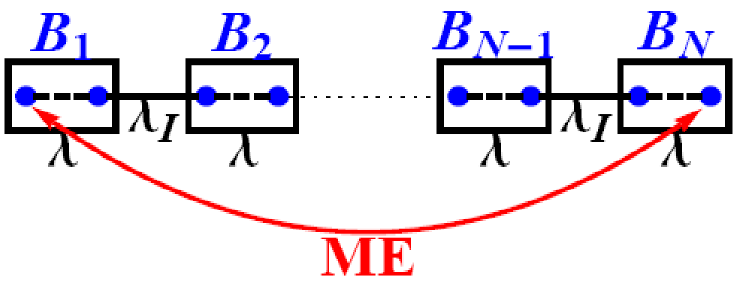
$$H_{T,2moduli} = H_{1,n}^{B_1} + H_{n+1,2n}^{B_2} + H_{n,n+1}^{I} \ , \tag{1}$$

with

$$H_{\gamma,\delta}^{\alpha} = \frac{1}{2}\sum_{i=\gamma}^{\delta-1} J_{i,i+1}(S_i^x S_{i+1}^x + S_i^y S_{i+1}^y) \ , \tag{2}$$

where $H_{n,n+1}^{I}$ is the interaction Hamiltonian between the two boundary qubits in B_1 and B_2.





Entanglement and quantum phase transitions in interacting quantum spin systems

Fundamental systems of interacting ***qubits*** (spin 1/2) undergoing quantum phase transitions at zero temperature. The XYZ Hamiltonian: It comprises the most important models of quantum spins coupled by exchange interaction terms, including the Ising, Heisenberg, XY, XX, XXZ models. Moreover, most relevant models of quantum spin chains for quantum information tasks (more on this in the following).

$$H_{XYZ} = \frac{1}{2}\sum_{\underline{i},\underline{j}} J_x^r S_{\underline{i}}^x S_{\underline{j}}^x + J_y^r S_{\underline{i}}^y S_{\underline{j}}^y + J_z^r S_{\underline{i}}^z S_{\underline{j}}^z - h\sum_{\underline{i}} S_{\underline{i}}^z$$

$$r = |\underline{i} - \underline{j}|$$

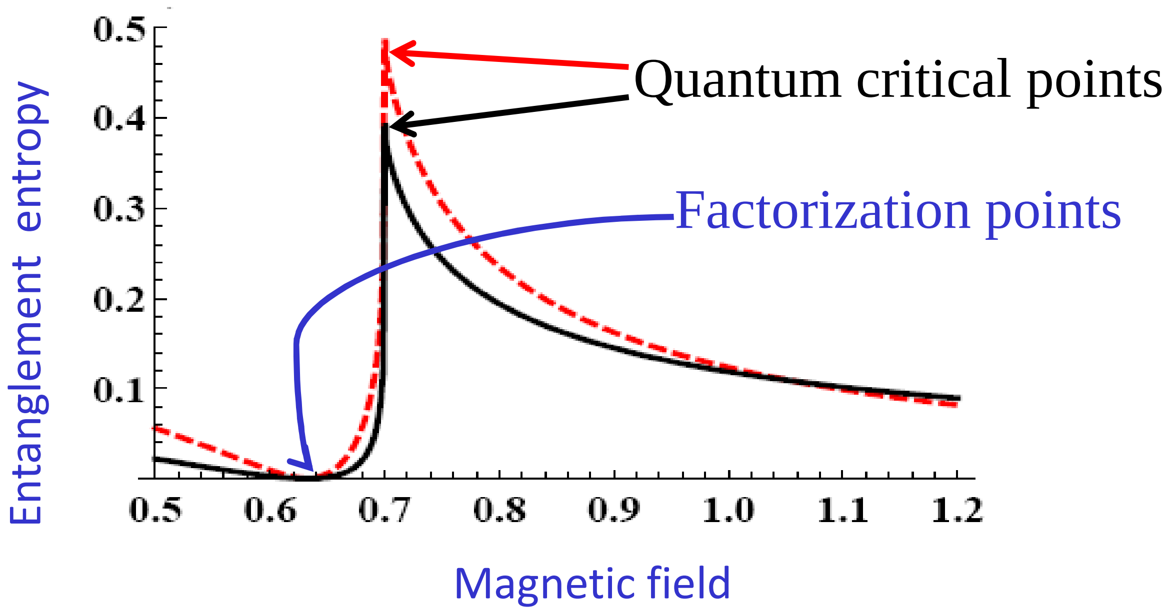
Includes models with short, finite, and long-range interactions. Most cases are non exactly solvable, with some notable exceptions, like the ***XY model***.

B1) Entanglement and quantum phase transitions in quantum spin chains

XY model: $$H_{XY} = -\sum_i S_i^x S_{i+1}^x + \Delta S_i^y S_{i+1}^y + h \sum_i S_i^z$$

Anisotropic spin model **(0 ≤ Δ < 1)**. Phase transition: A spontaneous magnetization develops along the ***x*** axis as the external transverse field ***h*** is varied. The **divergence** of the first derivative (with respect to ***h***) of the von Neumann block entanglement entropy between a single spin and the rest of the system signals the onset of the quantum phase transition:

Δ = 0.25 (--------)
Δ = 0.28 (——)



B2) Ground state factorization in systems of interacting qubits (quantum spin chains)

In the **XY model**, we have just seen an instance of a **factorization point**: The quantum entanglement vanishes and the ground state becomes factorized, even if the system is strongly interacting! Is this an acccidental phenomenon, a coincidence, or it hides a more profound physical picture?

Exploiting methods inspired by entanglement theory, it is possible to derive a general theory of **interaction balancing** and determine the conditions for the occurrence of full factorization of the many-body states in interacting quantum systems.

Main results of the analysis for quantum spin models of the **XYZ** type:

I) Factorization of the ground state is not a rare phenomenon.
II) It is due to a delicate balancing between spin-spin interactions and external fields.
III) It occurs irrespective of spatial dimension and interaction range.

