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Bilateral Barters and Marekt Equilibrium - an adaptive, behavioral approach

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Bilateral Barters and Market Equilibrium

- an adaptive, behavioral approach

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The plan

Consider an **exchange economy** with **transferable utility** and **resources.**

What is meant by **equilibrium**?

Can equilibrium be reached by **bilateral barters**?

by temporarily **relaxing individual constraints?**

when **enforcing those constraints?**

The setting

- There are several agents,
- each holding his endowment (= resource bundle = contingent claim).
- They proceed by repeated bilateral barters.

Issues:

- Will equilibrium obtain?
- Is there room for "simple" agents?
- Is optimization really needed?
- Must prices be announced?

DATA

- *agent* $i \in I$
- owns *endowment* $e_i \in \mathbb{X}$ (a vector space),
- *faces constraint* $x_i \in X_i$ closed convex $\subseteq \mathbb{X}$, and
- *wants* to maximize quasi-linear concave utility $u : \mathbb{X} \to \mathbb{R}$. **But** he
- *lacks* computational competence, perfect foresight, information, global vision, ...

- There is **no** coordination, **no** auctioneer, **no** market maker,...
- Moreover: bargaining, matching, search is not made explicit
- Nonetheless: Maybe holdings converge to equilibrium?
- Inspiration from Pareto:
  "*The economy is a great computing machine.*"

Equilibrium

- (x_i) an *allocation* iff $\sum_i x_i = \sum_i e_i =: e_I$
- *feasible allocation* iff moreover, $x_i \in X_i$ for each i.

Definition (Equilibrium) A feasible allocation (x_i) and a linear price $p : \mathbb{X} \to \mathbb{R}$ constitute and **equilibrium** iff

$$u_i(x_i) + p(e_i - x_i) \geq u_i(\chi) + p(e_i - \chi) \quad \text{for each } i \in I \text{ and } \chi \in X_i.$$

- Utility + net value of resource sale is maximal!

Intermezzo: the nature of equilibrium

- Recall that a linear $x^* : \mathbb{X} \to \mathbb{R}$ is a **supergradient** of the proper function $f : \mathbb{X} \to \mathbb{R} \cup \{-\infty\}$ at x, written $x^* \in \partial f(x)$, iff

$$f(\chi) \le f(x) + x^*(\chi - x) \quad \text{for all} \quad \chi \in \mathbb{X}.$$

- Also recall: a linear $n : \mathbb{X} \to \mathbb{R}$ is a **normal vector** to a proper subset $X \subseteq \mathbb{X}$ at $x \in X$ iff

$$n(\chi - x) \le 0 \quad \text{for all} \quad \chi \in X.$$

Proposition (On equilibrium) *A profile (x_i) alongside a price p constitutes an equilibrium iff*

$$p \in \partial u_i(x_i) - N_i(x_i) \text{ for each } i, \quad \text{and} \quad \sum_{i \in I} x_i = e. \quad \square$$

· that is: there is a *common* price
· price "=" marginal utility (modulo normal components)

More on the nature of equilibrium: Cooperative aspects

- Suppose coalition $C \subseteq I$ uses $e_C := \sum_{i \in C} e_i$ to get

$$u_C(e_C) := \sup \left\{ \sum_{i \in C} u_i(x_i) : \sum_{i \in C} x_i = e_C \ \ \& \ x_i \in X_i \right\}.$$

Recall that a payment scheme (π_i) is in the **core** of this transferable-utility game iff

$$\begin{cases} \text{Pareto efficient:} & \sum_{i \in I} \pi_i = u_I(e_I) \\ \text{stable against blocking:} & \sum_{i \in C} \pi_i \geq u_C(e_C) \quad \text{for each } C \subset I. \end{cases}$$

Proposition (Equilibrium as core solution) *For any equilibrium price p the payment scheme*

$$i \mapsto \pi_i := \sup \{ u_i(\chi) + p(e_i - \chi) : \chi \in X_i \}$$

is in the core. □

A first simple guideline for exchange:

