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Group Actions

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Let G be an affine algebraic group, $G \subset GL(n, \mathbb{C})$ for some n. Then G is said to be reductive if any one of the following equivalent conditions hold:

1) for any f.d. representation V of G, $V = \oplus V_i$, each V_i is a simple (irreducible) G-module.
2) The unipotent radical of G = (the largest closed, connected, normal unipotent subgroup of G) is trivial.

Examples: $G = GL(n,\mathbb{C})$, $SL(n,\mathbb{C})$
$SO(2n,\mathbb{C})$, $SO(2n+1,\mathbb{C})$, $Sp(2n,\mathbb{C})$
or the other "exceptional" groups.
Only concerned with $GL(n,\mathbb{C})$ and $SL(n,\mathbb{C})$.
If V is a possibly infinite-dimensional v.s. on which G acts, we say that V is a <u>rational</u> <u>G-module</u> if
$V = \bigcup_{n=1}^{\infty} V_n$, where V is an increasing union of the V_n's, where each V_n is a finite-dim. G-subspace of V.
Example: $G = GL(n)$ acting on $k[X_1 \cdots X_n]$ by substitution. Let now V be a rational G-module.

We can write $V = V^G \oplus V_G$
where
1) each of V^G and V_G are G submodules of V.
2) $(V_G)^G = 0$, i.e. V_G has no invariants. This complement V_G of V^G is <u>unique</u>.

Now let A be a f.g. $\mathbb{C}$-algebra and let G act on A by $\mathbb{C}$-algebra automorphisms. We write $A = A^G \oplus A_G$ as above. A^G is a $\mathbb{C}$-subalgebra of A. Let $R: A \to A^G$ be the projection of A onto A^G. One can prove $R(xy) = xR(y)$ if $x \in A^G$ and $y \in A$.

So $R: A \to A^G$ is a A^G module map. R is called the Reynold's operator.

We now prove A^G is also a f.g. $\mathbb{C}$-algebra (Hilbert's theorem):

First assume that G acts on a f.d. vector space V, and consider the action of G on $S^\cdot(V^*) := \bigoplus_d S^d(V^*)$.

We have $S^\cdot(V^*) = \mathbb{C}[X_1 \cdots X_n]$, $n = \dim V$.

Define $B = S^\cdot(V^*)^G$ and let R be the Reynolds operator: $S^\cdot(V^*) \to B$

As R is B-linear, $S^\cdot(V^*)$ is a module over B. For any ideal J of B, we have $J S^\cdot(V^*) \cap B = J$ itself.

But $S^{\bullet}(V^{\#}) = \mathbb{C}[X_1 \cdots X_n]$ is a Netherian Ring (Hilbert's Basis Theorem) & hence satisfies the A.C.C. So B also satisfies the A.C.C. But B is a graded ring, $B = \bigoplus_d B_d$, where each $B_d = S^d(V^{\#})^G$. Now a graded ring, which is Netherian is f.g. as an algebra over $\mathbb{C}$, hence $B = S^{\bullet}(V^{\#})^G$ is a f.g. algebra.

Now let A be any f.g. algebra over $\mathbb{C}$, on which G act. Choosing generators $f_1 \cdots f_n$ for A, we get a surjection of G-algebras,

$$\mathbb{C}[X_1 \cdots X_n] \twoheadrightarrow A \twoheadrightarrow 0$$

Taking G - invariants, (which preserve surjections), we get

$$\mathbb{C}[X_1 \cdots X_n]^G \twoheadrightarrow A^G \twoheadrightarrow 0.$$

f.g. , so A^G also f.g.

Properties of the map $\pi : \mathrm{Spec}(A) \to \mathrm{Spec}(A^G)$.

1) π is surjective, and affine, i.e. if $U \subset \mathrm{Spec}(A^G)$ is affine, then $\pi^{-1}(U)$ also affine and

$$\Gamma(U, \mathcal{O}_U) = \Gamma(\pi^{-1}(U), \mathcal{O}_{\pi^{-1}(U)})^G$$

2) If x and y are points of $X = \mathrm{Spec}(A)$,

then $\pi(x)=\pi(y)$ if and only if $\overline{O(x)} \cap \overline{O(y)} \neq \phi$.

(Very rarely is $\pi: X \to Y$ an orbit map, that is $\pi(x)=y \Leftrightarrow y=gx$ for some $g \in G$. There exist non-closed orbits!!)

In general, points in Y parametrize the <u>closed</u> <u>orbits</u> in X.

Now let G act on a vector space V let $P(V)$ = lines in V, the associated proj. space. For $x \in P(V)$, let $\hat{x}$ be any in V projecting to x. The following are equivalent:

1) There exists some $f \in S^d(V^\#)^G$,

$d \geq 1$, such that $f(\hat{x}) \neq 0$

2) The orbit closure $\overline{G(\hat{x})}$ in V does <u>not</u> contain 0, the origin in V.

Such a point $\hat{x}$ in V (or x in $P(V)$) is called <u>semi-stable</u> for the action of G on V.

Suppose $\exists$ $\hat{x}$ in V such that the orbit map $\pi_{\hat{x}} : G \to V$, $g \to g\hat{x}$ is <u>proper</u> as a map of Varieties. Then $\hat{x}$ (or x in $P(V)$) is called <u>properly stable</u> or just <u>stable</u>

Let G act on V, and hence on $P(V)$.

Theorem: (Mumford): 1) Both $P(V)^s$:= stable points and $P(V)^{ss}$:= semistable points are open and G-invariant in $P(V)$.