B2) Factorization in quantum spin systems: Single-qubit unitary operations (SQUOs)

Single Qubit Unitary Operations: $U_{A|B} = O_A \otimes I_B$

Hermitian: $O_A^\dagger = O_A$

Unitary: $O_A O_A^\dagger = O_A^2 = I_A$

Traceless: $\mathrm{Tr}\, O_A = 0$

Response of the system to controlled, nontrivial external perturbations.

B2) Factorization and SQUOs in quantum spin systems

Factorization:

$$|\psi\rangle_{AB}^{F} \equiv |\phi\rangle_{A} \otimes |\chi\rangle_{B}$$

Theorem:

$$|\psi\rangle_{AB} = |\psi\rangle_{AB}^{F} \quad \text{iff} \quad \exists\; U_{A|B}^{extr} : \quad U_{A|B}^{extr}|\psi\rangle_{AB} = |\psi\rangle_{AB}$$

Factorization $\Longleftrightarrow$ Invariance under the action of SQUOs

The "extremal" SQUO $U_{A|B}^{extr}$ is uniquely defined

B2) Factorization in quantum spin systems: SQUOs and entanglement

Pure State: $|\Psi\rangle_{AB}$

Transformed State: $|\tilde{\psi}\rangle_{AB} = U_{A|B} |\psi\rangle_{AB}$

Hilbert-space distance: $d\left(|\psi\rangle_{AB}, |\tilde{\psi}\rangle_{AB}\right) = \sqrt{1 - \left| {}_{AB}\langle \tilde{\psi} | \psi \rangle_{AB} \right|^2}$

Theorem: ***Distance ⇔ Entanglement (von Neumann entropy):***

$$S_E(\rho_A) = \min_{\{U_{A|B}\}} \left[d\left(|\psi\rangle_{AB}, |\tilde{\psi}\rangle_{AB}\right) \right]^2$$

B2) Quantifying factorization in quantum spin systems with SQUO-related observables

Observable estimators Q of separability under the action of SQUOs
[Bipartite system. Spin i = party A. Party B = all remaining spins N/i]

$$|\tilde{\psi}\rangle_{AB} = U_{A|B} |\psi\rangle_{AB} \qquad \Delta Q = {}_{AB}\langle\tilde{\psi}|\hat{Q}|\tilde{\psi}\rangle_{AB} - {}_{AB}\langle\psi|\hat{Q}|\psi\rangle_{AB}$$

If: 1) $\Delta Q \geq 0$, 2) $\left[U_{A|B}, \hat{Q}\right] \neq 0$

Then: $$\Delta Q = 0 \quad \Rightarrow \quad |\psi\rangle_{i|(N/i)} = |\phi\rangle_i \otimes |\chi\rangle_{(N/i)}$$

This in turn implies total (full) factorization in translational-invariant systems:

$$\Delta Q = 0 \quad \Rightarrow \quad |\psi\rangle = \bigotimes_i |\phi\rangle_i$$

B2) Determining ground-state factorization in quantum spin systems with entanglement excitation energies (EXEs)

Hamiltonian Structure: Entanglement excitation energies (EXEs) associated to extremal SQUOs

$$\text{Identification:} \quad \hat{Q} = \hat{H}$$

$$\text{Ground State:} \quad |\psi\rangle = |G\rangle$$

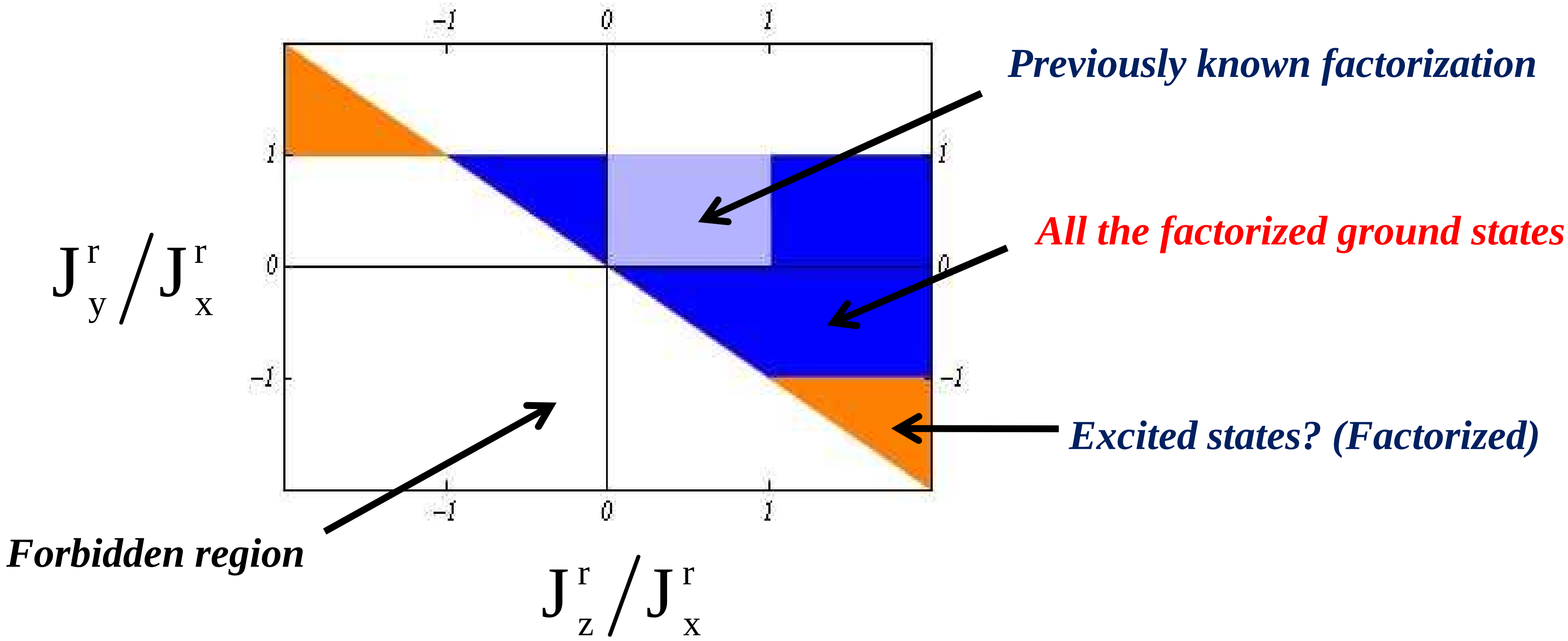
Entanglement Excitation Energy: $\Delta E = \langle \tilde{G} | \hat{H} | \tilde{G} \rangle - \langle G | \hat{H} | G \rangle$