- **A brave approach:** When agent i meets agent j, they compare gradients: Let

$$g_i = u_i'(x_i) \quad \text{and} \quad g_j = u_j'(x_j)$$

$$\text{Is} \quad g_i \quad \neq \quad g_j'?$$

- If yes, for suitable **step-size** $\sigma > 0$, **transfer**

$$\Delta x_i := \sigma \left[g_i - g_j \right] \quad \text{to } i \quad \text{from } j.$$

- Similarly, j gets

$$\Delta x_j := \sigma \left[g_j - g_i \right] = -\Delta x_i$$

- Note, no central coordination. Fully decentralized.

- The process *stops* when all gradients are equal.
- In equilibrium

$$\text{all gradients are equal} = \text{price } p.$$

Alas, it's not that simple: what about constraints & differentiability

Constraints $x_i \in X_i$? **Differentiability** of objectives?

- There are two strategies: *exact penalties* or *projection techniques*. **Here, exact penalties**, using *distance functions*

$$d_i(\chi) := \min \{ \|\chi - x_i\| : \ x_i \in X_i \}$$

- Replace original utility function $u_i(x)$ with

$$\hat{u}_i(x_i) = u_i(x_i) - c_i d_i(x_i), \quad \text{coefficient } c_i > 0 \text{ sufficiently large.}$$

- NB! $\hat{u}_i = u_i$ on X_i. The new function $\hat{u}_i$ remains concave.
- $\hat{u}_i$ is non-differentiable on the boundary of X_i

Exact penalty and market equilibrium

Proposition *Suppose* $c_i >$

$$\sup\left\{\|g_i - p\| : g_i \in \partial u_i(x_i) \text{ for some } x_i \in X_i, \text{ and } p = \text{ equilibrium price}\right\}$$

Then each equilibrium in the constrained economy $(u_i, X_i, e_i)_{i\in I}$ *is also one for the unconstrained one* $(\hat{u}_i, \mathbb{X}, e_i)_{i\in I}$.

Standing hypothesis: the penalty coefficients c_i are so large that

$$\begin{aligned} & \arg\max\left\{\sum_{i\in I}\hat{u}_i(x_i) : \sum_{i\in I} x_i = e_I\right\} \\ \subseteq \quad & \arg\max\left\{\sum_{i\in I} u_i(x_i) : \sum_{i\in I} x_i = e_I \ \& \ x_i \in X_i\right\} \end{aligned}$$

Bilateral exchange with temporary infeasibility

- **Start** any allocation: $\sum_{i \in I} x_i = e_I$.

- **Pick two agents** i, j in equiprobable manner. These hold x_i and x_j respectively.

- **Select supergradients** $\hat{g}_i \in \partial \hat{u}_i(x_i)$ and $\hat{g}_j \in \partial \hat{u}_j(x_j)$.

- **Update holdings**

$$x_i \leftarrow x_i + \sigma(\hat{g}_i - \hat{g}_j) \quad \text{and} \quad x_j \leftarrow x_j + \sigma(\hat{g}_j - \hat{g}_i)$$

Update step-size σ and **Continue** to **Pick two agents** until **Convergence.**

Convergence to market equilibrium

Theorem (Convergence to market equilibrium) *Under standing hypothesis, suppose all* u_i are *Lipschitz continuous, and*

$$\sum_{k=0}^{\infty} \sigma_k = +\infty \quad \& \quad \sum_{k=0}^{\infty} \sigma_k^2 < +\infty.$$

Then, for any initial allocation (x_i^0), *the process* (x_i^k) *converges with probability one to an equilibrium allocation.*

A simple example

3 commodities $c \in \mathbb{C} = \{1, 2, 3\}$, and 3 agents $i \in \{1, 2, 3\}$, $X_i := \mathbb{R}_+^{\mathbb{C}}$,

