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Wallcrossing for Moduli Spaces of Sheaves on Algebraic Surfaces and Donaldson Invariants

Lothar Goettsche
ICTP, Mathematics Group
Trieste
Italy

WALLCROSSING FOR MODULI SPACES OF SHEAVES ON ALGEBRAIC SURFACES AND DONALDSON INVARIANTS

Introduction

Let S be a projective algebraic surface and H an ample line bundle on S (you can think that S is embedded into some projective space $\mathbb{P}^N$, and H is the restriction of a hyperplane in $\mathbb{P}^N$ to S). The moduli space $M_S^H(2, C, n)$ parametrizes rank 2 H-stable torsion free sheaves E on S with Chern classes $c_1(E) = C$, $c_2(E) = n$. It depends by definition on H, and the aim of these lectures is to study this dependence. H varies in the vector space $H^2(S, \mathbb{R})$, and one finds that under this variation the moduli spaces $M_S^H(2, C, n)$ will usually stay constant, unless H crosses a wall, a hyperplane

$$\xi^\perp = \left\{a \in H^2(S, \mathbb{R}) \;\middle|\; (\xi \cdot a) = 0\right\}$$

defined by a class $\xi \in H^2(S, \mathbb{Z})$. We will study how $M_S^H(2, C, n)$ changes when H crosses a wall. We will then consider the generating functions of some invariants of the moduli spaces $M_S^H(2, C, n)$ and how they change under wallcrossing. For simplicity we will restrict our attention to two invariants:

(1) the topological Euler number (and the χ_y-genus which can be treated in a very similar way),
(2) the Donaldson invariants. Donaldson invariants are invariants of 4-manifolds X that originate from gauge theory, but if X is an algebraic surface S, then they are certain intersection numbers on the moduli spaces $M_S^H(2, C, n)$.

In both cases we will show a wallcrossing formula for their generating functions in terms of modular forms. The walls $\xi^\perp$ above are parametrized by the indefinite lattice $H^2(S, \mathbb{Z})$. We are thus lead to consider θ-functions for indefinite lattices to relate the generating functions of Euler numbers and Donaldson invariants to (Mock) modular forms.

In these lectures we will work over the complex numbers $\mathbb{C}$.

1. Lecture 1: Moduli spaces of sheaves and their wallcrossing

1.1. **Background and notation.** For a projective variety X of (complex) dimension d we denote $H^*(X,\mathbb{Z}) = \bigoplus_{i=0}^{2d} H^i(X,\mathbb{Z})$ its cohomology ring. The Euler number of X is $e(X) := \sum_{i=0}^{2d}(-1)^i \operatorname{rk}(H^i(X,\mathbb{Z}))$.

For a class $\alpha \in H^*(X,\mathbb{Z})$ we denote by $\int_X \alpha \in \mathbb{Z}$ the evaluation of α on the fundamental class of X. If S is a smooth projective surface, we usually use this to indentify $H^4(X,\mathbb{Z})$ with $\mathbb{Z}$. For classes $\alpha, \beta \in H^2(X,\mathbb{Z})$ we also write $\alpha\beta = (\alpha \cdot \beta) := \int_S \alpha\beta$.

Now let E be a coherent sheaf on X. There is an open dense subset of X on which E is locally free of some fixed rank r (i.e. a vector bundle). r is the *rank* of E and denoted $\operatorname{rk}(E)$. E has Chern classes $c_1(E), \dots, c_d(E)$, with $c_i(E) \in H^{2i}(X,\mathbb{Z})$. In particular, if $P(x_1, \dots, x_d) \in \mathbb{Z}[x_1, \dots, x_d]$ is a polynomial, then

$$\int_X P(c_1(E), \dots, c_d(E)) \in \mathbb{Z}$$

is the corresponding Chern number.

We denote by $H^i(X,E)$ the i-th coherent cohomology group of E; $H^0(X,E)$ is the space of global sections of E. We will write $h^i(X,E) := \dim(H^i(X,E))$. The *holomorphic Euler characteristic* of E is the alternating sum

$$\chi(X,E) := \sum_{i=0}^{d} (-1)^i h^i(X,E).$$

More generally for E, F two coherent sheaves on X, we denote by $\operatorname{Ext}^i(E,F)$ the ext groups. Then $\operatorname{Ext}^i(\mathcal{O}_X, F) = H^i(X,F)$, $\operatorname{Ext}^0(E,F) = \operatorname{Hom}(E,F)$ is the space of homomorphisms from E to F, and $\operatorname{Ext}^1(E,F)$ is the space of extensions

$$0 \to F \to G \to E \to 0$$

of E by F. Again we can form the alternating sum

$$\chi(E,F)=\sum_{i=0}^{2d}(-1)^i\dim(\mathrm{Ext}^i(E,F)).$$

We briefly review the Riemann-Roch theorem: Assume now that X is a nonsingular projective variety of dimension d and E is a vector bundle (locally free sheaf) of rank r on X. Let T_X be the tangent bundle on X, which is a vector bundle of rank d. We formally write

$$(1+c_1(E)+\ldots+\ldots c_r(E))=(1+x_1)\cdot\ldots\cdot(1+x_r).$$

Note that the x_i do not make sense as cohomology classes on X, however any symmetric polynomial in the x_i does, as it can be expressed in the $c_i(E)$. The *Chern character* of E is

$$\mathrm{ch}(E)=e^{x_1}+\ldots+e^{x_r},$$

where $e^x=1+x+\frac{x^2}{2}+\ldots$. The *Todd genus* of E is

$$\mathrm{td}(E)=\prod_{i=1}^{r}\frac{x_i}{1-e^{-x_i}}.$$

Then the celebrated Riemann-Roch theorem says that

$$\chi(E)=\chi(X,E)=\int_X \mathrm{ch}(E)\,\mathrm{td}(T_X).$$

For instance, if X is an algebraic surface, then

$$\chi(X,E)=\frac{c_1(E)(c_1(E)-K_X)}{2}-c_2(E)+r\chi(\mathcal{O}_X),$$

and $\chi(\mathcal{O}_X)=\frac{1}{12}(K_X^2+c_2(S))$. There is an extension to the case that E is not locally free, and a very similar version for the $\chi(E,F)$.

Let X be a smooth projective variety of complex dimension d. We consider the χ_y-genus

$$\chi_{-y}(X):=y^{-d/2}\sum_{i=0}^{d}(-y)^i\chi(\Omega_X^d).$$

Here Ω_X^d is the locally free sheaf of holomorphic d forms, the factor $y^{-d/2}$ is not standard, but usually used in the physics language. For instance $\chi_{-y}(\mathbb{P}^d)=\frac{y^{(d+1)/2}-y^{-(d+1)/2}}{y^{1/2}-y^{-1/2}}$.

1.2. **Review of moduli spaces of sheaves on surfaces.** In this whole course let S be a simply connected algebraic surface, and let H be an ample line bundle on S. We will often identify a line bundle L with its first Chern class $c_1(L) \in H^2(S,\mathbb{Z})$.

1.2.1. *Stability and moduli space.* A torsion-free coherent sheaf E on S is called H-semistable if for every subsheaf $F \subset E$ we have

$$\frac{\chi(F(n))}{\mathrm{rk}(F)} \leq \frac{\chi(E(n))}{\mathrm{rk}(E)}$$

holds for all sufficiently large n, where $E(n) = E \otimes H^{\otimes n}$. E is called H-stable if the strict equality holds.

Given r, $C \in H^2(S,Z)$ and $c_2 \in H^4(S,\mathbb{Z}) = \mathbb{Z}$ there exists a coarse moduli space $M_S^H(r,C,c_2)$ of rank r H-semistable torsion-free sheaves on S with Chern classes C, c_2. $M_S^H(r,C,c_2)$ is a projective scheme. We denote by $M_S^H(r,C,c_2)_s$ the open subscheme of H-stable sheaves. Under suitable conditions $M_S^H(r,C,c_2)$ is a fine moduli scheme, i.e. there is a universal sheaf $\mathcal{E}$ on $S \times M_S^H(r,C,c_2)$. In the future we will usually restrict attention to the case $r = 2$. We will write $M_S^H(C,d)$ for $M_S^H(2,C,c_2)$ with $d = c_2 - C^2/4$.

The tangent space to $M_S^H(C,d)_s$ at $[E]$ is $\mathrm{Ext}^1(E,E)_0$ and if $\mathrm{Ext}^2(E,E)_0 = 0$, then $M_S^H(C,d)_s$ is a nonsingular variety of dimension $4d - 3\chi(\mathcal{O}_S)$ (here the index 0 refers to the traceless part). This is true in particular if $-K_S$ is effective. Important examples of this are most rational surfaces (e.g. a blowup of $\mathbb{P}^2$ in a number of points). We will later most of the time restrict to this case.

1.2.2. *The Hilbert scheme of points on S.* The most fundamental example of a moduli space of sheaves on S is the Hilbert scheme of points. It can be used as a building block for constructing and understanding other moduli spaces and their wallcrossing.

Denote by $S^{[n]}$ the Hilbert scheme of n points on S. The points of $S^{[n]}$ are

$$\big\{Z \subset S \text{ finite subschemes of length } n\big\}.$$

Thus a general point of $S^{[n]}$ is just a set of n distinct points on S, however these points can come together and then they contain more information then just the support and the multiplicities.

$S^{[n]}$ is nonsingular and projective of dimension $2n$. There is a universal subscheme $Z_n(S) \subset S \times S^{[n]}$ given by

$$Z_n(S) = \big\{(x, Z) \in S \times S^{[n]} \bigm| x \in Z\big\}.$$

$S^{[n]}$ is a moduli space of rank 1 sheaves on S:

Remark 1. Let L be a line bundle on S. Then $M_S^H(1, L, n) = S^{[n]}$.

This is true because of the following: If $Z \in S^{[n]}$ is a subscheme of S of length n, and I_Z is its ideal sheaf, then $I_Z(L) := I_Z \otimes L$ is a rank 1 torsion free sheaf with Chern classes $c_1 = L$ and $c_2 = n$. Conversely if F is a rank 1 torsion free sheaf on S with Chern classes $c_1 = 0$, $c_2 = n$, then F is the ideal sheaf of a zero dimensional subscheme $Z \in S^{[n]}$.

We state the generating function for the topological Euler numbers $e(S^{[n]})$ [7], which is related to the Dirichlet η-function $\eta(\tau) = q^{1/24}\prod_{n=1}^{\infty}(1 - q^n)$

Theorem 2.

$$\sum_{n\geq 0} e(S^{[n]})q^n = \prod_{n=1}^{\infty}\left(\frac{1}{1-q^n}\right)^{e(S)}.$$

There are similar formulas for Betti numbers and Hodge numbers. We state the result for the χ_y-genus. Let $\sigma(S) = -\chi_1(S)$ be the signature of S (i.e. the number of positive eigenvalues of the intersection form on $H^2(X, \mathbb{R})$ minus the number of negative eigenvalues, the sign $-$, which is usually not there comes from the factor $y^{-d/2}$ we added to the χ_y-genus.). Then we can express the χ_y-genus of the $S^{[n]}$ in terms of the standard theta functions (see [9]): For $q = e^{2\pi i\tau}$, $y = e^{2\pi i z}$ with $\tau \in \mathcal{H} = \big\{\tau \in \mathbb{C} \bigm| \Im(\tau) > 0\big\}$ and $z \in \mathbb{C}$ let

$$\Theta_{11}(\tau, z) := q^{1/8}(y^{1/2} - y^{-1/2})\prod_{n\geq 1}(1 - q^n)(1 - q^n y)(1 - q^n/y)$$

be the Jacobi theta function.