$\Delta E = 0 \iff$ **Factorized ground state**

$\Delta E > 0 \iff$ **Entangled ground state**

B2) Applying the general theory to the determination of the factorization points of interacting quantum spin systems

Phase diagram for factorization in the XYZ models



B2) Summary on factorization in many-body systems

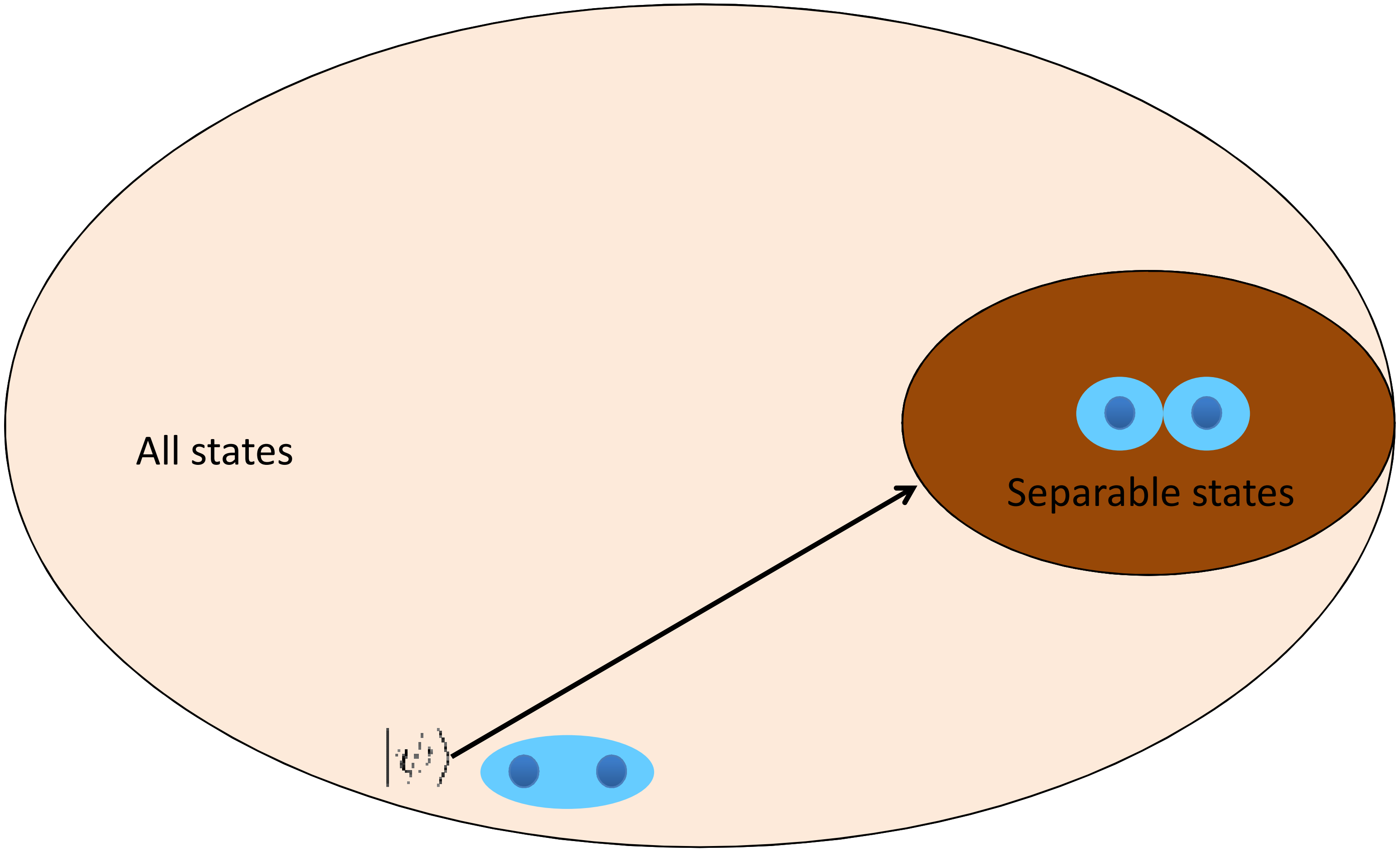
1) Formalism of local unitary operations. Operational- geometric approach to the characterization of separability and entanglement.

2) Theory of ground state factorization for (generally non exactly-solvable) quantum spin systems. Arbitrary lattice dimension and range of interaction. Classes of exact solutions, exploiting concepts and techniques of quantum information.

3) Factorization: A highly nontrivial balancing in strongly interacting many-body quantum systems.