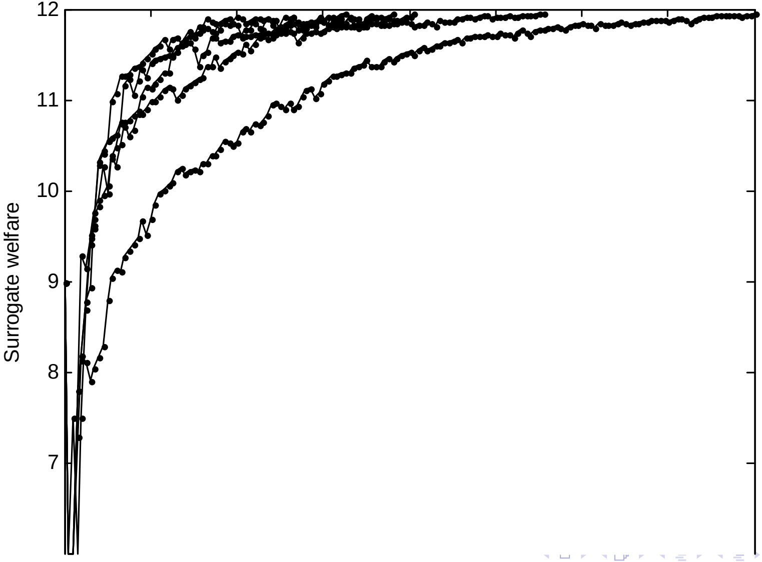
Agent	Payoff $u_i(x_i)$	Endowment e_i
1	$2x_{11} + 1x_{12} + 0x_{13}$	$(1, 1, 0)$
2	$0x_{21} + 2x_{22} + 1x_{23}$	$(0, 1, 1)$
3	$1x_{31} + 0x_{32} + 2x_{33}$	$(1, 0, 1)$

Pareto opt. allocation $[(2, 0, 0), (0, 2, 0), (0, 0, 2)]$, $u_I(e_I) = 12$, shadow price $p = (2, 2, 2)$.

Here $\delta_i := \|g_i - p\| = \sqrt{5}$, hence $c_i = 3$ suffices. $x = [x_c] \in \mathbb{X}$ has closest approximation $\bar{x} := [\max(x_c, 0)] \in \mathbb{R}_+^{\mathbb{C}}$.

Example illustrated:

$\sum_i \hat{u}_i \left(x_i^k\right)$, $\sigma_k = 1/k$.



Objections to bilateral exchange

$$x_i \leftarrow x_i + \sigma(\hat{g}_i - \hat{g}_j) \quad \text{and} \quad x_j \leftarrow x_j + \sigma(\hat{g}_j - \hat{g}_i)$$

Does not

1) distinguish between durable and perishable goods,
2) accommodate exogenous risk,
3) allow parallel encounters,
4) evoke explicit bargaining, deliberate search, or dynamic matching,
5) account for network features,
6) fit instances where some constraint(s) must be enforced throughout,
7) always ensure swift convergence,
8) compete very well qua algorithm.

Hereafter more on 6) hence projection techniques.

Maintaining feasibility throughout

Cone of feasible directions of X_i at $x_i \in X_i$

$$D_i(x_i) := \mathbb{R}_+(X_i - x_i).$$

Tangent cone

$$T_i(x_i) = clD_i(x_i).$$

Cone of common directions

$$T_{ij}(x_i, x_j) := T_i(x_i) \cap -T_j(x_j).$$

Maximal slope of joint improvement

$$\mathfrak{S}_{ij}(x_i, x_j) := \max_d \left\{ u_i'(x_i; d) + u_j'(x_j; -d) : d \in T_{ij}(x_i, x_j) \ \& \ \|d\| \leq 1 \right\}.$$

More on the maximal slope

$$\mathfrak{S}_{ij}(x_i, x_j) := \max_d \left\{ u_i'(x_i; d) + u_j'(x_j; -d) : d \in T_{ij}(x_i, x_j) \;\&\; \|d\| \leq 1 \right\}.$$