2) $P(V)^{ss}/G$ is a good quotient and $P(V)^{ss}/G$ is a projective variety. This means: denote $P(V)^{ss}/G$ by M. Then $P(V)^{ss} \to M$ is surjective, affine, G-invariant map. And for some $N >> 0$, the line bundles $\mathcal{O}_{P(V)}(N)$ descends to a line bundle L on M, hence M is projective.

$\exists$ an open subset M^s of M
s.t. if we take $\pi : P(V)^{ss} \to M$,
then $\pi^{-1}(M^s) = P(V)^s$, and
$P(V)^s \to M^s$ is a <u>geometric</u> <u>quotient</u>
i.e. it is a good quotient <u>and</u> the
orbits under G are <u>closed</u> in $P(V)^s$.
So M^s is just the space of all orbits in $P(V)^s$.

Hilbert-Mumford criterion: Let λ:
$G_m \to G$ be a 1-parameter subgroup
(1 PS). We write V as a direct
sum of eigenspaces for the action of λ

$V = \oplus V_i$, where on each V_i

$\lambda(t)v = t^i v$ for any $v \in V_i$.

Let $v \in V$ and $v = v_i + v_{i+1} \cdots\cdots$

be the decomposition of v into eigenvectors.

Define $\mu(\lambda, v) = i$.

Hilbert-Mumford: v is semistable (stable, properly stable) $\Longleftrightarrow$ for any non-trivial 1PS: $\lambda\ G_m \to G$, we have $\mu(\lambda, v) \leq 0\ (< 0)$.

Stable and semistable bundles on curves

Let X be a nonsingular projective curve of genus $g \geq 2$. Let V be a vector bundle on X of rank r. Define $\deg V = \deg \Lambda^r V$ ($\Lambda^r V$ line bundle on X), denoted by d.

Define <u>slope</u> of $V := \mu(V) = d/r$.

<u>Defn</u>: V is stable (semistable) $\Leftrightarrow$ for all proper subbundles W of V, we have $\mu(W) < (\leq)\ \mu(V)$.

L any line bundle of degree 1 on X, define $V(n) = V \otimes L^n$. And $\chi_V(n) =$

$$\chi[V(n)] = \deg V(n) + \operatorname{rk} V(n)(1-g)$$
$$= \deg V + rn + r(1-g)$$

Now let X be a nonsingular projective variety of dim d, H a very ample line

on X. Let V be a torsion-free sheaf on X
One defines a line bundle $c_1(V)$ on X
Define slope of $V := \mu(V) = \dfrac{c_1(V)\cdot H^{d-1}}{\operatorname{rk} V}$

Defn: V is <u>μ-stable</u> (<u>μ-semistable</u>) iff $\forall$ proper subsheaves W of V, we have $\mu(W) < (\leq)\ \mu(V)$.

If V is any torsion-free sheaf on X, define $\chi_V(n) = \chi[V \otimes H^n]$, a poly. of degree d in n.

$$\chi_V(n) = \sum_{i=0}^{d} a_i \binom{n+i}{i}, \; a_i \text{ integers}$$

Define V to be <u>χ-stable</u> (χ-semistable) iff $\forall$ proper subsheaves W of V, we have $\dfrac{\chi_W(n)}{\operatorname{rk} W} < (\leq)\ \dfrac{\chi_V(n)}{\operatorname{rk} V}$, $\forall n \gg 0$

μ-stable $\Rightarrow \chi$-stable $\Rightarrow \chi$ semistable $\Rightarrow \mu$ semistable, and no implication can be reversed.

Assume X is a curve on a polarized variety (X, H), with $\dim X \geq 2$. Let V be a vector-bundle (or a torsion free sheaf) which is not semistable (μ definition). Then $\exists !$ a filtration

$0 = V_0 \subset V_1 \cdots \subset V_{n-1} \subset V_n = V$

with 1) each $V_i | V_{i-1}$ μ-semistable

2) $\mu(V_i | V_{i-1}) > \mu(V_J | V_{J-1})$ if $i < J$, (slopes are strictly decreasing

3) each $V_i | V_{i-1}$ is torsion-free

Suppose V is not $\mathcal{X}$-semistable. Then there exists a unique filtration

$$0 = V_0 \subset V_1 \cdots \subset V_{n-1} \subset V_n = V$$

with 1) each V_i / V_{i-1} is torsion-free

2) each V_i / V_{i-1} is $\mathcal{X}$-semistable

3) $\mathcal{X}_{V_i/V_{i-1}}(n) / \operatorname{rk} V_i/V_{i-1} > \mathcal{X}_{V_J/V_{J-1}}(n) / \operatorname{rk} V_J / V_{J-1}$ if $i < J$.

These are called the Harder-Narasimhan filtrations for μ and $\mathcal{X}$ nonsemistability respectively.

Due to uniqueness, the H-N filtrations are rational: if V and X are defined over $k \neq \bar{k}$, and over $\bar{k}$ $\bar{V}$ is not semistable over $\bar{X}$, then $H.N.(\bar{V})$ is already defined over k (for both μ and $\mathcal{X}$).

Let X be a curve or a polarized variety (X, H) with $\dim X \geq 2$. Suppose we are given a collection of torsion-free sheaves $\{V\}$ on X. Then $\{V\}$ is said to be a <u>bounded family</u> on X if any one of the 3 equivalent conditions hold:

1) There exists a scheme T of finite type over $\mathbb{C}$ and a sheaf $\mathcal{V}$ on $X \times T$ such that for and $V \in \{V\}$, $\exists\, t \in T$ such that $V \simeq \mathcal{V}|_{X \times (t)}$.