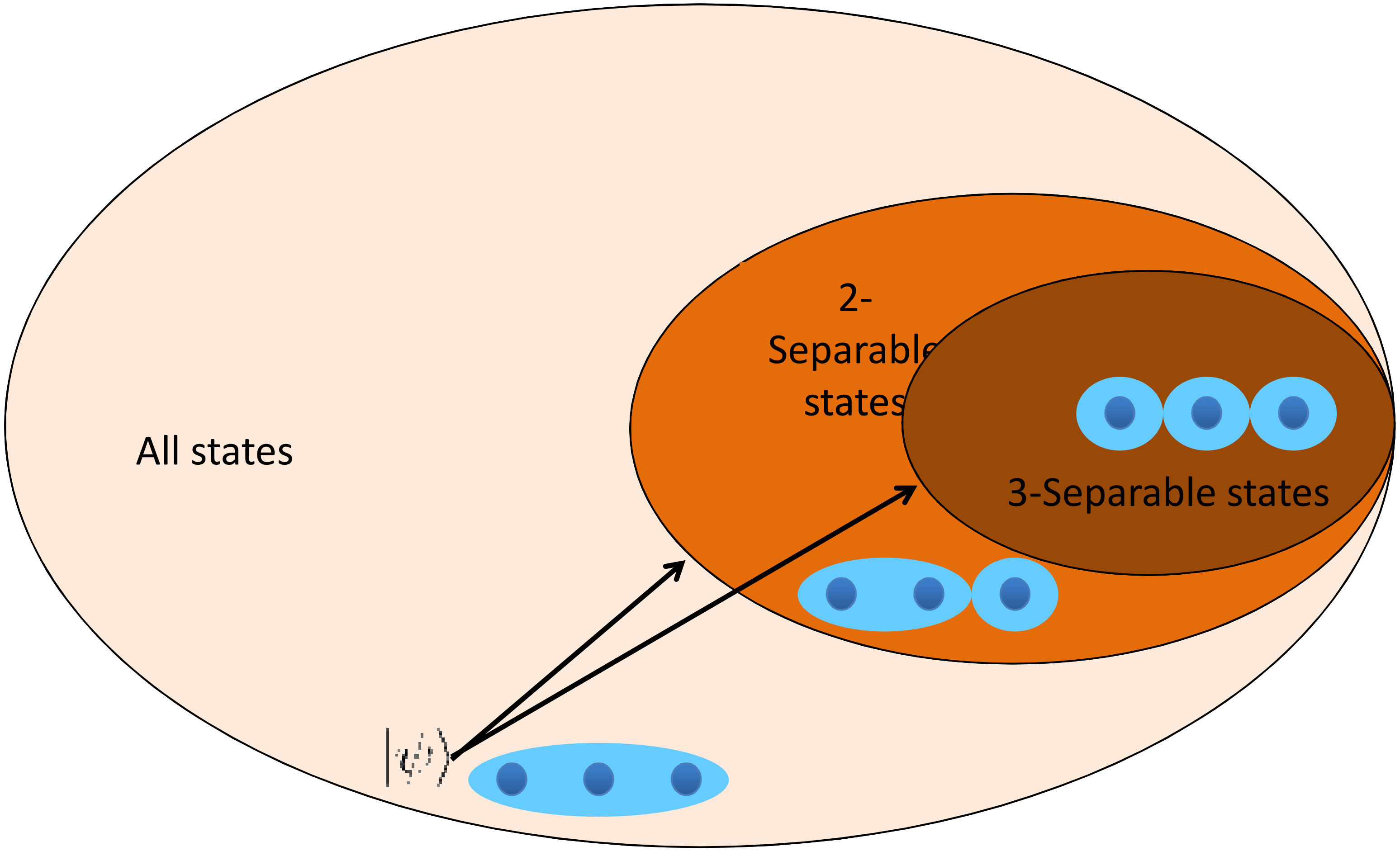
Geometric Entanglement

Bipartite States



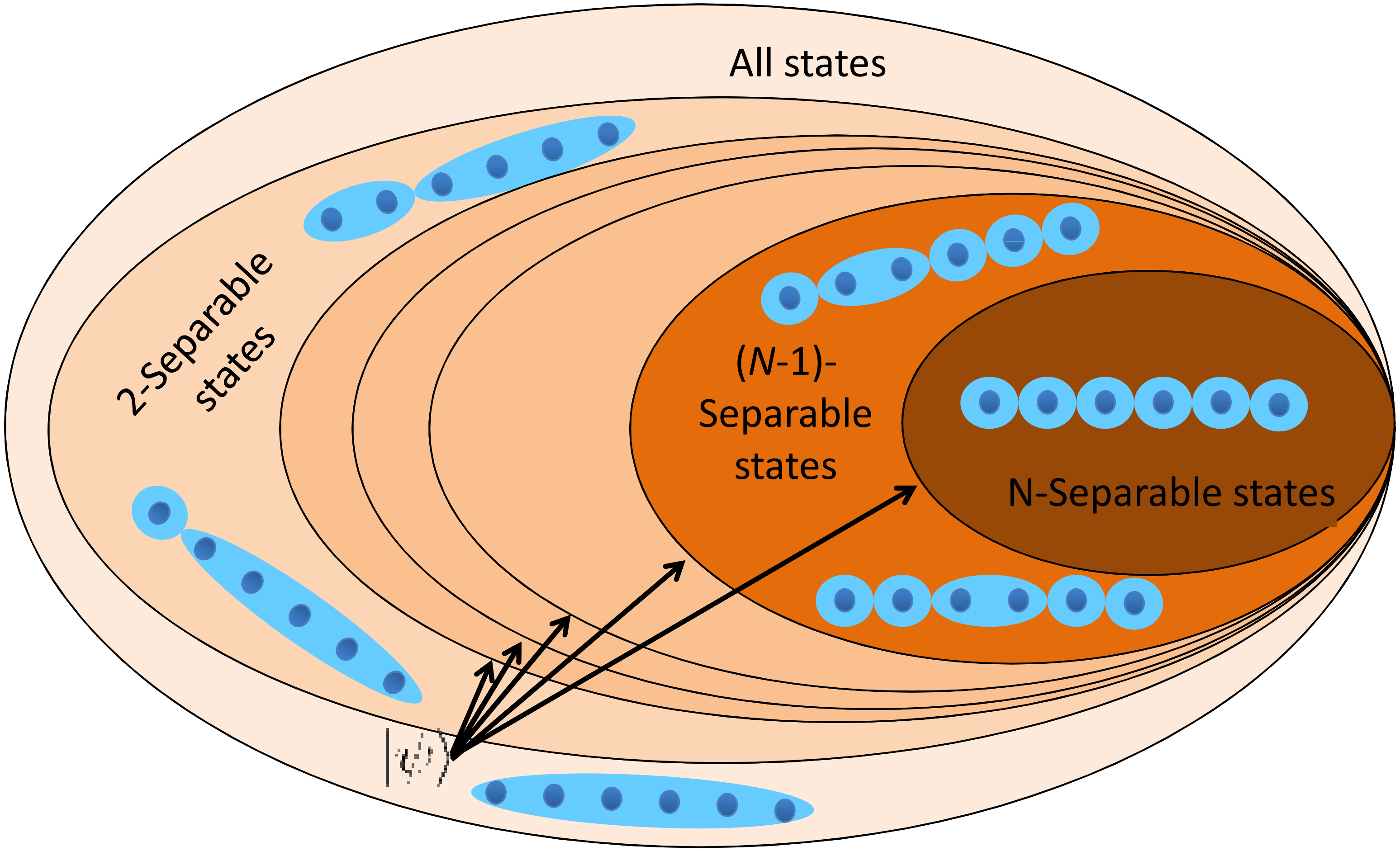
Geometric Entanglement

Tripartite states



Geometric Entanglement

Multipartite states



Hierarchies of Geometric Entanglement

- Take the Geometric Entanglement measure

$$E_G(|\psi\rangle) = 1 - \min_{|\phi\rangle \in \mathrm{Sep}} |\langle\phi|\psi\rangle|^2$$

and compute for the $|W\rangle$ and the $|GHZ\rangle$:

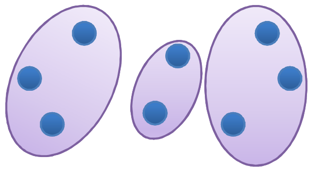
$$E_G(|GHZ\rangle) = \frac{1}{2}$$

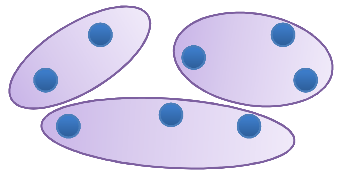
$$E_G(|W\rangle) = 1 \quad \left(\frac{N \quad 1}{N}\right)^{N-1} \to 0.632$$

LARGE N $E_G(|W\rangle) > E_G(|GHZ\rangle)$

Hierarchical Entanglement Measures

- Let $Q_1|Q_2|\ldots|Q_K$ be a K-partition of N:





- A state is K-separable if it separable by some K-partition
- Geometric Entanglement is defined as

$$E_G^{(K)}(|\psi\rangle) = 1 - \min_{|\phi\rangle \in K\text{-Sep}} |\langle\phi|\psi\rangle|^2$$

$$(K+1)\text{-Sep} \subset K\text{-Sep} \Rightarrow E_G^{(K+1)}(|\psi\rangle) \geq E_G^{(K)}(|\psi\rangle)$$

A useful tool

- Identifies and distinguishes several multipartite entanglement measures
- Can be extended by the convex roof
- Scaling laws can be obtained in the presence of permutation symmetry
- Can provide insight about the structure of levels of *K*-separability

Ground state physics of quantum spin models via a hierarchy of geometric measures of entanglement

- *Exactly solvable system:* XY model
- *Entanglement measure:* Hierarchical geometric measure

Hierarchical geometric measure of entanglement

1. **K-partition** of the system in K nonintersecting subsystems $\{Q_1, Q_2, \ldots, Q_K\}$: $Q_s \bigcap Q_{s'} = \emptyset$ for $s \neq s'$

2. The **K-th element of the hierarchy** of geometric measures of entanglement is defined as the **Euclidean distance from the set of K-separable states**:

$$\mathrm{E_G^{(K)}}(|\Psi\rangle) = 1 - \Lambda_\mathrm{K}^2(|\Psi\rangle)$$

$$\Lambda_\mathrm{K}^2(|\Psi\rangle) = \max_{|\varphi^{(K)}\rangle} \left|\langle\varphi^{(\mathrm{K})}|\Psi\rangle\right|^2$$

3. *For instance $\mathrm{E_G^{(2)}}$ measures the bipartite entanglement in the state $|\Psi\rangle$, $\mathrm{E_G^{(3)}}$ measures all the contributes up to tripartite entanglement, and so on up to $\mathrm{E_G^{(N)}}$ the original Wei-Goldbart global geometric entanglement that encompasses all the bipartite and multipartite contributions*

4. There follows the natural hierarchy $\mathrm{E_G^{(N)}} \geq \mathrm{E_G^{(N-1)}} \geq \cdots \geq \mathrm{E_G^{(3)}} \geq \mathrm{E_G^{(2)}}$

5. General result for the bipartite geometric entanglement:

$$\mathrm{E_G^{(2)}}(|\Psi\rangle) = 1 - \lambda_{\max}$$

$\lambda_{\max}$ maximum eigenvalue associated with the reduced block density matrix

Paradigmatic model: Anisotropic XY spin chain

$$H_{xy} = \sum_{i=1}^{N} \left[(1+\gamma) S_i^x S_{i+1}^x + (1-\gamma) S_i^y S_{i+1}^y\right] - h \sum_{i=1}^{N} S_j^z\,, \qquad S_{N+1}^\alpha \equiv S_1^\alpha$$

S_i^α $(\alpha = x, y, z)$ *spin-1/2 operators,* h *magnetic field,* γ *anisotropy*

1. Exactly solvable by Fermionization $(\{S_i^\alpha\} \to \{c_i\})$

2. The parity of the number of down spins is a constant of motion

$\Rightarrow$ Even (e) and Odd (o) fermion sectors

Lowest energy states

$|\Psi_\alpha\rangle \quad (\alpha = e, o)$

$$|\Psi_e\rangle = \prod_q \left(\cos\theta_q + \frac{i}{N}\sin\theta_q \sum_{h,k=1}^{N} e^{iq(h-k)} c_h^\dagger c_k^\dagger\right)|0\rangle\rangle$$

$$|\Psi_o\rangle = \frac{e^{i\frac{\pi}{4}}}{\sqrt{N}} \sum_{k=1}^{N} c_k^\dagger \prod_q \left(\cos\theta_q + \frac{i}{N}\sin\theta_q \sum_{h,k=1}^{N} e^{iq(h-k)} c_h^\dagger c_k^\dagger\right)|0\rangle\rangle$$