Theorem 3. *Let S be a simply connected projective algebraic surface, then*

$$\sum_{n\geq 0}\chi_{-y}(S^{[n]})q^n = q^{e(S)/24}(y^{1/2}-y^{-1/2})^{\chi(\mathcal{O}_S)}\frac{\eta(\tau)^{\sigma(S)-\chi(\mathcal{O}_S)}}{\Theta_{11}(\tau,z)^{\chi(\mathcal{O}_S)}}.$$

1.3. **Wallcrossing.** Now let S be a simply connected algebraic surface with $p_g = H^0(K_S) = 0$. Here K_S is the canonical line bundle, i.e. the bundle of holomorphic 2-forms. Thus S has no global holomorphic 2 form. We will also assume that $-K_S$ is effective. Under these assumptions the map sending a line bundle L on S to its first Chern class $c_1(L)$ is an isomorphism of the group of line bundles on S with $H^2(S,\mathbb{Z})$. Thus we will in future identify the two.

We want to know how the moduli spaces $M_S^H(C,d)$ depend on H. Thus let H_-, H_+ be two ample line bundles on S. How can it happen that $M_S^{H_+}(C,d)$ is different from $M_S^{H_-}(C,d)$?

Assume E is a torsion free rank 2 coherent sheaf on S, with Chern classes $c_1(E) = C$ and c_2, which is H_- stable but H_+ stable. We put $d := c_2 - C^2/4$. Then the Harder-Narasimhan filtration of E with respect to H_+ gives an exact sequence

$$0\to E_1\to E\to E_2\to 0, \tag{1}$$

with E_1, E_2 torsion free sheaves of rank 1 with

(1) $(1+c_1(E_1)+c_2(E_1))(1+c_1(E_2)+c_2(E_2)) = 1+c_1(E)+c_2(E)$,
(2) $\chi(E_1\otimes H_-^{\otimes N}) < \chi(E_2\otimes H_-^{\otimes N})$ and $\chi(E_1\otimes H_+^{\otimes N}) > \chi(E_2\otimes H_+^{\otimes N})$ for all N sufficiently large.

The first condition is the Whitney formula for the Chern classes, the second condition says that E is H_--stable, but that the sequence (1) violates the H_+-stability.

By Remark 1 we can rewrite the sequence (1) as

$$0\to I_{Z_1}(F)\to E\to I_{Z_2}(G)\to 0$$

Here F, G are line bundles on S, and I_{Z_1} and I_{Z_2} are the ideals of zero dimensional subschemes $Z_1 \in S^{[n]}$, $Z_2 \in S^{[m]}$. Then condition (1) translates into $F + G = C$ and $c_2(E) = FG + n + m$. We put $\xi := G - F \in H^2(S,\mathbb{Z})$. Then the condition on $c_2(E)$ can be rewritten as $d + \xi^2/4 = n + m \geq 0$. The condition (2) is by the Riemann-Roch formula equivalent to $\xi H_- < 0 < \xi H_+$. This leads us to the following definition.

Definition 4. We denote by $\mathcal{C} \subset H^2(S,\mathbb{R})$ the ample cone. This is the open subcone of $H^2(S,\mathbb{R})$ generated by the classes of ample line bundles. Let $C \in H^2(S,\mathbb{Z})$, $c_2 \in \mathbb{Z}$ and $d = c_2 - C^2/4$. Let $\xi \in H^2(S,\mathbb{Z})$ with $\xi \equiv C \mod 2H^2(S,\mathbb{Z})$. Denote

$$\xi^\perp = \left\{a \in H^2(S,\mathbb{R}) \;\middle|\; (a\cdot\xi) = 0\right\}.$$

ξ is called a class of type (C,d) and $\xi^\perp$ a wall of type (C,d) if

(1) $\xi^\perp \cap \mathcal{C} \neq \emptyset$
(2) $d + \xi^2/4 \geq 0$.

Note that (1) says that ξ is orthogonal to an ample divisor and by the Hodge index theorem this implies that $\xi^2 < 0$; condition (2) says it cannot be too negative. The walls $\xi^\perp$ of type (C,d) are locally finite in $\mathcal{C}$. The *chambers* of type (C,d) are the connected components of the complement of all the walls of type (C,d) in $\mathcal{C}$.

We have essentially proved the following:

Theorem 5. *Let H_-, H_+ be ample on S. If $M_S^{H_-}(C,d) \neq M_S^{H_+}(C,d)$, then there is a class ξ of type (C,d) with $\xi H_- < 0 < \xi H_+$. In particular $M_S^H(C,d)$ depends only on the chamber of type (C,d) in which H lies.*

In future, for any class $L \in \mathcal{C}$, which does not lie on a wall of type (C,d), we will put $M_S^L(C,d) = M_S^H(C,d)$, for any ample H in the same chamber.

Note that by definition $M_S^L(C,d) = M_S^{aL}(C,d)$ for all $a \in \mathbb{R}_{>0}$, i.e. it depends only one the point in $\mathcal{C}/\mathbb{R}_{>0}$.

Now we want to understand how the moduli spaces $M_S^H(C,d)$ change under wallcrossing. The answer is that extensions $0 \to I_{Z_1}(F) \to E \to I_{Z_2}(G) \to 0$ are replaced by extensions the other way round. Assume for simplicity that there is a unique class ξ of type (C,d) with $H_-\xi < 0 < H_+\xi$. We put $l := d + \xi^2/4 \in \mathbb{Z}_{\geq 0}$.

Definition 6. Let $n, m \in \mathbb{Z}_{\geq 0}$ with $n + m = l$. Let $E_\xi^{n,m}$ be the set of all sheaves E lying in extensions

$$0 \to I_{Z_1}(F) \to E \to I_{Z_2}(G) \to 0, \tag{2}$$

with $F+G=C$, $G-F=\xi$, $Z_1\in S^{[n]}$, $Z_2\in S^{[m]}$.

Note that for fixed Z_1, Z_2 the E lying in extentions (2) are parametrized by the projective space $\mathbb{P}(\mathrm{Ext}^1(I_{Z_2}(G),I_{Z_1}(F))$. We have made the assumption that $-K_S$ is effective; under this condition we have $\mathrm{Ext}^0(I_{Z_2}(G),I_{Z_1}(F))=\mathrm{Ext}^2(I_{Z_2}(G),I_{Z_1}(F))=0$ and the Riemann-Roch theorem gives

$$\dim(\mathrm{Ext}^1(I_{Z_2}(G),I_{Z_1}(F))=-\xi(\xi+K_S)/2+n+m-1.$$

We have a morphism $\pi: E_\xi^{n,m}\to S^{[n]}\times S^{[m]}, E\to(Z_1,Z_2)$. The $\mathrm{Ext}^1(I_{Z_2}(G),I_{Z_1}(F))$ are the fibres of a vector bundle $V_\xi^{n,m}$ on $S^{[n]}\times S^{[m]}$ of rank $-\xi(\xi-K_S)/2+n+m-1$, such that $E_\xi^{n,m}=\mathbb{P}(V_\xi^{n,m})$.

Theorem 7. (1) *There are projective varieties $M_{-1}=M_S^{H_-}(C,d)$, $M_0,\ldots,M_l=M_S^{H_+}(C,d)$, such that $M_i=M_{i-1}\setminus E_\xi^{i,l-i}\sqcup E_{-\xi}^{l-i,i}$.*
(2) *More precisely: M_i is obtained from M_{i-1} by blowing it up along $E_\xi^{i,l-i}$ and blowing down the exceptional divisor in another direction to obtain $E_{-\xi}^{l-i,i}$.*

We want to use this to compute the generating function for the wallcrossing of the Euler numbers of the moduli space $M_S^H(C,d)$. We will use the following well-known facts about the Euler number:

(1) (additivity) if X is variety, Y a closed subvariety and $U=X\setminus Y$, then $e(X)=e(Y)+e(U)$.
(2) (product) if $f:X\to Y$ is fibre bundle with fibre Z, then $e(X)=e(Y)e(Z)$.
(3) $e(\mathbb{P}^n)=n+1$.

Let H be an ample line bundle on S. We consider the generating function

$$\sum_{d\geq 0}e(M_S^H(C,d))q^d.$$

Theorem 8. *Let H_-, H_+ be ample on S (and not on a wall of type (C,d) for any d), then*

$$\sum_{d\geq 0}e(M_S^{H_+}(C,d))q^d-\sum_{d\geq 0}e(M_S^{H_-}(C,d))q^d=\sum_{\xi}(K_S\cdot\xi)q^{-\xi^2/4}\prod_{n\geq 0}\left(\frac{1}{1-q^n}\right)^{2e(S)}$$

Here ξ runs through all classes in $C+2H^2(S,\mathbb{Z})$ with $\xi H_-<0<\xi H_+$.

Proof. Fix d. By Theorem 7 we have (with ξ running through the classes of type (C,d) with $\xi H_- < 0 < \xi H_+$),

$$e(M_S^{H_+}(C,d)) - e(M_S^{H_-}(C,d)) = \sum_\xi \sum_{n+m=d+\xi^2/4} (e(E_{-\xi}^{n,m}) - e(E_\xi^{m,n}))$$
$$= \sum_\xi \sum_{n+m=d-\xi^2/4} e(S^{[n]})e(S^{[m]})(\xi K_S).$$

In the last step we use that by (3) above

$$e(E_\xi^{n,m}) = e(S^{[n]} \times S^{[m]})(-\xi(\xi + K_S)/2 + n + m - 1),$$
$$e(E_{-\xi}^{n,m}) = e(S^{[n]} \times S^{[m]})(-\xi(\xi - K_S)/2 + n + m - 1).$$

Putting this together we get that

$$\sum_{d\geq 0} e(M_S^{H_+}(C,d))q^d - \sum_{d\geq 0} e(M_S^{H_-}(C,d))q^d$$
$$= \sum_\xi (K_S\cdot\xi)q^{-\xi^2/4}\sum_{n,m} e(S^{[n]})e(S^{[m]})q^{n+m}$$
$$= \sum_\xi (K_S\cdot\xi)q^{-\xi^2/4}\prod_{n\geq 0}\left(\frac{1}{1-q^n}\right)^{2e(S)},$$

where in the last step we use Theorem 7. □

This argument can be generalized to the χ_y-genus. We defined the χ_y-genus only for smooth projective varieties. There is a generalized version of it for any quasiprojective variety, satisfying the analogue of properties (1), (2), (3) before

(1) (additivity) if X is variety, Y a closed subvariety and $U = X\setminus Y$, then $\chi_{-y}(X) = \chi_{-y}(Y) + \chi_{-y}(U)$.
(2) (product) if $f : X \to Y$ is fibre bundle with fibre Z, then $\chi_{-y}(X) = \chi_{-y}(Y)\chi_{-y}(Z)$.
(3) $\chi_{-y}(\mathbb{P}^n) = \frac{y^{(n+1)/2} - y^{-(n+1)/2}}{y^{1/2} - y^{-1/2}}$.

Using this, it is an exercise to modify the above proof to prove the following:

Theorem 9. *Let H_-, H_+ be ample on S (and not on a wall of type (C,d) for any d), then*

$$\sum_{d\ge 0}\chi_{-y}(M_S^{H_+}(C,d))q^d-\sum_{d\ge 0}\chi_{-y}(M_S^{H_-}(C,d))q^d$$
$$=q^{e(S)/12}(y^{1/2}-y^{-1/2})\frac{\eta(\tau)^{2\sigma(S)-2}}{\Theta_{11}(\tau,z)^2}\sum_{\xi}(y^{\xi K_S/2}-y^{-\xi K_S/2})q^{-\xi^2/4}.$$

Here ξ runs through all classes in $C+2H^2(S,\mathbb{Z})$ with $\xi H_-<0<\xi H_+$.

It is striking that the right hand side of both Theorem 8 and Theorem 9 looks like a theta function. We will see that the same happens for the Donaldson invariants.