2) There exists a fixed sheaf F on X such that every V is a quotient of X and the Hilbert Polynomials $\chi_V(n)$, $V \in \{V\}$ are <u>finite</u>.

3) $\exists$ a positive integer m_0 such that for all $m \geq m_0$ and for all V in $\{V\}$ we have $H^i(X, V(m)) = 0$ for all $i > 0$ and $H^0(X, V(m))$ generates $V(m)$. And the Hilbert Polynomials $\chi_V(n)$, $V \in \{V\}$ are <u>finite</u>.

Definition - Construction of Quot Schemes: Let (X, H) be any polarized variety. Let V be any coherent sheaf on X. We parametrize <u>all</u> <u>quotients</u> (or <u>subs</u>)

of V, $V \to F \to 0$, such that Hilbert poly. of F, $\chi_F(n)$ is <u>fixed</u>.

Fix a poly. $P(n)$ and let

$0 \to S \to V \to F \to 0$ be a

quotient of V, with $\chi_V(n) = P(n)$.

Then we show $\exists\, m_0$ such that for all $m \geq m_0$ and for all $i > 0$

1) $H^i(X, S(m)) = H^i(X, F(m)) =$

~~$H^i(X, F($~~ $H^i(X, V(m)) = 0$

and $H^0(S(m)$, $H^0(V(m))$ $H^0(F(m))$ <u>generate</u> $S(m)$, $V(m)$ and $F(m)$ resp.

As $\chi_V(n)$ and $\chi_F(n)$ are fixed, $\chi_S(n)$ is also fixed. So $H^0 S(m)$ and $H^0 F(m)$ are constant dimensional vector spaces for any quotient $V \to F$ and fixed $m \geq m_0$

Put $\dim \overset{0}{H} S(m) = m_1$ $\dim F(m) = m_2$

Then $\dim H^0 V(m) = m := m_1 + m_2$

Each quotient F of V with $\chi_V(n) = P(n)$ gives a point of $Gr(m, m_2)$, the Grassmannian of m_2 dimensional quotients of a fixed m-dimensional vector space. This we get a projective scheme $Quot(X, V, P)$ or just $Quot$ if X, V and P are fixed.

Properties: On $X \times Quot$, there exists a surjection : $p_1^{\#}(V) \to G$, with G a sheaf on $X \times Quot$, <u>flat</u> over $Quot$.

And for <u>any</u> scheme T, and for any surjection $p_1^{\#}(V) \to H$ on $X \times T$, with H flat over T, there exists a

<u>unique</u> morphism: $f : T \to \text{Quot}$
such that $(id_X \times f)^{\#} \mathcal{G} \simeq \mathcal{H}$.
$\text{Quot}(X, V, P)$ is a <u>representable</u>
functor.

Construction of Moduli Spaces of
Semistable Bundles on Curves:
Let X be a curve and let
(V) be the set of all semistable
bundles of rank r and degree d.
For any $V \in \{V\}$, $\chi_V(m) = \chi[V(m)]$
$= \deg V(m) + r(1-g) = d + rm + r(1-g)$.
We show that $\{V\}$ is a bounded
family: $\exists\, m_0$ s.t $\forall\, m \geq m_0$ and
all $V \in \{V\}$, we have

$H^1(V(m)) = 0$ and $H^0(V(m))$ generates $V(m)$. Then $n = \dim H^0(V(m))$ is constant for $V \in \{V\}$. Consider $\mathrm{Quot}(X, \mathcal{O}_X^n, P)$ where $P(m) = d + rm + r(1-g)$, the Hilbert Polynomial of $V \in \{V\}$

Let R^s (R^{ss}) be the open subset of Quot s.t. 1) if $x \in R^s$ (R^{ss}) then the quotient V_x of $\mathcal{O}_X^n$ is stable (semistable)

2) the canonical map $H^0(\mathcal{O}_X^n) = \mathbb{C}^n \longrightarrow H^0(V_x)$ is an isomorphism. Then $G := GL(n)$ acts on Quot and on R^s, R^{ss}

Let R be the open subset of Quot s.t $x \in R \iff$

$$H^0(\mathcal{O}_X^n) \to H^0(V_x) \text{ is an iso.}$$

(no condition of stability for $x \in R$)

$\exists$ a map C ("the covariant")

$C : R \longrightarrow P(W)$, W some vector space on which G acts s.t.

1) C is G inv.

2) If $x \in R^s$, then $C(x) \in P(W)^s$

3) For $x \in R$, then $C(x) \in P(W)^{ss}$

$\iff x \in R^{ss}$.

3) shows that
$$R^{ss} \hookrightarrow P(W)^{ss}$$
$P(W)^{ss}/G$ exists as a good quotient. So R^{ss}/G exists as a good quotient.

2) also shows that, since $P(W)^{s}/G$ exists as a geometric quotient (orbit space), R^{s}/G exists as a geometric quotient (orbit space)

Defn: $R^{ss}/G = M(r, d+rm)$
the coarse moduli space of semistable bundles of rank r and degree $d+rm$
$R^{s}/G = M^{s}(r, d+rm)$, the coarse

moduli space of stable bundles of rank r and degree $d+rm$.

Properties: Let $\mathcal{V}$ on $X \times T$ be a family of stable bundles of rank r and degree $d+rm$. We get a set-theoretic map: $c \; T \longrightarrow M^s$ given by $t \Rightarrow$ iso. class of $\mathcal{V}/X \times \{t\}$.