$(|0\rangle\rangle \equiv |0\rangle_1 \ldots |0\rangle_N \;\Leftrightarrow\; |\uparrow\rangle_1 \ldots |\uparrow\rangle_N)$

Bipartite and multipartite geometric entanglement in the ground state of the XY model

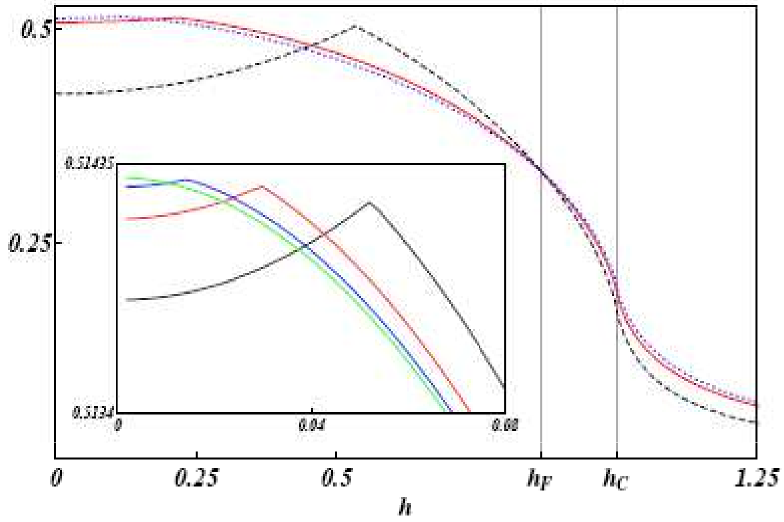
Novel information on the physics of the ground state can be obtained by analyzing the following measures:

- $\mathrm{E_G^{(2)}(M|N-M)}$
- $\mathrm{E_G^{(3)}(M_i|M_j|N-M_i-M_j)}$
- $\mathrm{E_G^{(4)}(M_i|M_j|M_k|N-M_i-M_j-M_k)}$

The sets M, M_i, M_j, M_k denote blocks of either contiguous or non-contiguous spins, so that the internal distances between the spins play an important role

Ground state bipartite geometric entanglement $(\mathrm{M}=2)$

$$\mathrm{E}^{(2)}_{\mathrm{G},2}(\mathrm{r}) \equiv \mathrm{E}^{(2)}_{\mathrm{G}}(\mathrm{i}, \mathrm{i}+\mathrm{r}|1, \ldots, \widehat{\mathrm{i}}, \ldots, \widehat{\mathrm{i}+\mathrm{r}}, \ldots, \mathrm{N})$$



$E^{(2)}_{G,2}(r)$ as a function of h with $\gamma = 0.5$ for spin-spin distances $r = 1$, $r = 2$, $r = 3$ in the main plot, $r = 4$, $r = 5$, $r = 6$, $r = 10$ in the inset

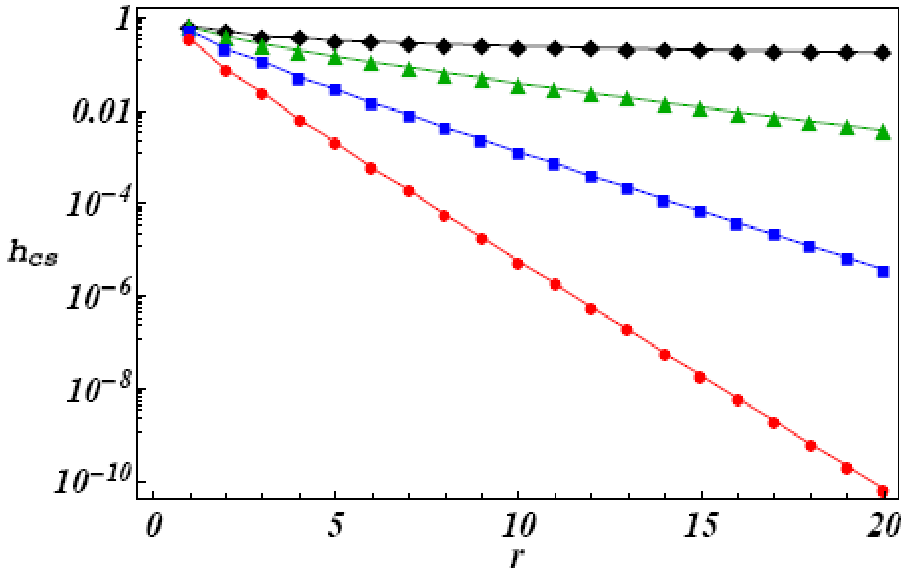
- *Inflection point at the critical field* h_C (divergence of the first derivative)
- *Intersection point at the factorizing field* $(h_F = \sqrt{1-\gamma^2})$
- *Novel transition points h_{cp}, precursors of factorization h_F and criticality $h_C = 1$. They are points of parity inversion in the local domains of magnetic order, as well as points of transmutation/conversion of multipartite into bipartite entanglement (points of partial disentanglement)*

Physical meaning of the parity inversion points

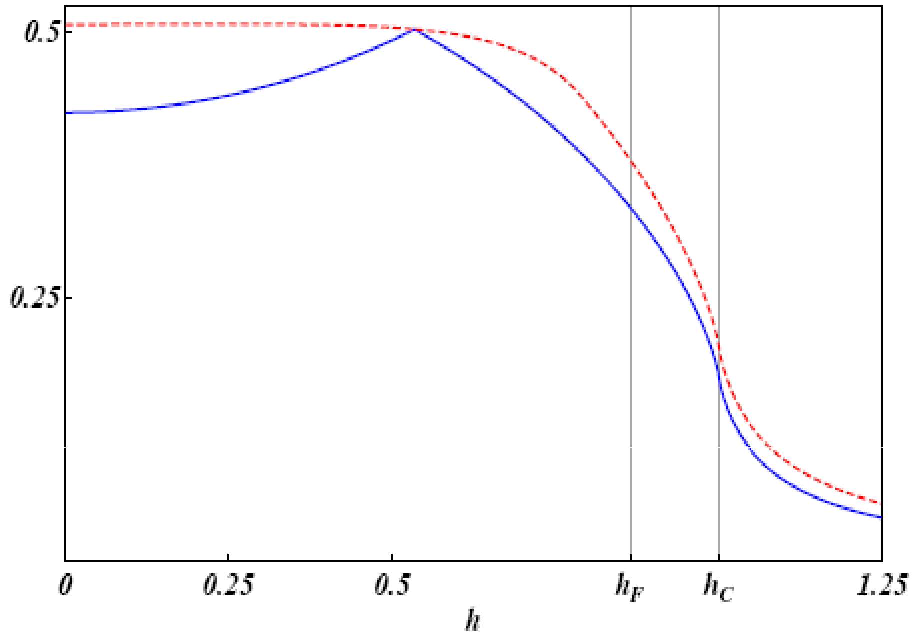
The parity inversion point in $E_G^{(2)}$ detects a transition between (probabilistic) odd and even local (block) magnetic orders, identified by the change of the maximum eigenvalue (probability) of the block reduced density matrix ρ_M