2. Lecture 2: Donaldson invariants and their wallcrossing

Let X be a simply connected oriented compact 4-manifold. We denote by $b_+(X)$ the number of positive eigenvalues on the intersection form on $H^2(X,\mathbb{R})$.

The Donaldson invariants of X are defined via gauge theory as intersection numbers on moduli spaces of antiselfdual connections on principal $SU(2)$ or $SO(3)$ bundles on X. The Donaldson invariants depend by definition on the choice of a Riemannian metric g on X, however, if $b_+(X)>1$, they are in fact independent of g. If $b_+(X)=1$ they depend only on the period point $\omega(g)$ in the positive cone

$$\big\{\alpha\in H^2(X,\mathbb{R})\;\big|\;\alpha^2>0\big\}/\mathbb{R}_{>0}\subset H^2(X,\mathbb{R})/\mathbb{R}_{>0}.$$

If X is an algebraic surface S we can also define and study them as intersection numbers on the moduli spaces $M_S^H(C,d)$ we introduced in lecture 1. In fact we will only use the definition in algebraic geometry and will not give the definition in gauge theory. A brief introduction of the gauge theory definition, and the relation of this to the definition in algebraic geometry can be found e.g. in [10].

2.1. **Donaldson invariants in algebraic geometry.** Let S be a simply connected projective algebraic surface and let H be ample on S. Fix $C\in H^2(S,\mathbb{Z})$, $c_2\in\mathbb{Z}$ and put $d:=c_2-C^2/4$. Denote by

$M := M_S^H(C,d) = M_S^H(2,C,c_2)$ the corresponding moduli space of H semistable torsion free sheaves on S. Let

$$e := 4c_2 - C^2 - 3\chi(\mathcal{O}_X) = 4d - 3\chi(\mathcal{O}_X).$$

e is called the expected dimension of M. If $\mathrm{Ext}^2(E,E)_0 = 0$, then $M_S^H(C,d)_s \subset M_S^H(C,d)$ will be nonsingular of dimension e. We assume for simplicity that there is a universal sheaf $\mathcal{E}$ on $S \times M$. It is known that such a sheaf exists if

(1) H is general wrt (C,c_2), i.e. $M_S^H(C,d) = M_S^H(C,d)_s$.
(2) $C \notin 2H^2(S,\mathbb{Z})$ or c_2 is odd.

We write $H_i(X) := H_i(X,\mathbb{Q})$, $H^i(X) := H^i(X,\mathbb{Q})$.

Definition 10. Let $\mu : H_i(X) \to H^{4-i}(M)$ be defined by $\mu(\alpha) := (c_2(\mathcal{E}) - \frac{1}{4}c_1(\mathcal{E})^2)/\alpha$. Here / is the slant product: i.e. if we write $c_2(\mathcal{E}) - \frac{1}{4}c_1(\mathcal{E})^2 = \sum_i \beta_i \otimes \gamma_i$ with $\beta_i \in H^*(X,\mathbb{Q})$, $\gamma_i \in H^*(M,\mathbb{Q})$, then $\mu(\alpha) = \sum_i \langle \beta_i, \alpha\rangle \gamma_i$, where $\langle , \rangle$ is the pairing between cohomology and homology. If M is nonsingular, we can identify homology and cohomology via Poincaré duality, in this case

$$\mu(\alpha) = p_{2*}\big(p_1^*\alpha(c_2(\mathcal{E}) - c_1(\mathcal{E})^2/4)\big).$$

μ is independent of the choice of the universal sheaf $\mathcal{E}$, and can be defined even if no universal sheaf exists.

Assume now that M has as dimension the expected dimension e. For $\alpha \in H_2(S,\mathbb{Z})$ and $p \in H_0(S,\mathbb{Z})$ the class of a point (s.th. $\mu(\alpha) \in H^2(M)$, $\mu(p) \in H^4(M)$), we define the Donaldson invariants

$$D^{S,H}_{C,e}(\alpha^{e-2k}p^k) := \int_M \mu(\alpha)^{e-2k}\mu(p)^k.$$

We will consider the generating function

$$D^{S,H}_C(\exp(\alpha z + px)) := \sum_{n,m\geq 0} D^{S,H}_{C,n+2m}(\alpha^n p^m)\frac{z^n}{n!}\frac{x^m}{m!}.$$

If M does not have the expected dimension as dimension, one can still define the Donaldson invariants using the $M_S^H(C,d)$, by using the formalism of virtual fundamental classes. This is done and also applied to wallcrossing formulas in [19]. We will not attempt to explain the details which are very complicated.

2.2. **The wallcrossing term.** Let $p_g(S) = h^0(K_S)$ be the geometric genus of S. Then $b_+(S) = 1 + 2p_g(S)$. The ample line bundle H on S determines as Riemannian metric the Fubini-Study metric $g(H)$. By the above it follows that if $p_g(S) > 0$, then the Donaldson invariants $D^{S,H}_{C,e}$ are independent of H.

For the rest of this lecture let S be a simply connected projective algebraic surface with $p_g(S) = 0$. Fix $C \in H^2(X, \mathbb{Z})$ and $d = c_2 - C^2/4$, $e = 4d - 3$. We know from lecture 1 that for H, H' ample on S we have $M^H_S(C, d) = M^{H'}_S(C, d)$ if H and H' lie in the same chamber of type (C, d), and we can take the same universal sheaf on both. Thus $D^{S,H}_{C,e}$ depends only on the chamber of type (C, d) of H. For every class ξ of type (C, d) we want to compute the wallcrossing term $\delta^S_{\xi,e}$ such that if H_-, H_+ are ample on S we have

$$D^{S,H_+}_{C,e} - D^{S,H_-}_{C,e} = \sum_{\xi} \delta^S_{\xi,e}, \tag{3}$$

for ξ running throught the classes of type (C, d) with $\xi H_- < 0 < \xi H_+$.

By definition, if ξ is a class of type (C, d), then it is also a class of type $(C, d + n)$ for all $n \geq 0$, thus we want to compute the generating function

$$\delta^S_{\xi}(\exp(\alpha z + px)) := \sum_{n,m \geq 0} \delta^S_{\xi,n+2m}(\alpha^n p^m)\frac{z^n}{n!}\frac{x^m}{m!}. \tag{4}$$

We now state the wallcrossing formula, which is the main result of [11]. We introduce the following modular forms.

Let S be a simply connected surface with $p_g = 0$.

Let $\mathcal{H} := \{\tau \in \mathbb{C} \mid \Im(\tau) > 0\}$ be the complex upper half plane. Let $q = e^{2\pi i \tau}$ for $\tau \in \mathcal{H}$. We introduce the θ constants

$$\theta_{00}(\tau) = \sum_{n \in \mathbb{Z}} q^{n^2/2},\ \theta_{01}(\tau) = \sum_{n \in \mathbb{Z}} (-1)^n q^{n^2/2}, \theta_{10}(\tau) = \sum_{n \in \mathbb{Z}} q^{\frac{1}{2}(n+1/2)^2}.$$

Let $G_2(\tau) := -\frac{1}{24} + \sum_{n>0} \sigma_1(n) q^n$ be the Eisenstein series of weight 2. We put

$$u := -\frac{\theta_{00}^4 + \theta_{10}^4}{\theta_{00}^2\theta_{10}^2},\ h := \frac{2i}{\theta_{00}\theta_{10}},\ T := h^2 G_2 - \frac{u}{6}.$$

Theorem 11. *For $\alpha \in H_2(S,\mathbb{Z})$, put*

$$\delta_\xi^S(\exp(\alpha z + px)) := i^{(\xi\cdot K_S)-1}\Big[q^{-\xi^2/8}\exp\big(h\langle\xi/2,\alpha\rangle z + T\langle\alpha^2\rangle - ux\big)(ih)^3\theta_{01}^{K_S^2}\Big].$$

Then if H_-, H_+ are ample on S, (3) *and* (4) *hold.*

We now want to briefly explain the steps of the proof and the ingredients that go into it.

(1) Reduction to Hilbert schemes of points,
(2) comparison with the Nekrasov partition function,
(3) application of the Nekrasov conjecture to prove the wallcrossing formula for toric surfaces,
(4) generalization to general surfaces by cobordism invariance.

2.2.1. *Reduction to Hilbert schemes of points.* Let H_-, H_+ be ample on S and assume that ξ is the unique class of type (C,d) with $H_-\xi < 0 < H_+\xi$. Let $l := d + \xi^2/4 \in \mathbb{Z}_{\ge 0}$. Then as seen above we have $M_S^{H_-}(C,d) = M_{-1}, M_0, \ldots, M_l = M_S^{H_+}(C,d)$, such that M_i is obtained from M_{i-1} by blowup along $E_\xi^{i,l-i}$ and blowdown of the exceptional divisor $D_{i,i-1}$ in another direction to obtain $E_{-\xi}^{l-i,i}$, furthermore $E_\xi^{n,m}$ is the projective space bundle over $S^{[n]}\times S^{[m]}$ associated to an explicit vector bundle $V_\xi^{n,m}$. Then we get

$$\begin{aligned} D_{C,e}^{X,H_+}(\alpha^{e-2k}p^k) - D_{C,e}^{X,H_+}(\alpha^{e-2k}p^k) &= \sum_{i=0}^{l}\Big(\int_{M_i} - \int_{M_{i-1}}\Big)\mu(\alpha)^{e-2k}\mu(p)^k \\ &= \sum_{i=0}^{l}\int_{D_{i,i-1}} B(\alpha^{e-2k},p^k) \\ &= \sum_{i=0}^{l}\int_{S^{[i]}\times S^{[l-i]}} \pi_* B(\alpha^{e-2k},p^k). \end{aligned}$$

The first line is obvious. In the second line one has to compute the difference of the pullbacks of $\mu(\alpha)^{e-2k}\mu(p)^k$ from M_i and M_{i-1} to $\widehat{M}_{i,i-1}$, which is both the blowup of M_{i-1} along $E_\xi^{i,l-i}$ and M_i along $E_{-\xi}^{l-i,i}$, and show that it is the pushforward of a suitable class $B(\alpha^{e-2k},p^k)$ in $H^*(D_{i,i-1},\mathbb{Q})$. $D_{i,i-1}$ is a fibre bundle over $S^{[i]}\times S^{[l-i]}$ with fibre a product of projective spaces (in fact one proves that

$$D_{i,i-1} = E_\xi^{i,l-i}\times_{S^{[i]}\times S^{[l-i]}} E_{-\xi}^{l-i,i}.$$

In the last step we push down along the projection $\pi : D_{i,i-1} \to S^{[i]} \times S^{[l-i]}$ to get an explicit intersection number of so-called "tautological classes" on the $S^{[i]} \times S^{[l-i]}$.