1) c is a morphism $T \longrightarrow M^s$, functorially in T

2) pts. of M^s classify the iso. classes of stable bundles of rank r and degree $d+rm$

Define $\underline{F}$: Schemes $\rightarrow$ Sets

$\underline{F}(T)$ = families of stable bundles of rank r, degree $d+rm$ on $X \times T$

$\underline{G}(T) = \mathrm{Hom}(T, M^s)$

a) We get a <u>morphism of functors</u>:

$\underline{F} \longrightarrow \underline{G}$

that is, for every T, $\underline{F}(T) \to \underline{G}(T)$ functorially in T.

b) And for $T = \mathbb{C}$, $\underline{F}(\mathbb{C})$ is <u>bijective</u> with $\underline{G}(\mathbb{C})$.

For the whole M, semistable of rank r and degree $d + rm$ we get maps $\underline{F}(T) \longrightarrow \underline{G}(T)$.

<u>But</u> $\underline{F}(\mathbb{C}) \to \underline{G}(\mathbb{C})$ is <u>not</u> <u>a</u> <u>bijection</u>!

Does there exist a "universal family" $\mathcal{U}$ on $X \times M^s$ s.t: for every T, and for every family $\mathcal{V}$ on $X \times T$

$\mathcal{V}$ on $X \times T = (id_X \times c)^\#(\mathcal{U})$, where $c : T \to M^s$ is the classifying morphism.

This is true $\iff (r, d) = 1 \iff M = M^s$, (every semistable is stable).

$\iff$ if there exists a $GL(n)$ line bundle L on R^s such that the centre $\{\lambda\, Id\}$ of $GL(n)$ acts by weight 1 on the fibres of the universal bundle on $X \times R^s$.

References: Chapters 3 and 5 of:

Peter Newstead: "Lectures on Introduction to Moduli Problems and Orbit Spaces", Tata Institute of Fundamental Research, Bombay, 1978

Construction of Moduli of semistable sheaves on Surfaces

Let (X, H) a smooth projective polarized over $\mathbb{C}$. H is a very ample line bundle on X. Let $\{V\}$ be the set of all χ-semistable torsion free sheaves on X. Recall that V is χ-stable (χ-semistable) $\Leftrightarrow$ for every

$$W \subset V, \quad \frac{\chi_W(n)}{rk\, W} < \left(\leq \frac{\chi_V(n)}{rk\, V} \right) \forall n >> 0.$$

We may assume that each $V \in \{V\}$ has been twisted with $m >> 0$ so that

1) $H^0(V)$ generates V
2) $H^i(V) = 0$ for $i = 1, 2$.

Then as $(rk\, V, c_1, c_2)$ are fixed, we $H^0(V)$ is constant, $\forall\, V \in \{V\}$

Let $n = \dim H^0(V)$ for $V \in \{V\}$

We further assume that $c_1(V)$ is a constant line bundle, for $V \in \{V\}$, call this L. As each V is globally generated, we get a map

$$T(V) : \Lambda^r[H^0(V)] \longrightarrow H^0(\det V) = H^0(L)$$

for each V. Note that $H^0(V)$ is a constant vector space, say W, of dim n. So we get $T(V) : \Lambda^r(W) \to H^0(L)$ for each V

The group $G = SL(r)$ [or $GL(r)$] acts on $\mathrm{Hom}(\Lambda^r(W), H^0(L))$, Let Z be the projectivization of this space.

Let $H = \cancel{\mathrm{Hilb}}\ \mathrm{Quot}(X, \mathcal{O}_X^r, P)$ be the Quotient scheme of all quotients of $\mathcal{O}_X^r$, with Hilbert polynomial $P = \chi_V(n)$

Let $R \supset R^{ss} \supset R^{s}$ be as before i.e. $R = x \in \mathrm{Quot}$ such that $\mathcal{O}_X^r \to F_x \to 0$, F_x is torsion-free and $H^0(\mathcal{O}_X^r) \simeq H^0(F_x)$. For R^{ss} (resp. R^s), we further demand that F_x is χ-semistable (χ-stable).

So $T : R \to Z$.

Basic properties of T

1) T is proper, injective map.

2) $T(R^s) \subset Z^s$

3) $T(V) \in Z^{ss}$ if and only if V is χ-semistable.

Again, by general theorems, the good quotient $R^{ss}/\mathcal{G}$ exists and the geometric quotient $R^s/\mathcal{G}$ exists, and $R^s/\mathcal{G}$ is open in $R^{ss}/\mathcal{G}$. Here $R^s/\mathcal{G}$ parametrizes the set of isomorphism classes of $\mathcal{X}$-stable sheaves with fixed rk r, fixed det. L and fixed c_2. Suppose V is strictly semistable, i.e. semistable but not stable: Then $\exists$ a filtration

$$0 = V_0 \subset V_1 \cdots \subset V_{m-1} \subset V_m = V$$

such that

1) Each V_i/V_{i-1} is torsion-free

2) for each i, $\chi_{V_i/V_{i-1}}(n)/\operatorname{rk} V_i/V_{i-1}$

$= \chi_V(n)/\operatorname{rk} V$. Such a filtration is called the <u>stable filtration</u> or the <u>Jordan-Holder filtration</u>. It is <u>not</u> <u>unique</u>, but $gr(V) := \bigoplus_i (V_i/V_{i-1})$ is <u>unique</u>. 2 sheaves V_1 and V_2 are called <u>S-equivalent</u> if $gr(V_1) \simeq gr(V_2)$ as sheaves on X. R^{ss}/G parametrizes the set of S-equivalence classes of $\mathcal{X}$ - semistable sheaves on X. Note that V is stable $\Leftrightarrow V \simeq gr(V)$.