$$|\lambda_{max}\rangle_{i,i+r} = \begin{cases} \frac{1}{\sqrt{2}}\left(|\uparrow\downarrow\rangle \pm |\downarrow\uparrow\rangle\right)\ , & h < h_{cp}, \quad \text{odd \# of down spins} \\ \cos\varphi|\uparrow\uparrow\rangle + \sin\varphi|\downarrow\downarrow\rangle\ , & h > h_{cp}, \quad \text{even \# of down spins} \end{cases}$$

Behavior of the parity inversion point as a function of the distance r between the two spins of the block for $\gamma = 0$, $\gamma = 0.2$, $\gamma = 0.5$, $\gamma = 0.8$



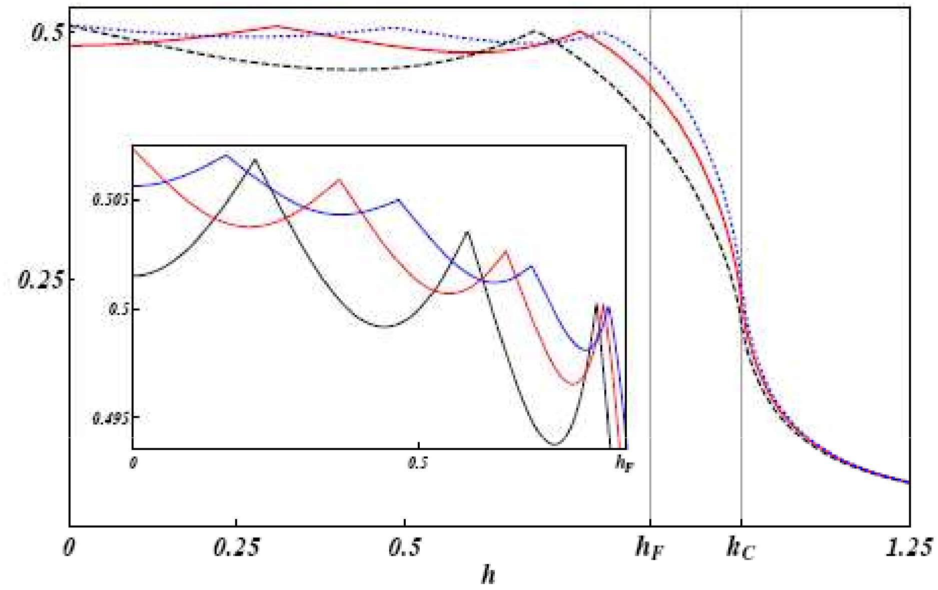
$M = 2$: Comparing $E_G^{(2)}(2|N-2)$ and $E_G^{(3)}(1|1|N-2)$



At the parity inversion point the tripartite entanglement is suppressed as the two spins in the block become decorrelated (transition between the only two possible entangled orders, even and odd). Therefore at the cusp point $E_G^{(3)} = E_G^{(2)}$ since the only contribution remaining in $E_G^{(3)}$ at the cusp is the bipartite one between $M = 2$ and the rest of the chain $N - 2$. There occurs a conversion of tripartite to bipartite entanglement.

Bipartite geometric measure: M-spin partition

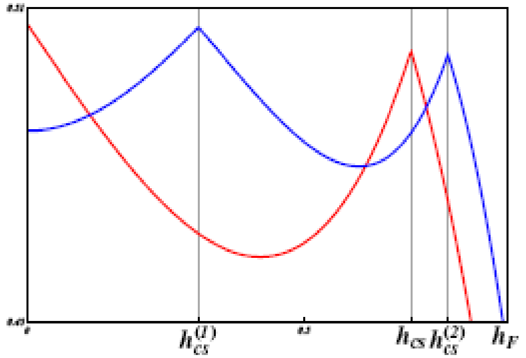
$$E_{G,M}^{(2)} \equiv E_G^{(2)}(i, i+1, \ldots, i+M | 1, \ldots \widehat{i}, \widehat{i+1}, \ldots, \widehat{i+M}, \ldots, N)$$



$E_{G,M}^{(2)}$ as a function of h with $\gamma = 0.5$ for blocks of $M = 3$, $M = 4$, $M = 5$ spins in the main plot, and blocks of $M = 6$, $M = 7$, $M = 8$ spins in the inset

- For the bipartite entanglement $E_{G,M}^{(2)}$ there occur multiple parity inversion points in the range $[0, h_F]$. At fixed M, the number of parity inversion points is M (mirror symmetry $h \to -h$). As M increases, the rightmost parity inversion point tends to the factorization point h_F
  $\Rightarrow$ (h_F is naturally interpreted as the last precursor of the transition)

Bipartite geometric measure: blocks $M = 3$ and $M = 4$



Structure of the transitions between the different local block magnetic orders:

1. $M = 3$-spin partition $(i, i+1, i+2|1, \ldots, \widehat{i}, \widehat{i+1}, \widehat{i+2} \ldots, N)$:
   single transition (odd - even)