For completeness we list the final outcome

$$D^{X,H_+}_{C,e}(\alpha^e) - D^{X,H_+}_{C,e}(\alpha^e)$$
$$= \left[\sum_{i=0}^{l} \int_{S^{[i]}\times S^{[l-i]}} \frac{\big((c_2 - c_1^2/4)\big(\mathcal{I}_1 \oplus \mathcal{I}_2(-\xi)(t)\big)\big)/\alpha\big)^e}{c_{top}\big(\operatorname{Ext}^1_\pi(\mathcal{I}_2, \mathcal{I}_1(\xi))(-t)\big) c_{top}\big(\operatorname{Ext}^1_\pi(\mathcal{I}_1, \mathcal{I}_2(-\xi))(t)\big)}\right]_{t^{-1}}.$$

Here $\mathcal{I}_1$ is the pullback of the ideal sheaf $I_{Z_i(S)}$ of the universal subscheme $Z_i(S) \subset S \times S^{[i]}$ to $S \times S^{[i]} \times S^{[l-i]}$, and similarly $\mathcal{I}_2$ is the pullback of the ideal sheaf $I_{Z_{l-i}(S)}$. $\mathcal{I}_1(\xi))$ is the tensor product of $\mathcal{I}_1$ with the pullback of ξ from S to $S \times S^{[i]} \times S^{[l-i]}$.

$$\pi : S \times S^{[i]} \times S^{[l-i]} \to S^{[i]} \times S^{[l-i]}$$

is the projection and $\operatorname{Ext}^1_\pi(\mathcal{I}_2, \mathcal{I}_1(\xi))$ is the relative ext group. t is a variable, and for a vector bundle E of rank r we define the $c_i(E(t))$ by means of a formal splitting

$$1 + c_1(E) + c_2(E) + \ldots + c_r(E) := (1 + x_1) \ldots (1 + x_r)$$

by

$$1 + c_1(E(t)) + c_2(E(t)) + \ldots + c_r(E(t)) := (1 + x_1 + t) \ldots (1 + x_r + t).$$

(They can be interpreted as the equivariant Chern classes of E with a trivial $\mathbb{C}^*$ action). Finally $[\ ]_{t^{-1}}$ means the coefficient of t^{-1}.

2.2.2. *Equivariant localization.* Let X be a smooth projective variety of dimension d with an action of $T = (\mathbb{C}^*)^k$. Assume for simplicity that the fixlocus X^T is a finite set $X^T = \{p_1, \ldots, p_n\}$. Equivariant localization tells us that intersection numbers of Chern classes of equivariant vector bundles V on X can be computed in terms of the action of T on the fibres $V(p_i)$ at the fixpoints.

Let V be an equivariant vector bundle of rank r on X, i.e. the action of T on X lifts to an action of V. Then the fibres $V(p_i)$ at the fixpoints are vector spaces with a T-action. Thus they split into eigenspaces

$$V(p_i) = \sum_{j=1}^{r} \mathbb{C} s_j,$$

where the action of $t = (t_1, \dots, t_r)$ on the eigenvector s_j is given by

$$(t_1, \dots, t_r) \cdot s_j = t_1^{n_1(j)} \dots t_r^{n_k(j)} s_j,$$

with $n_i(j) \in \mathbb{Z}$. We define $c_k(V(p_i))$ by

$$\begin{aligned} c(V(p_i)) &= 1 + c_1(V(p_i)) + \dots + c_r(V(p_i)) \\ &:= \prod_{j=1}^{r} (1 + n_1(j)\varepsilon_1 + \dots + n_r(j)\varepsilon_r) \in \mathbb{Z}[\varepsilon_1, \dots, \varepsilon_r]. \end{aligned} \tag{5}$$

Let T_X be the tangent bundle on X which is equivariant. Let $V_1, \dots V_s$ be equivariant vector bundles on X.

Theorem 12. *(Bott residue theorem) Let $f(\{c_j(V_1)\}_j, \dots, \{c_j(V_s)\}_j)$ be a polynomial in the Chern classes of $V_1, \dots, V_s$. Then*

$$\begin{aligned} &\int_X f(\{c_j(V_1)\}_j, \dots, \{c_j(V_s)\}_j) \\ &\quad = \lim_{\varepsilon_1, \dots, \varepsilon_k \to 0} \sum_{i=1}^{n} \frac{f(\{c_j(V_1(p_i))\}_j, \dots, \{c_j(V_s(p_i))\}_j)}{c_d(T_X(p_i))} \in \mathbb{Q}. \end{aligned}$$

Background: Equivariant cohomology and equivariant localization. Let a group G act on a manifold X. Then *equivariant cohomology groups* $H^*_G(X)$ are defined as follows. Let BG be the classifying space for G and $EG \to BG$ be the universal fibre bundle. This means that EG is contractible with a free G-action and BG is the quotient. Then $H^*_G(X) = H^*(X \times_G EG)$, where $X \times_G EG$ is the quotient of $X \times EG$ modulo the diagonal action of G. By definition $H^*_G(X)$ is a module over $H^*_G(pt) = H^*(BG)$. If V is a G-equivariant vector bundle over X, then it has Chern classes $c_i(V) \in H^{2i}_G(X)$.

Now let us go back to the previous situation. Let X be a smooth projective variety of dimension d with an action of $T = (\mathbb{C}^*)^k$. Assume for simplicity that the fixlocus X^T is a finite set $X^T = \{p_1, \dots, p_n\}$. Then for the morphism $X \to pt$ to a point we have the equivariant pushforward $\int_X^{eq} : H^*_T(X) \to H^*_T(pt) = \mathbb{Z}[\varepsilon_1, \dots, \varepsilon_k]$. Let again be $V_1, \dots, V_s$ be equivariant vector bundles.

Theorem 13. *(Atiyah-Bott localization) Let $f(\{c_j(V_1)\}_j,\ldots,\{c_j(V_s)\}_j)$ be a polynomial in the Chern classes of $V_1,\ldots,V_s$. Then*

$$\begin{aligned}\int_X^{eq} & f(\{c_j(V_1)\}_j,\ldots,\{c_j(V_s)\}_j)\\ &=\sum_{i=1}^n \frac{f(\{c_j(V_1(p_i))\}_j,\ldots,\{c_j(V_s(p_i))\}_j)}{c_d(T_X(p_i))}\in \mathbb{Z}[\varepsilon_1,\ldots,\varepsilon_n],\end{aligned} \tag{6}$$

and the usual intersection number is

$$\int_X f(\{c_j(V_1)\}_j,\ldots,\{c_j(V_s)\}_j)=\lim_{\varepsilon_1,\ldots,\varepsilon_k\to 0}\int_X^{eq} f(\{c_j(V_1)\}_j,\ldots,\{c_j(V_s)\}_j).$$

2.2.3. *Nekrasov partition function.* The Nekrasov partition function "is" the generating function for the equivariant Donaldson invariants of $\mathbb{A}^2$. We can write $\mathbb{P}^2$ as $\mathbb{A}^2\cup l_\infty$, where l_∞ is the line at infinity. Let $M(n)$ be the moduli space of pairs (E,ϕ), where E is a rank 2 torsion free coherent sheaf on $\mathbb{P}^2$, with $c_2(E)=n$ and ϕ is an isomorphism of $E|_{l_\infty}$ with the trivial bundle $\mathcal{O}^2$. $M(n)$ is a smooth quasiprojective variety of dimension $4n$, however it is not compact.

$T=\mathbb{C}^3$ acts on $M(n)$. First $(\mathbb{C}^*)^2$ acts on $\mathbb{P}^2$ fixing l_∞ by

$$(t_1,t_2)\cdot(z_0:z_1:z_2)=(z_0:t_1z_1:t_2z_2).$$

An extra factor $\mathbb{C}^*$ acts on $M(n)$ by acting on the trivialization:

$$s\cdot(E,\phi)=(E,diag(s^{-1},s)\circ\phi).$$

The fixpoint locus is finite:

$$M(n)^T=\big\{(I_{Z_1}\oplus I_{Z_2},id)\;\big|\;Z_i\in(\mathbb{A}^2,0)^{[n_i]}\text{ monomial, }\ n_1+n_2=n\big\}.$$

Here $(\mathbb{A}^2,0)^{[n_i]}\subset(\mathbb{A}^2)^{[n_i]}$ are the subschemes with support the origin 0. and $Z\in(\mathbb{A}^2,0)^{[n]}$ is called monomial if I_Z is generated by monomials in the coordiates x,y of $\mathbb{A}^2$. It follows that

$$I_Z=(y^{n_0},y^{n_1}x,\ldots,y^{n_k}x^k,x^{k+1}),$$

where $(n_0,\ldots,n_k)$ is a partition of n. Thus the fixpoints of $M(n)$ correspond to pairs of partitions.

We denote by $\varepsilon_1, \varepsilon_2, a$ the variables associated to t_1, t_2, s, in the localization formula. The *Nekrasov partition function* is

$$Z(\varepsilon_1, \varepsilon_2, a, t) := \sum_{n \geq 0} \Big(\int_{M(n)}^{eq} 1 \Big) \Lambda^{4n} \in \mathbb{Q}(\varepsilon_1, \varepsilon_2, a)[[\Lambda]],$$

Here we formally apply localization, i.e. if for any pair of partitions (ν_1, ν_2) of numbers n_1, n_2 with $n_1 + n_2 = n$, P_{ν_1,ν_2} is the corresponding fixpoint on $M(n)$, we have

$$\int_{M(n)}^{eq} 1 = \sum_{\nu_1,\nu_2} \frac{1}{c_{4n}(T_{M(n)}(P_{\nu_1,\nu_2}))},$$

which is given by an explicit, if very complicated, combinatorial formula. (Strictly speaking this is only the instanton part, the full partition function is obtained by multiplying with the perturbation part, and explicit function of the same variables). We put

$$F(\varepsilon_1, \varepsilon_2, a, \Lambda) := \log Z(\varepsilon_1, \varepsilon_2, a, \Lambda).$$

The Nekrasov conjecture, (proved by many people, see e.g. [25][21]) says that

(1) $\varepsilon_1\varepsilon_2 F$ is regular at $\varepsilon_1, \varepsilon_2 = 0$,

(2)
$$F_0(a, \Lambda) := \big(\varepsilon_1\varepsilon_2 F(\varepsilon_1, \varepsilon_2, a, \Lambda)\big)|_{\varepsilon_1=\varepsilon_2=0}$$
is the so called Seiberg-Witten prepotential. It is given by the relation of two period integrals on the family of elliptic curves
$$y^2 = (z^2 - u)^2 - 4\Lambda^4.$$
The coefficients of $F(\varepsilon_1, \varepsilon_2, a, \Lambda)$ of degree at most 2 in $\varepsilon_1, \varepsilon_2$ have also been determined.

2.2.4. *Relating the wallcrossing formula to the Nekrasov partition function.* Now let S be a smooth projective toric surface, i.e. S has an action of $(\mathbb{C}^*)^2$ with finitely many fixpoints $S^{(\mathbb{C}^*)^2} = \{p_1, \ldots, p_s\}$. Then at each fixpoint p_i there are coordinates x_i, y_i which are eigenvectors for the $(\mathbb{C}^*)^2$ action, i.e. such that

$$(t_1, t_2) \cdot x_i = t_1^{n_1^i} t_2^{n_2^i} x_i, \quad (t_1, t_2) \cdot y_i = t_1^{m_1^i} t_2^{m_2^i} y_i,$$

With n^i_j, m^i_j in $\mathbb{Z}$. Let

$$w(x_i) = n^i_1\varepsilon_1 + n^i_2\varepsilon_2, \quad w(y_i) = m^i_1\varepsilon_1 + m^i_2\varepsilon_2,$$

be the weight of the action on the x_i, y_i.

The action of $(\mathbb{C}^*)^2$ lifts an action on $S^{[n]}$, still with finitely many fixpoints. A subscheme $Z \in S^{[n]}$ is a fixpoint, if we can write $Z = Z_1 \cup \ldots \cup Z_s$, with Z_i supported at p_i and I_{Z_i,p_i} generated by monomials in the x_i, y_i. Thus the fixpoints in $S^{[n]}$ are in one-one correspondence with s-tuples of partitions, and the fixpoints in $\bigcup_{n_1+n_2=l} S^{[n_1]} \times S^{[n_2]}$ with $2s$-tuples of partitions.

One then proves the following identity.

(7)
$$\delta^S_\xi(\exp(\alpha z + px))$$
$$= \Big[\exp\Big(\sum_{i=1}^s F(w(x_i), w(y_i), \frac{t-\xi(p_i)}{2}, \Lambda e^{(\alpha(p_i)z + p(p_i)x)/4}\Big)_{\varepsilon_1,\varepsilon_2=0}\Big]_{t^{-1}}|_{\Lambda=1}.$$

For the right hand side of this formula we realize ξ and α as first Chern classes of equivariant line bundles and p as the second Chern class of an equivariant vector bundle so that $\xi(p_i)$, $\alpha(p_i)$ $p(p_i)$ are defined above.