$R^{ss} / G := M$ is the coarse moduli space of $\mathcal{X}$-semistable sheaves on X, with num. invariants r, $\det V$, c_2. Let T be any scheme and $\mathcal{V}$ a sheaf on $X \times T$ such for each $t \in T$, $\mathcal{V}_t$ is semistable with num. invariants r, $\det V$, c_2. Then we get a map $c : T \to M$, the classifying map. Similiarly we get a map $c : T \to M^s$ for each family of stable sheaves on X, parametrized by T,

Existence of Universal-families on M^s

Consider M^s with r, c_1 and c_2 fixed.
Write $\chi_V(n) = a_0 + a_1\binom{n+1}{1} + a_2\binom{n+2}{2}$
with the a_i integers.

Prop: If $g.c.d.(a_0, a_1, a_2) = 1$
then there exists a universal or
a Poincaré family $\mathcal{V}$ on $X \times M^s$.

M^s is a fine moduli space.
Unlike the situation for curves, the
converse does not hold in general
(not known).

Local Properties of M^s.

We write $M^s = R^s / G$. Let V be a stable bundle in R^s and denote by V again its image in M^s. By Luna's étale slice theorem $\exists$ $S \subset R^s$, with $V \in S$ and $T \subset M^s$, with $V \in T$ and a map $\pi : S \to T$, which is étale at V i.e. $\pi^{\#}$ induces an isomorphism:

$$(\mathcal{O}_{T,V})^{\wedge} = (\mathcal{O}_{M^s,V})^{\wedge} \simeq (\mathcal{O}_{S,V})^{\wedge}$$

Put $R := (\mathcal{O}_{M^s,V})^{\wedge} \simeq (\mathcal{O}_{S,V})^{\wedge}$

If $\mathcal{U}$ is the universal family on $X \times R^s$ denote by $\mathcal{U}$ again the restriction to $X \times S$.

So we get a family on $X \times \mathcal{O}_{S,V}$

and hence on $X \times (\mathcal{O}_{S,v})^{\wedge} = X \times^{\mathrm{Spec}}_{n} R$.

Question: When is R <u>smooth</u> or <u>non-singular</u>

R is smooth $\Longleftrightarrow$ for all Artin local rings C, for any ideal $J \subset C$, any map $\bar{f} : R \to \bar{C} := C/J$ <u>lifts</u> to a map $f : R \to C$

$$\begin{tikzcd} R \arrow[r] \arrow[dr, "\bar{f}"'] & C \arrow[d] \\ & C/J) \end{tikzcd}$$

$\bar{f}$ gives a map $\mathrm{Spec}(C/J) \to \mathrm{Spec}\, R$

So a family $\bar{f}^{\#}(\mathcal{U})$ on $X \times \mathrm{Spec}(C/J)$

So $\bar{f}$ extends to a map $f : R \to C$

$\Longleftrightarrow$ $\bar{f}^{\#}(\mathcal{U})$ extends to a family say W, on $X \times \mathrm{Spec}(\mathbb{C})$. In general, the <u>obstructions</u> to lifting bundles "modulo nilpotents" lie in the obstruction space $H^2(X, \mathrm{End}(V))$.

So if $H^2(X, \mathrm{End}(V)) = 0$, then $\mathcal{M}^s$ is smooth at V, or $\mathcal{O}_{\mathcal{M}^s, V}$ is <u>regular</u>.

Similarly, we may consider $\mathcal{M}^s_{\det \text{ fixed}}$, the <u>fixed</u> - <u>determinant</u> moduli space. Now the obstructions to lifting fixed-determinant families of vector bundles "modulo nilpotents" lies in $H^2(X, \mathrm{End}^0(V))$, where $\mathrm{End}^{\circ}(V)$ is the bundle of trace 0 endomorphisms.

We have

$$0 \to \mathrm{End}^{\circ}(V) \to \mathrm{End}(V) \xrightarrow{\mathrm{Tr}} \mathcal{O}_X \to 0$$

(this sequence splits on X).

Now in general, the Zariski tangent space to V in M^s is given by $H^1(X, \mathrm{End}\, V)$. Similarly, the tangent space to V in $M^s_{\text{fixed, det}}$ is $H^1(X, \mathrm{End}^{\circ} V)$. Now consider the map: $\det : M^s \to \mathrm{Pic}(X)$, given by $V \to \det V$

The fibre over $L \in \mathrm{Pic}(X)$ is $\mathcal{M}^s_L = \{ V \in \mathcal{M}^s \mid \det V = L \}$.

A similar reasoning shows that if $H^2(X, \mathrm{End}^0 V) = 0$ then $\mathcal{M}^s_L$ is smooth at V. Suppose this holds, i.e. $H^2(X, \mathrm{End}^0 V) = 0$. Then $\mathcal{M}^s_L$ is smooth at V. But $\mathrm{Pic}(X)$ is always smooth. So we have

Theorem: If $H^2(X, \mathrm{End}^0 V) = 0$ $\forall V \in \mathcal{M}^s$, then $\mathcal{M}^s$ is smooth at all points. In general we have:

$\dim H^1(\mathrm{End}\, V) \geq \dim(\mathcal{M}^s \text{ at } V)$

$\geq \dim H^1(\mathrm{End}\, V) - \dim H^2(\mathrm{End}\, V)$.