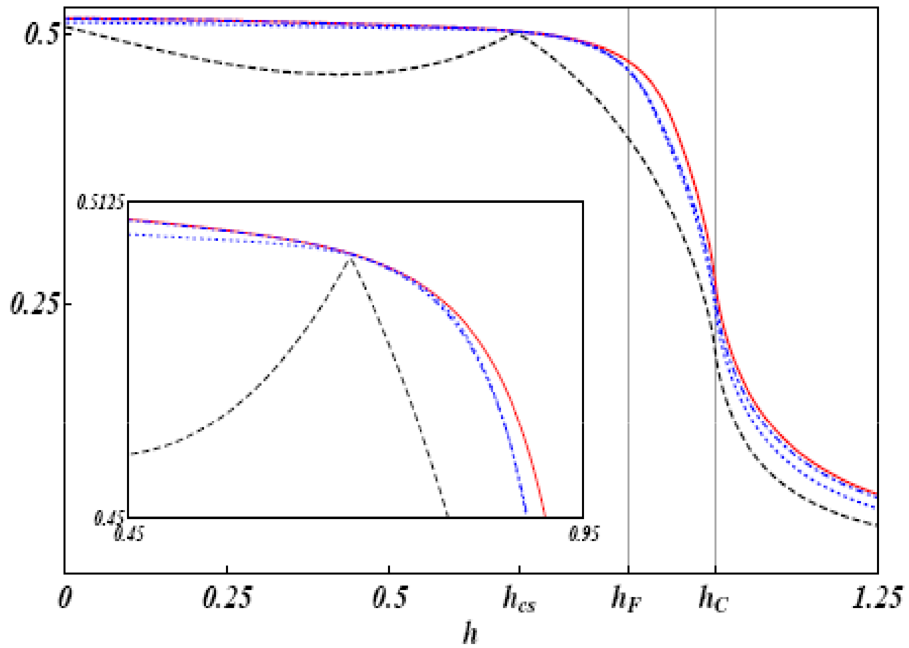
$$|\lambda_{max}\rangle_{i,i+1,i+2} = \begin{cases} \alpha_1(|\uparrow\uparrow\downarrow\rangle + |\downarrow\uparrow\uparrow\rangle) - \beta_1|\uparrow\downarrow\uparrow\rangle + \gamma_1|\downarrow\downarrow\downarrow\rangle\,, & h < h_{cp} \\ \alpha_2(|\uparrow\downarrow\downarrow\rangle + |\downarrow\downarrow\uparrow\rangle) - \beta_2|\downarrow\uparrow\downarrow\rangle + \gamma_2|\uparrow\uparrow\uparrow\rangle\,, & h > h_{cp} \end{cases}$$

2. $M = 4$-spin partition $(i, i+1, i+2, i+3|1, \ldots, \widehat{i}, \widehat{i+1}, \widehat{i+2}, \widehat{i+3}, \ldots, N)$:
   double transition (even - odd - even)

$$|\lambda_{max}\rangle_{i,i+1,i+2,i+3} =$$

$$\begin{cases} \xi_1(|\downarrow\downarrow\uparrow\uparrow\rangle + |\uparrow\uparrow\downarrow\downarrow\rangle) + \xi_2(|\downarrow\uparrow\downarrow\uparrow\rangle + |\uparrow\downarrow\uparrow\downarrow\rangle) + \\ \xi_3|\downarrow\uparrow\uparrow\downarrow\rangle + \xi_4|\uparrow\downarrow\downarrow\uparrow\rangle + \xi_5|\downarrow\downarrow\downarrow\downarrow\rangle + \xi_6|\uparrow\uparrow\uparrow\uparrow\rangle\,, & 0 < h < h_{cp}^{(1)}, \quad h > h_{cp}^{(2)} \\ \varepsilon_1(|\downarrow\uparrow\uparrow\uparrow\rangle \pm |\uparrow\uparrow\uparrow\downarrow\rangle) + \varepsilon_2(|\uparrow\uparrow\downarrow\uparrow\rangle \pm |\uparrow\downarrow\uparrow\uparrow\rangle) + \\ \varepsilon_3(|\uparrow\downarrow\downarrow\downarrow\rangle \pm |\downarrow\downarrow\downarrow\uparrow\rangle) + \varepsilon_4(|\downarrow\uparrow\downarrow\downarrow\rangle \pm |\downarrow\downarrow\uparrow\downarrow\rangle)\,, & h_{cp}^{(1)} < h < h_{cp}^{(2)} \end{cases}$$

$\mathrm{M} = 3$: Comparing $\mathrm{E}_{\mathrm{G}}^{(2)}(3|\mathrm{N}-3)$, $\mathrm{E}_{\mathrm{G}}^{(3)}(1|2|\mathrm{N}-3)$, and $\mathrm{E}_{\mathrm{G}}^{(4)}(1|1|1|\mathrm{N}-3)$



At the parity inversion point the tripartite and quadripartite entanglement are suppressed and converted to bipartite entanglement according to more complex patterns among the different blocks (enhanced freedom). As a consequence, the conversion of the multipartite components is not total, and, at the cusp point, one has an approximate equivalence: $\mathrm{E}_{\mathrm{G}}^{(4)} \gtrsim \mathrm{E}_{\mathrm{G}}^{(3)} \gtrsim \mathrm{E}_{\mathrm{G}}^{(2)}$

Conclusions

- Hierarchical entanglement allows to gain insight in the physical structure of the ground state that is complementary to that provided by the n-point correlation functions. It reveals the existence of a structure of locally (probabilistically) ordered domains as the quantum critical point is approached from the broken symmetry phase. The sequence is determined by the successive oscillations in the parity.

- Meaning of the factorization point as last precursor of a quantum phase transition

- Phenomenon of conversion/transmutation of multipartite entanglement to bipartite entanglement at the parity inversion points

Outlook

- *Preliminary studies suggest the existence of similar structures in XYZ and Heisenberg-like systems*

- *Structure of correlation functions and minimization of statistical mechanics quantities associated to the structure of locally ordered domains*

- *Extension of the analysis to higher-dimensional systems and different universality classes*

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