How can this formula be true? We compute the left hand side by localization, the right hand side is already defined by localization. The left hand side is an intersection number on $\bigcup_{n_1+n_2} S^{[n_1]} \times S^{[n_2]}$. Thus the fixpoints are parametrised by $2s = s \cdot 2$-tuples of partitions. The right hand side is a product over s copies of the Nekrasov partition function, thus the fixpoints are parametrized by $2s = 2 \cdot s$-tuples of partitions. Thus we sum over the same fixpoints on both sides and one proves that the contribution at each fixpoint is the same. Thus the final results are equal. Using the Nekrasov conjecture, one can use localization on S to compute the right hand side of (7). The final outcome after some computation is the formula of Theorem 11.

2.2.5. *General case.* Let now S be a general surface with $p_g = 0$. Then S is not necessarily toric. However by an argument of [5] "tautological" intersection numbers on Hilbert schemes of points $S^{[n]}$ are given by universal formulas in the intersection numbers of S. It follows that the wallcrossing formula is determined by its value on toric surfaces.

2.3. **The Donaldson invariants of $\mathbb{P}^2$.** In the future we will use Poincaré duality on S, and for a class $A \in H^2(S,\mathbb{Z})$ we also write $D_C^{S,H}(\exp(Az+px)) := D_C^{S,H}(\exp \alpha z + px))$, where $\alpha \in H_2(S,\mathbb{Z})$ is the Poincaré dual homology class.

The projective plane $\mathbb{P}^2$ is in some sense the simplest projective algebraic surface, thus it is a measure of our understanding of the Donaldson invariants whether we can compute them for $\mathbb{P}^2$. We want to compute the Donaldson invariants of $\mathbb{P}^2$ as an application of the wallcrossing formula. This seems impossible because $H^2(\mathbb{P}^2,\mathbb{Z}) = \mathbb{Z}H$ where H is the hyperplane class, and there are no walls.

We will use the following strategy. Let $\widehat{\mathbb{P}}^2$ be the blowup of $\mathbb{P}^2$ in a point. Let L be ample on $\widehat{\mathbb{P}}^2$. For suitable L (close to the pullback of H from $\mathbb{P}^2$) the Donaldson invariants of $\mathbb{P}^2$ can be expressed in terms of those of $\widehat{\mathbb{P}}^2$, and for other suitable L (close to the fibre F of the ruling of $\widehat{\mathbb{P}}^2$) the Donaldson invariants of $\widehat{\mathbb{P}}^2$ vanish. This allows to determine the Donaldson invariants of $\mathbb{P}^2$ by wallcrossing.

Thus we use the following two results, which also work for the other invariants of the moduli spaces.

(1) The vanishing result.

Proposition 14. *(*[15, Cor. 5.3.3, Cor. 5.3.4]*) Let S be a surface with a morphism $\pi : S \to \mathbb{P}^1$, such that the general fibre of π is $\mathbb{P}^1$. Let $F \in H^2(S,\mathbb{Z})$ be the class of a fibre of π. Let $C \in H^2(X,\mathbb{Z})$ with $(C \cdot F)$ odd. Fix $c_2 \in H^4(X,\mathbb{Z})$ and put $d = c_2 - C^2/4$. Let L be ample on S Then for all n sufficiently large $M_S^{L+nF}(C,d) = \emptyset$. In particular we have*

(1) $e(M_S^{L+nL}(C,d)) = 0$,
(2) $\chi_{-y}(M_S^{L+nL}(C,d)) = 0$,
(3) $D_C^{S,L+nF}(\alpha^{d-3-2k}p^{2k}) = 0$.

(2) The blowup formula. Let S be an algebraic surface and $\widehat{S}$ the blowup of X in a point. Let $E \in H^2(\widehat{S},\mathbb{Z})$ be the class of the exceptional divisor. We denote by the same letter a class on $H^2(S,\mathbb{Z})$ and its pullback to $\widehat{S}$. Then any class $H^2(\widehat{S},\mathbb{Z})$ can be written as $L + nE$ with $n \in \mathbb{Z}$, $L \in H^2(S,\mathbb{Z})$.

Theorem 15. *Let S be an algebraic surface, Let $C \in H^2(S,\mathbb{Z})$. Let $\widehat{S}$ be the blowup of S in a point. Let H be ample on S. Then for all $\varepsilon > 0$ sufficiently small:*

(1) $D_C^{\widehat{S},H-\varepsilon E}(L^{e-2k}p^k) = D_C^{S,H}(L^{e-2k}p^k)$,
(2) $D_{C+E}^{\widehat{S},H-\varepsilon E}(EL^{e-2k}p^k) = D_C^{S,H}(L^{e-2k}p^k)$.

There is also a blowup formula for Euler numbers and χ_y-genera, which we will use later.

Theorem 16. *Let S be an algebraic surface, Let $C \in H^2(S,\mathbb{Z})$. Let $\widehat{S}$ be the blowup of S in a point. Let H be ample on S. Then for all $\varepsilon > 0$ sufficiently small:*

$$\sum_{d\ge 0}\chi_{-y}(M_{\widehat{S}}^{H-\varepsilon E}(C,d))q^d = q^{1/12}\frac{\Theta_{00}(2\tau,z)}{\eta(\tau)^2}\sum_{d\ge 0}\chi_{-y}(M_S^H(C,d))q^d$$

$$\sum_{d\ge 0}\chi_{-y}(M_{\widehat{S}}^{H-\varepsilon E}(C+E,d))q^d = q^{1/12}\frac{\Theta_{10}(2\tau,z)}{\eta(\tau)^2}\sum_{d\ge 0}\chi_{-y}(M_S^H(C,d))q^d$$

$$\sum_{d\ge 0}e(M_{\widehat{S}}^{H-\varepsilon E}(C,d))q^d = q^{1/12}\frac{\theta_{00}(2\tau)}{\eta(\tau)^2}\sum_{d\ge 0}e(M_S^H(C,d))q^d$$

$$\sum_{d\ge 0}e(M_{\widehat{S}}^{H-\varepsilon E}(C+E,d))q^d = q^{1/12}\frac{\theta_{10}(2\tau)}{\eta(\tau)^2}\sum_{d\ge 0}e(M_S^H(C,d))q^d$$

We want to prove the following theorem:

Theorem 17. (1)

$$D_H^{\mathbb{P}^2,H}(\exp(Hz+px))$$
$$= \left[\sum_{\substack{m>n>0\\ m\text{ even},\ n\text{ odd}}}(-1)^{(n+m-1)/2}q^{(m^2-n^2)/8}\exp\left(\frac{n}{2}hz+Tz^2-ux\right)(ih)^3\theta_{01}^8\right]_{q^0},$$

(2)

$$D_0^{\mathbb{P}^2,H}(\exp(Hz+px))$$
$$= \sum_{\substack{m>n>0\\ m\text{ odd},\ n\text{ even}}}(-1)^{(n+m-2)/2}\frac{m}{2}q^{(m^2-n^2)/8}\exp\left(\frac{n}{2}hz+Tz^2-ux\right)(ih)^4\theta_{01}^8\Bigg]_{q^0},$$

.

Proof. Let $\widehat{\mathbb{P}}^2$ be the blowup of $\mathbb{P}^2$ in a point, E the exceptional divisor. Then $\widehat{\mathbb{P}}^2$ is a ruled surface with $H-E$ the class of a fibre. (1) We have $H(H-E)=1$; Thus $D_H^{\widehat{\mathbb{P}}^2,H-E}(\exp(Hz+px))=0$. Thus by the blowup formula

$$D_H^{\mathbb{P}^2,H}(\exp(Hz+px)) = D_H^{\widehat{\mathbb{P}}^2,H}(\exp(Hz+px)) = \sum_{\xi} \delta_\xi^{\widehat{\mathbb{P}}^2}(\exp(Hz+px)).$$

Here ξ runs through the classes in $H+2H^2(\widehat{\mathbb{P}}^2,\mathbb{Z})$ with $H\xi>0>(H-E)\xi$, i.e. through the set

$$\big\{nH-mE \bigm| m>n>0,\ m \text{ even}, n \text{ odd}\big\}.$$

As $K_{\widehat{\mathbb{P}}^2}^2=8$, $(nH-mE)^2=n^2-m^2$, $(nH-mE)H=n$, the claim follows by the wallcrossing formula. (2) is very similar, now we use that $D_E^{\widehat{\mathbb{P}}^2,H}(E\exp(Hz+px))=D_0^{\mathbb{P}^2,H}(\exp(Hz+px))$. □

3. Lecture 3: Theta functions for indefinite lattices

Let S again a simply connected projective surface with $p_g=0$. In lectures 1 and 2 we have seen that the generating functions for the Euler numbers, the χ_y-genera of the moduli spaces $M_S^H(C,d)$ and the Donaldson invariants $D_C^{S,H}(\exp(\alpha z+px))$ are subject to wallcrossing when the ample class H is varied. The walls are parametrized by the (shifted) lattice $C+2H^2(S,Z)$ and the wallcrossing term for each class ξ looks like the contribution of the lattice vector ξ to the theta function of the lattice, if one takes as inner product the *negative* of the intersection form in $H^2(S,\mathbb{Z})$. Write $r:=rk(H^2(S,\mathbb{Z}))$. Then the intersection form on $H^2(S,\mathbb{Z})$ is indefinite of signature $(1,r-1)$, thus this lattice has signature $(r-1,1)$. We are interested in the differences $D_C^{S,H}(\exp(\alpha z+px))-D_C^{S,H'}(\exp(\alpha z+px))$ for two classes H, H' on S. For this we have to sum over all classes ξ satisfying linear inequalities $\xi H_+>0>\xi H_-$.

In order to study the properties of these generating functions we therefore want to introduce and study theta functions for lattices of signature $(r-1,1)$, which are obtained by summing over all lattice vectors satisfying such linear inequalities.

3.1. **Definition of the theta functions.** We will first introduce these abstractly and then apply them to invariants of moduli spaces.

Notation 18. A lattice is a free $\mathbb{Z}$-module Γ together with a bilinear form B on Γ, which is nondegenerate and $\mathbb{Z}$-valued. We write $Q(v) = B(v,v)/2$ for $v \in \Gamma$. We write $\Gamma_{\mathbb{C}} := \Gamma \otimes \mathbb{C}$ and $\Gamma_{\mathbb{R}} = \Gamma \otimes \mathbb{R}$, and denote by B, Q also their extensions to $\Gamma_{\mathbb{C}}$, $\Gamma_{\mathbb{R}}$. The type of Γ is the pair $(r-s,s)$ where r is the rank of Γ and s is the largest rank of a sublattice on which Q is negative definite.