Construction of Bundles on Surfaces

Let X be a surface, V a globally generated vector bundle of rank r, with $r > 2 = \dim X$. Then we have

1) $0 \to \mathcal{O}_X^{r-2} \to V \to V' \to 0$

2) For a general choice of $r-1$ global sections in $H^0(X, V)$ we have

$$0 \to \mathcal{O}_X^{r-1} \to V \to V^1 \to 0$$

with $V' = L \otimes I_Z$, $L \in \mathrm{Pic}(X)$

Z codim 2 subscheme.

3) For a general choice of r global sections in $H^0(X, L)$

we have

$$0 \to \mathcal{O}_X^{r} \to V \to L \to 0$$

with L a line bundle on a

smooth curve $C \subset X$

The Cayley - Bacharach property

Let X be a smooth projective surface, L and $M \in \mathrm{Pic}(X)$, $Z = \{z_1, \dots, z_n\}$ a finite set of points on X. Then $\exists$ an extension:

$$0 \to L \to V \to M \otimes I_Z$$

with V a vector bundle if and only if: every $s \in H^0(L^* \otimes M \otimes K)$ vanishing at all but 1 point of $\{z_1 \dots z_n\}$ also vanishes at that point.

Example: Let C_1 and C_2 be 2 cubics in $\mathbb{P}^2$, meeting at $z_1, \dots, z_9$. Then any other cubic D passing through 8 of the 9 points

$(z_1 \cdots z_9)$ also passes through the ninth point.
Apply theorem to $L = \mathcal{O}_{\mathbb{P}^2}(-3)$
$M = \mathcal{O}_{\mathbb{P}^2}(3)$. So $L^{\#} \otimes M \otimes K_{\mathbb{P}^2}$
$= \mathcal{O}_{\mathbb{P}^2}(3)$!!

Example: Suppose x is on a surface X and <u>every</u> $s \in H^0(L^* \otimes M \otimes K)$ vanishes at x. Then $(L^{\#} \otimes M \otimes K_X, x)$ has the C.B. property.

Example: If x and y are 2 pts on X which <u>cannot be seperated</u> by $H^0(L^{\#} \otimes M \otimes K_X)$, then $L^{\#} \otimes M \otimes K_X, \{x, y\}$ has the C.B. property

Theorem: X be smooth, $L \in \mathrm{Pic}(X)$
Then $\exists\ c_0$ s.t for all $c \geq c_0$,

$\exists$ a μ-stable bundle V of rank 2, with $\det(V) = L$ and $c_2(V) = c_2$.

Elementary Transformations

Suppose X is a surface, $C \subset X$ a smooth curve. V a bundle on X and W a bundle on C.

Then $\exists$ a sequence

$$0 \to V_1 \to V \to W \to 0$$

on X, V_1 is an <u>elementary</u> transformation of V along W.

Example: If $C \subset X$ are above, let $\Omega_X(\log C)$ ~~be~~ the 1 forms with logarithmic poles along C. Then $\exists$

$$0 \to \Omega_X(\log C) \to \Omega_X(C) \to \Omega_C(C) \to 0$$

Now let V be a rk. r bundle on X. Then $E^*(nH)$ is globally generated for $\forall\, n >> 0$. We get

$$0 \to \mathcal{O}_X^r \to E^{\#}(nH) \to M \to 0$$

M a line bundle on some curve $C \subset X$. Dualize and twist by (nH) to get

$$0 \to V \to \mathcal{O}_X(nH)^{\oplus r} \to L \to 0$$

where $L \in \mathrm{Pic}(C)$.

Theorem: Every V of rank r is an elementary transformation of $\mathcal{O}_X(nH)^{\oplus r}$ along a line bundle on some curve.

Theorem: Given $L \in \mathrm{Pic}(X)$, and integer c_0, $\exists$ a μ-stable rk r V with $\det(V) \simeq L$ and $c_2(V) \geq c_0$.

Examples of Moduli Spaces

1) Let $X \subset \mathbb{P}^3$ be a general hypersurface of degree 4.

Then X is a K-3 surface:

i) $H^1(X, \mathcal{O}_X) = 0$

K_X is trivial, hence $H^2(X, \mathcal{O}_X)$ has dim. 1. $\mathrm{Pic}(X)$ generated by $\mathcal{O}_X(1)$ [restriction of $\mathcal{O}_{\mathbb{P}^3}(1)$ to X].

Let $M = M(2, \mathcal{O}_X(-1), c_2)$ be the moduli space of μ-stable sheaves of rank 2, det $\mathcal{O}_X(-1)$ and $c_2(V) = c_2$.

Example: $M(2, \mathcal{O}_X(-1), 3) \cong X$

Sketch: Let $V \in M$. Then $\exists$ $x \in X$ and an exact sequence

$$0 \to V \to \mathcal{O}_X^3 \to I_x(1) \to 0$$

where I_x is the ideal sheaf of x
So V determines $x \in X$.

Take any $x \in X$, let V_x be the kernel of the surjection

$$H^0(X, I_x(1)) \otimes \mathcal{O}_X \twoheadrightarrow I_x(1)$$

we check: V_x is stable, $\mathrm{rk}\, V = 2$ and $c_2(V) = 3$.

By writing a universal family on $X \times M$, we see that each is the moduli space of the other.