From now on assume that $s=1$, i.e. Γ has type $(r-1,1)$. Then the set of vectors $h \in \Gamma_{\mathbb{R}}$ with $Q(h)<0$ has two connected components. Fix a vector $h_0 \in \Gamma_{\mathbb{R}}$ with $Q(h_0)<0$, and let

$$C_\Gamma := \big\{h \in \Gamma_{\mathbb{R}} \bigm| Q(h)<0,\ B(h,h_0)<0\big\},$$

be the connected component containing h_0. Let

$$S_\Gamma := \big\{f \in \Gamma \bigm| f \text{ primitive},\, Q(h)<0,\ B(h,h_0)<0\big\}.$$

$C_\Gamma/\mathbb{R}_+$ is an $(r-1)$-dimensional hyperbolic space and S_Γ is a set of representatives of the corresponding cusps

$$\big\{h \in \Gamma_{\mathbb{Q}} \bigm| Q(h)=0,\ B(h,h_0)<0\big\}/\mathbb{Q}_+.$$

For $h \in C_\Gamma$ put $D(h) := \mathcal{H} \times \Gamma_{\mathbb{C}}$, and for $h \in S_L$ put

$$D(h) := \big\{(\tau,x) \in \mathcal{H} \times \Gamma_{\mathbb{C}} \bigm| 0 < \Im(B(h,x)) < \Im(\tau)\big\}.$$

For $t \in \mathbb{R}$ put $\mu(t) := \begin{cases} 1 & t \geq 0, \\ 0 & t<0 \end{cases}.$

Definition 19. Let $f,g \in C_\Gamma \cup S_\Gamma$. For $(\tau,x) \in D(f)\cap D(g)$, write $q=e^{2\pi i\tau}$. Define the theta function of Γ with respect to Γ by

$$\Theta_\Gamma^{f,g}(\tau,x) := \sum_{v\in\Gamma} \big(\mu(B(v,f)) - \mu(B(v,g))q^{Q(v)}e^{2\pi i B(\xi,x)}.$$

More generally we define for $c,b \in \Gamma$ and $(\tau,x) \in D(f)\cap D(g)$

$$\Theta_{\Gamma,c,b}^{f,g}(\tau,x) := \sum_{v\in\Gamma+c/2} \big(\mu(B(v,f)) - \mu(B(v,g))\big)q^{Q(v)}e^{2\pi i B(\xi,x+b/2)}. \tag{8}$$

Lemma 20. *This sum* (8) *converges absolutely and locally uniformly.*

We will later adress the modular properties of these theta function. First we want to establish their connection to generating functions of Euler numbers and χ_y-genera of the moduli spaces $M_S^H(C,d)$ and Donaldson invariants.

3.2. **Relation to invariants of moduli spaces.** Let S be a simply connected projective algebraic surface with $-K_S$ effective and $p_g(S)=0$. Let $\Gamma = H^2(S,\mathbb{Z})$ with inner product $B(C,D) := -(C\cdot D) = -\int_S CG$ the negative of the intersection form. Let H_0 be ample on S, and define the connected component C_Γ by the condition $B(H,H_0)<0$, i.e. C_Γ is the connected component containing ample classes. We write $C_S = C_\Gamma$, $S_S = S_\Gamma$, $\overline{C}_S = C_S \cup S_S$. Write $y = e^{2\pi i z}$ for $z\in\mathbb{C}$.

Proposition 21. *Let $C \in H^2(S,\mathbb{Z})$. Let $H, L \in \overline{C}_S$ not on a wall of type (C,d) for any d. Then*

(1)
$$\sum_{d\geq 0}\big(\chi_{-y}(M_S^H(C,d)) - \chi_{-y}(M_S^L(C,d))\big)q^{d-e(S)/12}$$
$$= \frac{(y^{1/2}-y^{-1/2})\eta(\tau)^{2\sigma(S)-2}}{\Theta_{11}(\tau,z)^2}\Theta^{L,H}_{\Gamma,C,0}(2\tau, K_S z),$$

(2)
$$\sum_{d\geq 0}\big(e(M_S^H(C,d)) - e(M_S^L(C,d))\big)q^{d-e(S)/12}$$
$$= \frac{1}{\eta(\tau)^{2e(S)}}\operatorname*{Coeff}_{2\pi i z}\Theta^{L,H}_{\Gamma,C,0}(2\tau, K_S z).$$

Proof. We only prove (1) and leave (2) as an exercise. By Theorem 9 we have, putting $v = \xi/2$ for $\xi \in C + 2H^2(X,\mathbb{Z})$.
$$\frac{\Theta_{11}(\tau,z)^2}{(y^{1/2}-y^{-1/2})\eta(\tau)^{2\sigma(S)-2}}\sum_{d\geq 0}\big(\chi_{-y}(M_S^H(C,d)) - \chi_{-y}(M_S^L(C,d))\big)q^{d-e(S)/12}$$
$$= \sum_{\substack{v\in C/2+H^2(S,\mathbb{Z})\\ (v\cdot H)>0>(v\cdot L)}} q^{-v^2}\big(y^{(v\cdot K_S)} - y^{-(v\cdot K_S)}\big)$$
$$= \sum_{v\in C/2+\Gamma} q^{2Q(v)}\big(\mu(B(v\cdot L) - \mu(B(v\cdot H))\big)y^{B(v,K_S)} = \Theta^{L,H}_{\Gamma,C,0}(2\tau, K_S z).$$

$\square$

Now we want to express the Donaldson invariants in terms of these indefinite theta functions.

Proposition 22. *Let $A \in H^2(S,\mathbb{Z})$, then*

$$\begin{aligned} &D_C^{S,H}(\exp(Az+px)) - D_C^{S,L}(\exp(Az+px)) \\ &\quad = \frac{i}{2}\Big[\Theta^{L,H}_{C,K_S}(\tau,\frac{Azh}{2\pi i})\exp(-2Q(A)Tz^2-ux)(ih)^3\theta_{01}^8\Big]_{q^0}. \end{aligned}$$

Proof. By definition $\delta^S_{-\xi} = -\delta^S_\xi$. Thus we can replace in Theorem 11 δ_ξ^S by $\frac{1}{2}(\delta^S_\xi - \delta^S_{-\xi})$. This means, putting $v = \xi/2$, we get that $-i^{(\xi\cdot K_S)}q^{-\xi^2/8}\exp(h(\xi/2\cdot A))$ gets replaced by

$$\begin{aligned} &-i(-1)^{B(-v,K_S)}q^{Q(v)}e^{hB(-v,A)z} + i(-1)^{B(v,K_S)}q^{Q(v)}e^{hB(v,A)z} \\ &\qquad = iq^{Q(v)}e^{hB(v,A+K_S)z} - iq^{Q(v)}e^{hB(-v,A+K_S)z}. \end{aligned}$$

Thus

$$\begin{aligned} &D_C^{S,H}(\exp(Az+px)) - D_C^{S,L}(\exp(Az+px)) \\ &\quad = \frac{i}{2}\Big[\sum_{\substack{v\in C/2+H^2(S,\mathbb{Z})\\ (v\cdot H)>0>(v\cdot L)}}\big(q^{Q(v)}e^{hB(v,A+K_S)z} - q^{Q(v)}e^{hB(-v,A+K_S)z}\big) \\ &\qquad \cdot \exp(-2Q(A)Tz^2-ux)(ih)^3\theta_{01}^8\Big]_{q^0} \\ &\quad = \frac{i}{2}\Big[\Theta^{L,H}_{C,K_S}(\tau,\frac{Azh}{2\pi i})\exp(-2Q(A)Tz^2-ux)(ih)^3\theta_{01}^8\Big]_{q^0}. \end{aligned}$$

□

Now let S be a rational surface with $-K_S$ effective. After possibly blowing up S in a point, there will be a morphism $S \to \mathbb{P}^1$ whose general fibre is $\mathbb{P}^1$. Thus we can use the blowup formulas Theorem 15, Theorem 16 and the vanishing result Proposition 14 to compute not just the the differences of the Donaldson invariants, Euler numbers and χ_y-genera at different ample classes, but the actual invariants.

Corollary 23. *Let S be a rational surface with $-K_S$ effective. Assume that there is a morphism $S \to \mathbb{P}^1$ whose general fibre is isomorphic to $\mathbb{P}^1$. Let F be the class of a fibre and assume $(F\cdot C)$ is odd. Then we have for any ample line bundle H on S*

(1)

$$\sum_{d\ge 0}\chi_{-y}(M_S^H(C,d))q^{d-e(S)/12}$$
$$=\frac{(y^{1/2}-y^{-1/2})\eta(\tau)^{2\sigma(S)-2}}{\Theta_{11}(\tau,z)^2}\Theta^{F,H}_{\Gamma,C,0}(2\tau,K_Sz),$$

(2)

$$\sum_{d\ge 0}e(M_S^H(C,d))q^{d-e(S)/12}$$
$$=\frac{1}{\eta(\tau)^{2e(S)}}\operatorname*{Coeff}_{2\pi iz}\Theta^{F,H}_{\Gamma,C,0}(2\tau,K_Sz).$$

(3)

$$D_C^{S,H}(\exp(Az+px))$$
$$=\frac{i}{2}\Big[\Theta^{F,H}_{C,K_S}(\tau,\frac{Azh}{2\pi i})\exp(-2Q(A)Tz^2-ux)(ih)^3\theta_{01}^{K_S^2}\Big]_{q^0}.$$

3.3. **Modular properties at the boundary.** Assume that f and g are both in S_Γ. We want to show that the $\Theta^{f,g}_{\Gamma,c,b}$ have modular properties.

The idea is the following. Let H be the rank two hyperbolic lattice generated by two vectors f,g with $Q(f)=Q(g)=0$, $B(f,g)=-1$. Then $\Theta_H^{f,g}$ is a known function. If $\Gamma=H\oplus\Lambda$, with Λ positive definite, then $\Theta_\Gamma^{f,g}=\Theta_H^{f,g}\Theta_\Lambda$, where Θ_Λ is the usual theta function. We want to reduce the general case to this.

Definition 24. For $\tau\in\mathbb{C}$ and $0<\Im(u)/\Im(\tau)<1$, $0<\Im(v)/\Im(\tau)<1$ let

$$F(\tau,u,v):=\Big(\sum_{n\ge 0,m>0}-\sum_{n<0,m\le 0}\Big)q^{nm}e^{2\pi i(nu+mv)}.$$

Then F has the following properties [29]

(1) $F(\tau,u,v)$ has a meromorphic extension to $\mathcal{H}\times\mathbb{C}^2$,

(2) F transforms as a Jacobi form: for $\begin{pmatrix}a&b\\c&d\end{pmatrix}$ it satisfies the transformation formula

$$F\Big(\frac{a\tau+b}{c\tau+d},\frac{u}{c\tau+d},\frac{v}{c\tau+d}\Big)=(c\tau+d)e^{\frac{2\pi i(cuv)}{c\tau+d}}F(\tau,u,v),$$

(3) For $|\Im(u)/\Im(\tau)| < 1$, $|\Im(v)/\Im(\tau)| < 1$ we have the Fourier development

$$F(\tau,u,v) = -\frac{1}{1-e^{2\pi i u}} + \frac{1}{1-e^{-2\pi i v}} - 2\sum_{n,m>0} \sinh(2\pi i(nu+mv))q^{nm},$$

(4) It can be expressed in terms of classical theta functions

$$F(\tau,u,v) = \frac{\eta(\tau)^3\Theta_{11}(\tau,u,v)}{\Theta_{11}(\tau,u)\Theta_{11}(\tau,v)}.$$

Notation 25. For any vector $v \in \Gamma_{\mathbb{R}}$, and any meromorphic function $J : \mathcal{H} \times \Gamma_{\mathbb{C}} \to \mathbb{C}$ we define $J|_v\mathcal{H} \times \Gamma_{\mathbb{C}} \to \mathbb{C}$ by

$$J|v(\tau,x) := q^{Q(v)}e^{2\pi i B(v,x)}J(\tau,x+v\tau).$$

(Note that for instance $\Theta_{10} = \Theta_{00}|_{1/2}$). For any $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in Sl(2,\mathbb{Z})$, and any $k \in \mathbb{Z}$ we put

$$J_kA(\tau,x) := (c\tau+d)^{-k}e^{\frac{-2\pi i c Q(x)}{c\tau+d}}J(\Big(\frac{a\tau+b}{c\tau+d},\frac{x}{c\tau+d}\Big).$$

We denote $T := \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$, $S := \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ be the generators of $Sl(2,\mathbb{Z})$.

Theorem 26. *Let Γ be a unimodular lattice of type $(r-1,1)$. Let $f,g \in S_\Gamma$. Let $c,b \in \Gamma$.*

(1) *$\Theta^{f,g}_{\Gamma,c,b}$ has a meromorphic continuation to $\mathcal{H} \times \Lambda_{\mathbb{C}}$.*
(2) *For $|\Im(B(f,x)/\Im(\tau)| < 1$, $|\Im(B(g,x)/\Im(\tau)| < 1$ we have the Fourier development*

$$\Theta^{f,g}_{\Gamma,c,b}(\tau,x) = \frac{1}{1-e^{2\pi i B(f,x+b/2)}}\sum_{\substack{B(v,f)=0\\ B(f,g)\le B(v,g)<0}} q^{Q(v)}e^{2\pi i B(v,x+b/2)}$$
$$-\frac{1}{1-e^{2\pi i B(g,x+b/2)}}\sum_{\substack{B(v,g)=0\\ B(f,g)\le B(v,f)0<0}} q^{Q(v)}e^{2\pi i B(v,x+b/2)}$$
$$+2\sum_{B(v,g)<0<B(v,f)} q^{Q(v)}\sinh(2\pi i B(v,x+b/2)$$

Here v runs through $\Gamma + c/2$.

(3) *Let w be a characteristic element in Γ, e.g. $w = K_S$.*

$$\begin{aligned}\frac{\Theta^{f,g}_{c,b}}{\theta^{\sigma(\Gamma)}_{00}}|_1 S &= (-1)^{-B(b,c)/2}\frac{\Theta^{f,g}_{b,c}}{\theta^{\sigma(\Gamma)}_{00}},\\ \frac{\Theta^{f,g}_{c,b}}{\theta^{\sigma(\Gamma)}_{00}}|_1 T &= (-1)^{B(3c-2w,c)/2}\frac{\Theta^{f,g}_{c,b-c+w}}{\theta^{\sigma(\Gamma)}_{01}}\end{aligned} \tag{9}$$

Proof. We give a rough sketch of the main ideas. For simplicity we only consider the case $c = b = 0$, i.e. $\Theta^{f,g}_{\Gamma}$. Let H be the hyperbolic lattice of type (1,1) with generators f, g with $Q(f) = Q(g) = 0$, $B(f,g) = -1$. Then

$$\Theta^{f,g}_{H}(\tau, x) = F(\tau, -B(f,x), B(g,x)).$$

Now let $f, g \in \Gamma$ be two linear independent vectors with $Q(f) = Q(g) = 0$, $B(f,g) = -N$.

Let $\Lambda := \langle f, g\rangle \oplus \langle f, g\rangle^{\perp} \subset \Gamma$, then

$$\Theta^{f,g}_{\Lambda}(\tau, x) = F(N\tau, -B(f,x), B(g,x))\Theta_{\langle f,g\rangle^{\perp}}(\tau, x_{\perp}),$$

Here $x_{\perp}$ is the projection of x to $\langle f, g\rangle^{\perp}$ and

$$\Theta_{\langle f,g\rangle^{\perp}}(\tau, x) = \sum_{v\in\langle f,g\rangle^{\perp}} q^{Q(v)}e^{2\pi i B(v,x)}.$$

In general let P be a set of representatives of Γ modulo $\langle f, g\rangle \oplus \langle f, g\rangle^{\perp}$ Then we can write $\Theta^{f,g}_{\Gamma} := \sum_{p\in P}\Theta^{f,g}_{\Lambda}|_p$. The claim follows from the modular properties of F, $\Theta_{\langle f,g\rangle^{\perp}}$ and the properties of the operation $|_p$. □

3.4. **Applications to Donaldson invariants.** The original motivation for introducing the indefinite theta functions $\Theta^{F,G}_{\Gamma,c,b}$ in [14] was to prove an analogue of the structure theorems [16] for the Donaldson invariants of 4-manifolds with $b_+ > 1$ in the case of rational surfaces (which have $b_+ = 1$).

Kronheimer and Mrowka in [16] introduce the notion of simple type. A 4-manifold X is called of *simple type* if $D^X_C(\alpha^k p^r(p^2 - 4)) = 0$. for all C, all $\alpha \in H_2(S, \mathbb{Z})$, and all k, r. For manifolds of simple type they prove the following structure theorem.

Theorem 27. *Let X be a simply connected compact oriented 4-manifold with $b_+ > 1$ of simply type. There are finitely many cohomology classes $K_1, \ldots, K_s \in H^2(X, \mathbb{Z})$, and $a_1, \ldots, a_s \in \mathbb{Q}$ such that for all $\alpha \in H_2(X, \mathbb{Z})$,*

$$D_C^X(exp(\alpha z)(1 + p/2)) = e^{\langle\alpha^2\rangle/2} \sum_{i=1}^{s} (-1)^{C(C+K_s)/2} a_i e^{\langle K_s \cdot \alpha\rangle}. \tag{10}$$

The Witten conjecture then relates the a_i and K_i to the Seiberg-Witten invariants.

We want to see that something similar also holds for S a rational algebraic surface. In this case the left hand side of (10) depends on the choice of an ample class H and is subject to wallcrossing. Thus the formula cannot be true for all possible H. We want to see that it holds for F the limit of ample classes with $F^2 = 0$.

We will prove the following version of the structure theorem.

Theorem 28. *Let S be a rational algebraic surface. Assume there exists a morphism $S \to \mathbb{P}^1$ whose general fibre G is $\mathbb{P}^1$ (we can always reduce to this case by using the blowup formula). Then $G^2 = 0$. Let $F \in H^2(X, \mathbb{Z})$ be a limit of ample classes with $F^2 = 0$. Let*

$$B := \left\{W \in H^2(X, \mathbb{Z}) \;\middle|\; W \equiv K_X \mod 2,\ W^2 \geq K_S^2,\ WF \geq 0 \geq WG\right\}$$

(Basic classes), and put $M := \max\left\{W^2 \;\middle|\; W \in B\right\}$. Note that by definition $M \leq 0$. Let $k := (M - K_X^2)/8 + 1$. Let $\alpha \in H_2(S, \mathbb{Z})$, $C \in H^2(S, \mathbb{Z})$.

1. *$D_C^{S,F}(\alpha\,(p^2 - 4)^k) = 0$ (k-th order simple type condition).*
2. *There is a formula similar to (10) where the K_i are replaced by the elements of M.*

We give a rough idea of the proof. Put

$$\phi(\tau, z, x) := \frac{i}{2}\Theta_{C,K_S}^{G,F}(\tau, \frac{Azh}{2\pi i}) \exp(-2Q(A)Tz^2 - ux)(ih)^3 \theta_{01}^{K_S^2}.$$

Then we know that

$$D_C^{S,F}(\exp(Az + px)) = \left[\phi(\tau, z, x)\right]_{q^0}.$$

The main step is the following:

Proposition 29. *$\phi(\tau,z,x)\in M_2^!(\Gamma(2))[[z,x]]$, i.e. $\phi(\tau,z,x)$ is a power series in z, x whose coefficients are almost holomorphic modular forms of weight 2 on $\Gamma(2)$.*

Proof. From the definition we see that the coefficients of the power series development $\phi(\tau,z,x)=\sum_{n,m}a_{n,m}z^nx^m$ are holomorphic on $\mathcal{H}$. By the transformation behaviour (9) of $\Theta_{C,K_S}^{G,F}$ and the transformation behaviour $G_2\big(\frac{a\tau+b}{c\tau+d}\big)=(c\tau+d)^2G_2(\tau)-\frac{c(c\tau+d)}{4\pi i}$, it follows that

$$\phi(\frac{a\tau+b}{c\tau+d},z,x)=(c\tau+d)^2\phi(\tau,z,x).$$

and the claim follows. □

Proof. (of Theorem 28) Thus $\phi(\tau,z,x)$ is a two form on $\overline{\mathcal{H}}/\Gamma(2)=\mathbb{P}^1$, holomorphic outside the three cusps $0,1,\infty$. $D_C^{S,F}(\exp(Az+px))$ is the residue of this 2-form at ∞. By the residue theorem this is minus the sum of the residues over the 2 other cusps 0 and 1. To compute the residues at 0 we apply S (for 1 things are similar). Then $\Theta_{C,K_S}^{G,F}$ is replaced by $\Theta_{K_S,C}^{G,F}$, i.e. now the sum is over the lattice $2H^2(S,\mathbb{Z})+K_S/2$, and we see that h, u, T are holomorphic at the other cusps. u has the values 2 and -2 at the other two cusps so that u^2-4 vanishes at both cusps. The only term which can have a pole at 0 and 1 is $\theta_{01}^{K_S^2}$ which starts with $q^{K_S^2/8}$. On the other hand the theta function $\Theta_{K_S,C}^{G,F}$ has a zero of order at least $M/8$. Thus we see that

$$D_C^{S,F}(\exp(Az+px)(p^2-4)^k)=\big[\phi(\tau,z,x)(u^2-4)^k\big]_{q^0}=0,$$

because 1 form is holomorphic both at 1 and 0, and $D_C^{S,F}(\exp(Az+px))$ is expressed in terms of the summands in $\Theta_{K_S,C}^{G,F}$ corresponding to elements of M. The claim follows. □

Corollary 30. (1) *Let S be the blowup of $\mathbb{P}^2$ in at most 8 points and let $F\in H^2(S,\mathbb{Z})$ be a limit of ample classes with $F^2=0$. Then for all $C\in H^2(X,\mathbb{Z})$, $\alpha\in H_2(S,\mathbb{Z})$ we have $D_C^{S,F}(\exp(\alpha z+px))=0$.*

(2) *Let S be the blowup of S in 9 points and let $F=3H-E_1-\ldots-E_9$. Let $C\in H^2(S,\mathbb{Z})$ with $\langle C\cdot F\rangle$ odd. Then*

$$D_C^{S,F}\big(\exp(\alpha z)(1+p/2)\big)=-(-1)^{C(C+F)/2}\frac{\exp(\frac{\langle\alpha^2\rangle}{2}z^2)}{\cosh(\langle F\cdot\alpha\rangle z)}.$$

3.5. **Applications to invariants of moduli spaces.** We give two examples of applications to Euler numbers and χ_y-genera of moduli spaces of sheaves on rational surfaces.

Let S be a rational ruled surface. Let F be the class of of a fibre of the ruling. Denote $e(M_S^{F+}(C,d)) := e(M_S^H(C,d))$, for H an ample class on S such that there is no wall of type (C,d) between H and F.

Proposition 31.

$$\sum_{d\geq 0} e(M_S^{F+}(F,d))q^{d-1/3} = \frac{2G_2(\tau)+\frac{1}{12}}{\eta(\tau)^3}.$$

Proof. By a sequence of blowups and blowdowns we can reduce to the case that $S=\mathbb{P}^1\times\mathbb{P}^1$. Let G be the class of the fibre of the other projection to $\mathbb{P}^1$. Then $\sum_{d\geq 0} e(M_S^{G+}(F,d))q^{d-1/3}=0$ by Proposition 14, and thus $\sum_{d\geq 0} e(M_S^{F+}(F,d))q^{d-1/3}$ can be computed in terms of $\Theta_{H,F,0}^{G,F}$. $\square$

Now let S be the blowup of $\mathbb{P}^2$ in 9 points (in fact in the 9 intersection points of two general curves of degree 3 in $\mathbb{P}^2$). Then S has a fibration $S\to\mathbb{P}^1$ whose general fibre F is an elliptic curve. Let H be (the pullback of) the hyperplane clase on $\mathbb{P}^2$, and let $E_1,\ldots,E_9$ be the exceptional divisors. Then $F=3H-E_1-\ldots-E_9$.