Example: Let $\pi : X \to \mathbb{P}^1$ be an elliptic K-3 surface with a section $\sigma : \mathbb{P}^1 \to X$. Put $H = \sigma + 3f$. Put $H_m = H + mf$, for $m \gg 0$

Let $M = M(2, \mathcal{O}_X(\sigma - f), 1$

Then $M \cong X$.

We check there is a unique non-split extension

$$0 \to \mathcal{O}_X(f) \to G \to \mathcal{O}_X(\sigma - f) \to 0$$

Fix $x \in X$, and put $F := \pi^{-1}(\pi(x))$

The above sequence restricted to F does not split, we check $\exists!$

$\phi : G \to I_{F,x}(2\sigma)$, where $I_{F,x}$ is the ideal sheaf of x in F.

$E_x := \ker \phi$.

Check that E_x is stable with the correct Chern classes.

To see that $X \cong M$, we write down a universal family on $X \times M$.

Construction of stable bundles on $\mathbb{P}^n$.

On $\mathbb{P}^n$ we have a sequence the Euler sequence

$$0 \to \mathcal{O} \to \mathcal{O}(1)^{n+1} \to T_{\mathbb{P}^n} \to 0.$$

This gives on $\mathbb{P}^n \times \mathbb{P}^n$ a vector bundle W and a section $s \in H^0(W)$ s.t. $Z(s) = \text{diagonal } \Delta \subset \mathbb{P}^n \times \mathbb{P}^n$

This in turn gives a spectral sequence, for sheaf E on $\mathbb{P}^n$,

$$E_1^{p,q} = H^q(\mathbb{P}^n, E(p)) \otimes \Omega^{-p}(-p)$$

with $E_\infty^{p,q} = 0$ if $p+q \neq 0$ and $\bigoplus_0^n E_\infty^{-p,p}$ is the associated graded of a filtration of E.

Example: Let E be a rk. 2 stable bundle on $\mathbb{P}^2$ with $c_1(E) = -1$ and $c_2(E) = 1$.

$$0 \to E_1^{-2,1} \to E_1^{-1,1} \to E_1^{0,1} \to 0$$

This is a Monad on $\mathbb{P}$:

a sequence of bundles
$0 \to A \to B \to C \to 0$

NOT EXACT but

A a subbundle of B, and

$A \subset \mathrm{Ker}(B \to C) := K$.

K/A is called the cohomology bundle

of the monad.

Continuing with our example,
we compute $E_1^{-2,1}$, $E_1^{-1,1}$ and $E_1^{0,1}$
we get $E \simeq \Omega^1_{\mathbb{P}^2}(1)$.

Example: E be rank 2 stable on $\mathbb{P}^2$, $c_1(E)=0$

and $c_2(E)=2$

we get E as the cohomology of a monad

$$\mathcal{O}(-1)^{\oplus 2} \longrightarrow \Omega^1(1)^{\oplus 2} \longrightarrow \mathcal{O}$$

Example: Let E be a rk. r stable on $\mathbb{P}^2$

Then E is the cohomology of

$$0 \to H \otimes \mathcal{O}_{\mathbb{P}^2}(-1) \to K \otimes \mathcal{O}_{\mathbb{P}^2} \to L \otimes \mathcal{O}_{\mathbb{P}^2}(1) \to 0$$

where H, K, L are f.d. $\mathbb{C}$ vector spaces. Here $c_1(E) = 0$.

$\underline{M_{\mathbb{P}^2}(0, n)}$

Let V be a 3 dim. space over $\mathbb{C}$ and $\mathbb{P} = \mathbb{P}(V)$. Let E be a rank 2 bundle on $\mathbb{P}$ with $c_1(E) = 0$, $c_2(E) = n$. Again we get E is the cohomology of a monad

$$0 \to H \otimes \mathcal{O}_{\mathbb{P}}(-1) \xrightarrow{a} K \otimes \mathcal{O}_{\mathbb{P}} \xrightarrow{b} H' \otimes \mathcal{O}_{\mathbb{P}}(1) \to 0$$

$H = H^1(\mathbb{P}, E(-2))$, $K = H^1(\mathbb{P}, E \otimes \Omega^1)$

$H' = H^1(\mathbb{P}, E(-1)) \simeq H^*$

Now E is a rk 2 bundle, with $\det E \simeq \mathcal{O}_{\mathbb{P}}$. So $\exists$ an iso. $f : E \to E^*$ with $f^* = -f$.

The space K carries a non-deg. symplectic 2 form $q: K \to K^*$ with $q^\# = -q$. So E is the cohomology of a self-dual monad

$$0 \to H \times \mathcal{O}_{\mathbb{P}}(-1) \xrightarrow{a} K \otimes \mathcal{O}_{\mathbb{P}} \xrightarrow{a^\# \circ (q \otimes id)} H^\# \otimes \mathcal{O}_{\mathbb{P}}(1) \to 0$$

The map $a : H \otimes \mathcal{O}_{\mathbb{P}}(-1) \to K \otimes \mathcal{O}_{\mathbb{P}}$ is in $H^0(\mathbb{P}, H^* \otimes \mathcal{O}_{\mathbb{P}}(1) \otimes K =$ $H^* \otimes K \otimes V^\#$. Regard this a as an element $\alpha : V \to L(H, K)$

So for $v \in V$, we get $\alpha(v) : H \to K$

$\alpha(v)^t : K^* \to H^* = K \to H^*$

Satisfying 3 conditions

(E 1): $\alpha(v) : H \to K$ is injective for $v \neq 0$

(E 2): $\alpha(v)^t \circ \alpha(v) : H \to H^*$ is zero

(E.3): $\hat{\alpha} : V \otimes H \to K$ is surjective

$$\hat{\alpha}(v \otimes h) = \alpha(v) h$$

Now take V, H, K be vector spaces of dim 3, n and $2n+2$. On K fix a non-degenerate symplectic form $q: K \to K^{\#}$, $q^{\#} = -q$. Let $Sp(q)$ be the associated group. Then $G := GL(H) \times Sp(q)$ acts on $L(V, L(H,K))$. Let $P = \alpha \in L(V, L(H,K))$, α has properties (E1), (E2) (E3). P is a G invariant subset. Then P/G is isomorphic to $M_{\mathbb{P}^2}(2,0,n)$.

Moduli Spaces of Sheaves on K-3 Surfaces

Let X be a K-3 surface.

We define the Euler characteristic of the pair of sheaves E, F by :=

$$\chi(E,F) = \sum_i (-1)^i \dim \mathrm{Ext}^i(E,F)$$

Defn: If $v = \oplus v_i \in H^{ev}(X, \mathbb{Z})$

$= \oplus_i H^{2i}(X, \mathbb{Z})$, then $v^{\vee} := \oplus (-1)^i v_i$

For example if $v = v_0 + v_1 + v_2$, $v_i \in H^{2i}$,

then $v^{\vee} = v_0 - v_1 + v_2$

Lemma: $\chi(E, F) = \int_X (ch\, E)^{\vee} ch(F) td\, X$

Definition: Let E be a coherent sheaf on X. Then the Mukai vector of E

$= ch(E) . \sqrt{td(x)}$. Recall

that $ch(E) = rk\, E + c_1 + \frac{1}{2}(c_1^2 - 2c_2)$

Definition Define (v, w) by

$$(v, w) = -\int v^{\vee} . w$$

$\langle v, w)$ is a bilinear form on $H^{ev}(X, \mathbb{Q})$

For sheaves E and F, we have $\chi(E, F)$

$= -(v(E), v(F))$.

Definition: For a K-3 Surface X,

if E is a torsion-free sheaf, ~~then~~

with $\mathrm{rk}\, E = r$, $c_1(E) = c_1$, $c_2(E) = c_2$, define $v(E) := (r, c_1, c_1^2/2 - c_2 + r)$

We use the notation $M(v)$ for the moduli space of semistable torsion-free sheaves of rank r, $c_1(E) = c_1$, $c_2(E) = c_2$.

Note M^s, the moduli of stable sheaves is always smooth on a K-3 surface!!

Theorem: If $(v,v) + 2 = 0$, or equivalently $(v,v) = -2$, then if M^s is nonempty, then M consists of a single point, locally free and stable. And $M = M^s$

Theorem: Suppose $(v,v) = 0$. If M^s has a complete irreducible component M_1, then $M_1 = M^s = M$, i.e. M is irreducible and all sheaves are stable [and $\overset{s}{M}$ is smooth].

Definition: Let Y be any surface. Define $\widetilde{H}(Y, \mathbb{Z})$ (or $\widetilde{H}(Y, \mathbb{Q})$) natural weight 2 Hodge structure on $\widetilde{H}(Y, \mathbb{Z})$ given by

$$\widetilde{H}^{2,0}(Y, \mathbb{C}) = H^{2,0}(Y, \mathbb{C}),$$

$$\widetilde{H}^{0,2}(Y, \mathbb{C}) = H^{0,2}(Y, \mathbb{C})$$

and $\widetilde{H}^{1,1}(Y, \mathbb{C}) = H^0(Y, \mathbb{C}) \oplus H^{1,1}(Y, \mathbb{C}) \oplus H^4(Y, \mathbb{C})$.

$\widetilde{H}(Y, \mathbb{Z})$ has a pairing given by $v, w \to (v, w) := -\int_X \check{v} \cdot w$

$H^2(Y, \mathbb{Z}) \subset \widetilde{H}(Y, \mathbb{Z})$ is compatible with the Hodge structure.

The Mukai vector $M(v)$ is an element of $\widetilde{H}(Y, \mathbb{Z})$ of type $(1,1)$

The expected dimension of M^s is 2 $\Leftrightarrow$ v is isotropic i.e. $(v,v) = 0$

Assume now that v is an isotropic vector s.t. M^s has a complete component. Then $M^s = M$.

Let $\mathcal{E}$ be a quasi-universal family on $X \times M$, of rank $s \cdot r$

Definition: Let $f : H^*(X, \mathbb{Q}) \to H^*(M, \mathbb{Q})$ and $f' : H^*(M, \mathbb{Q}) \to H^*(X, \mathbb{Q})$ be defined by

$f(c) = q_\#(\mu \cdot p^\#(c))$ and

$f'(c) = p_\#(\nu \cdot q^\#(c))$

where $\mu := v^\vee(\mathcal{E})/s$ and

$\nu = v(\mathcal{E})/s$

Theorem: Let v be isotropic and assume $M = M^s$. Assume $\exists$ a universal family on $X \times M$.

Let $\mathcal{E}$ be the universal family.

Then:

1) M is a K-3 Surface

2) $f \circ f^{1} = 1$

References for pages 28 - 57

1) Okonek, Schneider and Spindler, "Vector Bundles on Complex Projective Spaces" Progress in Mathematics, Birkhauser 1980, Chapter 2, Sections 3 and 4

2) Huybrechts and Lehn, "The Geometry of Moduli Spaces of Sheaves", Second Edition, Cambridge University Press, 2010, Part 2, Sections 5 and 6.