Proposition 32. *Let $i\in\{1,\ldots,9\}$. Then*

(1)

$$\sum_{d\geq 0}\big(\chi_{-y}(M_S^{F+}(H,d))+\chi_{-y}(M_S^{F+}(E_i,d))\big)q^{d-1/2} = \frac{y^{1/2}-y^{-1/2}}{\Theta_{11}(\tau/2,z)\eta(\tau/2)^9},$$

(2)

$$\sum_{d\geq 0}\big(e(M_S^{F+}(H,d))+e(M_S^{F+}(E_i,d))\big)q^{d-1/2} = \frac{1}{\eta(\tau/2)^{12}}.$$

Note that $e(M_S^{F+}(H,d)=0$ unless $d\equiv -1/4 \mod 1$ and $e(M_S^{F+}(E_i,d))=0$ unless $d\equiv 1/4 \mod 1$. Thus both the $e(M_S^{F+}(H,d))$ and the $e(M_S^{F+}(E_i,d))$ are determined by this formula.

Proof. Let $G = H - E_i$, then $F \cdot G = 1$ amd G is the fibre of fibration over $\mathbb{P}^1$ with fibre $\mathbb{P}^1$. It follows with $\Gamma = H^2(S,\mathbb{Z})$ that

$$\sum_{d\geq 0}\big(\chi_{-y}(M_S^{F+}(H,d)) + \chi_{-y}(M_S^{F+}(E_i,d))\big)q^{d-1/2}$$
$$= \frac{y^{1/2}-y^{-1/2}}{\eta(\tau)^{18}\Theta_{11}(\tau,z)^2}\big(\Theta^{G,F}_{\Gamma,H,0}(2\tau,K_Sz) + \Theta^{G,F}_{\Gamma,E_i,0}(2\tau,K_Sz)\big).$$

These now can be computed using the arguments in the proof of the modularity properties of these theta functions, i.e. reducing to $\Theta^{F,G}_{\Lambda}$ with $\Lambda = \langle F,G\rangle \oplus \langle F,G\rangle^{\perp}$, using the operation $|_p$. □

4. Lecture 4: Relation to Zwegers theta functions

In this lecture I want briefly indicate how the theta functions $\Theta^{f,h}_{\Gamma,c,b}$ are related to the topic of this school: Mock modular forms and Mock Jacobi forms. Assume that $f \in S_\Gamma$ and $h \in C_\Gamma \cap \Gamma$, then $\Theta^{f,h}_{\Gamma,c,b}$ will in general no longer have modular properties. But it turns out that it is the holomorphic part of the nonholomorphic theta functions introduced by Zwegers in [30], and thus in a suitable sense it is a Mock Jacobi form. It follows that the generating functions for Euler numbers, χ_y-genera and Donaldson invariants for any ample class H can be expressed in terms of mock modular forms and mock Jacobi forms.

4.1. **Relation to Zwegers Theta functions and Mock modular properties.** We have seen that the indefinite theta functions $\Theta^{f,g}_{\Gamma,c,b}$ are Jacobi forms if $Q(f) = Q(g) = 0$. Otherwise the function do not have any evident modular properties. However it turns out that for f,h arbitrary the $\Theta^{f,h}_{\Gamma,c,b}$ are the holomorphic part of the Zwegers theta functions, which are real analytic Jacobi forms. In other forms, the $\Theta^{f,h}_{\Gamma,c,b}$ are Mock Jacobi forms.

Notice that by definition $\Theta^{f,h}_{\Gamma,c,b}(\tau,x) = \Theta^{f,h}_{\Gamma,c}(\tau,x{+}b/2)$, with $\Theta^{f,h}_{\Gamma,c}\tau,x) = \Theta^{f,h}_{\Gamma,c,0}(\tau,x)$. We assume for simplicity that $B(f,v) \neq 0 \neq B(h,v)$, for all $v \in \Gamma + c/2$. Then the we can rewrite the definition of $\Theta^{f,h}_{\Gamma,c}$ as

$$\Theta^{f,h}_{\Gamma,c}(\tau,x) := \frac{1}{2}\sum_{v\in\Gamma+c/2}\big(\operatorname{sgn}(B(f,v)) - \operatorname{sgn}(B(h,v))\big)q^{Q(v)}e^{2\pi i B(v,x)}.$$

We will write $y := \Im(\tau)$. Following Zwegers this indefinite Theta function can be made modular by adding a nonholomorphic part.

Definition 33.

$$\widehat{\Theta}^{f,h}_{\Gamma,c}(\tau,x) := \frac{1}{2}\sum_{v\in\Gamma+c/2}\left(E\left(\frac{B(f,v+\frac{\Im(x)}{y})\sqrt{y}}{\sqrt{-Q(f)}}\right)\right.$$
$$\left.-E\left(\frac{B(g,v+\frac{\Im(x)}{y})\sqrt{y}}{\sqrt{-Q(g)}}\right)\right)q^{Q(v)}e^{2\pi iB(v,x)},$$

and $\Theta^{f,h}_{\Gamma,c,b}(\tau,x) = \Theta^{f,h}_{\Gamma,c,b}(\tau,x+b/2)$. Here E denotes the incomplete error function

$$E(x) = 2\int_0^x e^{-\pi u^2}du.$$

In fact using Zweger's notation from [30] we have that

$$\widehat{\Theta}^{f,h}_{\Gamma,c}(\tau,x) = q^{-B(\alpha,\alpha)}e^{-B(\alpha,\beta)}\vartheta^{f,h}_{\alpha+c,\beta}(\tau),$$

where $x = \alpha\tau+\beta$. It is known that

$$E(x) = sgn(x)(1-\beta(x^2)), \tag{11}$$

where $\beta(t) = \int_t^\infty u^{-1/2}e^{-\pi u}$. In particular if $f\in S_\Gamma$, we see that $E\left(\frac{B(f,v+\frac{\Im(x)}{y})\sqrt{y}}{\sqrt{-Q(f)}}\right) = \mathrm{sgn}(B(x,v))$, thus $\widehat{\Theta}^{f,g}_{\Gamma,c}(\tau,x) = \Theta^{f,g}_{\Gamma,c,0}(\tau,x)$ in case $f,g\in S_\Gamma$.

From now on assume that $f\in S_\Gamma$ and $h\in C_\Gamma\cap\Gamma$. Then we have

$$\widehat{\Theta}^{f,h}_{\Gamma,c}(\tau,x) := \Theta^{f,h}_{\Gamma,c}(\tau,x) - \Phi^h_c(\tau,x).$$

where

$$\Phi^h_c(\tau,x) = \frac{1}{2}\sum_{v\in\Gamma+c/2}\left(sign(B(v,h) - E\left(\frac{B(h,v+\Im(x)/y)\sqrt{y}}{\sqrt{-Q(h)}}\right)\right)e^{2\pi iB(v,x)}q^Q(v).$$

We have assumed that $B(h,v)\neq 0$, i.e. $|B(h,v)|\geq 1/2$ for all $v\in\Gamma+c/2$. Thus if $|\Im(x)/y|<1/2$, we can rewrite this as

$$\Phi^h_c(\tau,x) = \frac{1}{2}\sum_{v\in\Gamma+c/2}\left(sign(B(v,h)\beta\left(\frac{yB(h,v+\Im(x)/y)^2}{-Q(h)}\right)e^{2\pi iB(v,x)}q^Q(v).$$

$\Theta^{f,h}_{\Gamma,c}(\tau,x)$ is the Mock part and $\Phi^h_c(\tau,x)$ is the nonholomorphic part of $\widehat{\Theta}^{f,h}_{\Gamma,c}(\tau,x)$. We can write $\Phi^h_c(\tau,x)$ as a finite sum $\Phi^h_c(\tau,x) = R_j\theta_j$, where R_j is a nonholomorphic unary theta function for the negative definite lattice $\langle h\rangle$ and θ_j is a standard theta function for $\langle h\rangle^\perp$. A typical example for R_j would be

$$R(\tau,z):=\sum_{v\in\mathbb{Z}+\frac{1}{2}}(-1)^{v-1/2}\big(\operatorname{sgn}(v)-E\big((v+\Im(z)/y)\sqrt{2y}\big)\big)e^{-2\pi ivz}q^{-v^2},$$

for $z\in\mathbb{C}$, which plays a role in Zweger's work.

From the results of [30] it follows that the theta functions $\widehat\Theta^{f,h}_{\Gamma,c}(\tau,x)$ have modular properties. For simplicity we only list the case $c=0$.

Proposition 34. (1) $\widehat\Theta^{f,h}_{\Gamma}(\tau+2,x)=\widehat\Theta^{f,h}_{\Gamma}(\tau,x)$
(2) $\widehat\Theta^{f,h}_{\Gamma}(-\frac{1}{\tau},\frac{x}{\tau})=i(i\tau)^{r/2}\widehat\Theta^{f,h}_{\Gamma}(\tau,x)$.

Note that by definition $\sum_{c\in\Gamma/2\Gamma}\widehat\Theta^{f,h}_{\Gamma,c}(\tau,x)=\widehat\Theta^{f,h}_{\Gamma}(\tau/4,x)$.

Thus roughly we can say that for $h\in C_\Gamma\cap\Gamma$, $f\in S_\Gamma$, the theta functions $\Theta^{f,h}_{\Gamma,c}(\tau,x)$ can be expressed in terms of Mock Jacobi forms.

4.2. **Applications to moduli of sheaves.** There are many applications of this and more generally of mock modular forms to invariants of moduli spaces (see e.g. [2],[26],[18],[1]). We will list only some very simple examples.

Let S be a rational surface with $-K_S$ effective. Assume there is a morphism $S\to\mathbb{P}^1$ whose general fibre is isomorphic to $\mathbb{P}^1$. Let F be the class of a fibre and assume $(F\cdot C)$ odd. Then Corollary 23 expresses the generating functions $\sum_{d\ge 0}\chi_{-y}(M^H_S(C,d))q^{d-e(S)/12}$, $\sum_{d\ge0}e(M^H_S(C,d))q^{d-e(S)/12}$, $D^{S,H}_C(\exp(Az+px))$ in terms of $\Theta^{F,H}_{\Gamma,c,b}(\tau,x)$ where now Γ is $H^2(S,\mathbb{Z})$ with the negative of the intersection form. If there is no such morphism $S\to\mathbb{P}^1$, we can reduce to this situation by the use of blowup formulas. By the above we will get that these generating functions can be expressed in terms of Mock modular forms and Mock Jacobi forms.

Below we will list (without proofs) a some examples of this.

We restrict attention to the case of $\mathbb{P}^2$.

We denote by $\sigma(X)$ the signature of the intersection form of X on the middle cohomology. With our definition of χ_{-y} we have $\sigma(X)=(-1)^{dim(X)/2}\chi_1(X)$.

Let $H(n)$ be the Hurwitz class number, i.e. the number of equivalence classes of quadratic forms Q of discriminant $-n$ (counted with

$1/|Aut(Q)|$). Put

$$\mathcal{H}_1(\tau) := \sum_{n\geq 0} H(4n+3)q^{n+3/4}.$$

We also consider

$$h_1(\tau) := \sum_{n\geq 0} \frac{\prod_{k=1}^{2n}(1+q^k)}{\prod_{k=1}^{n+1}(1-q^{2k-1})}q^n.$$

This is a Mock modular form considered by Ramanujan in his "lost" notebook.

Proposition 35. (1)

$$\sum_{d\geq 0} e(M_{\mathbb{P}^2}^H(H,d))q^{d-1/4} = \frac{3\mathcal{H}_1(\tau)}{\eta(\tau)^6}.$$

(2)

$$\sum_{d\geq 0} \sigma(M_{\mathbb{P}^2}^H(H,d))q^{d-1/4} = qh_1(\tau)\frac{\eta(\tau)^2}{\eta(2\tau)^4}$$

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