P_{ij} = orthogonal projection onto $T_{ij}(x_i, x_j)$:

$$\mathfrak{S}_{ij}(x_i, x_j) = \min \left\{ \| P_{ij}\left[g_i - g_j\right] \| : g_i \in \partial u_i(x_i), \; g_j \in \partial u_j(x_j) \right\}.$$

Also, with $dist(C_i, C_j) := \inf \| C_i - C_j \|$,

$$\mathfrak{S}_{ij}(x_i, x_j) = dist\left[\partial u_i(x_i) - N_i(x_i), \partial u_j(x_j) - N_j(x_j)\right].$$

Bilateral exchange with feasibility maintained throughout

- **Start** any feasible allocation: $\sum_{i \in I} x_i = e_I$.

- **Pick two agents** i, j in equiprobable manner. These hold $x_i \in X_i$ and $x_j \in X_j$ respectively.

- **Select a common direction** $d = P_{ij}[g_i - g_j]$ where $g_i \in \partial u_i(x_i)$, $g_j \in \partial u_j(x_j)$ yields

$$\mathfrak{S}_{ij}(x_i, x_j) = \|P_{ij}[g_i - g_j]\|$$

- **Choose suitable step-size $\sigma > 0$ and update holdings**

$$x_i \leftarrow x_i + \sigma d \quad \text{and} \quad x_j \leftarrow x_j - \sigma d$$

Continue to **Pick two agents** until **Convergence.**

On common price and joint improvement

Recall that in equilibrium

$$p \in \cap_{i \in I} \left[\partial u_i(x_i) - N_i(x_i) \right].$$

We say agents i, j **see a common price** iff

$$[\partial u_i(x_i) - N_i(x_i)] \cap [\partial u_j(x_j) - N_j(x_j)] \neq \varnothing.$$

Proposition *i, j see a common price iff*

$$\mathfrak{S}_{ij}(x_i, x_j) = 0. \quad \square$$

Thus joint improvement is possible as long as $\mathfrak{S}_{ij}(x_i, x_j) > 0$.

Complete trade (no more joint improvement)

Trade is complete in the set

$$\mathbb{C} := \{\text{feasible allocations } (x_i) : \text{ each } \mathfrak{S}_{ij}(x_i, x_j) = 0\}$$

Two questions:
1) Will (x_i^k) "converge" to $\mathbb{C}$?

2) Will each profile $(x_i) \in \mathbb{C}$ be an equilibrium?

Standing hypotheses now: *The set of feasible allocations is bounded*, and

$$T_{ij}(x_i, x_j) = cl\left[D_i(x_i) \cap -D_j(x_j)\right].$$

Convergence and equilibrium

Proposition (On convergence) *Suppose each* $D_i(x_i) = \mathbb{R}_+(X_i - x_s)$ *is closed, and that those trade for which the slope* $\mathfrak{S}_{ij}(x_i, x_j)$ *is largest. Then* (x_i^k) *clusters to the set*

$$\mathbb{C} := \{\text{feasible allocations } (x_i) : \text{ each } \mathfrak{S}_{ij}(x_i, x_j) = 0\}.$$

Proposition (On equilibrium) Each profile $(x_i) \in \mathbb{C}$ is an equilibrium if
1) $\mathbb{X}$ is one-dimensional or
2) Some agent i has $x_i \in intX_i$ and u_i differentiable at x_i-

Concluding remarks on equilibrium and dynamics?

- How can players arrive at equilibrium - if any?
- While underway, how much competence, coordination, and foresight is required?
- What are the roles of cognition and perception?

Lacunes in economic theory

only concerned with equilibrium,

modestly interested in computation, uniqueness, stability, or attainability

most often out-of-equilibrium behavior gets no mention,

cognition and perception are hardly in focus.

Bilateral barters as viewed here

requires no coordination, experience, foresigth, or optimization,..

It's totally decentralized.

Fully driven by low-complexity adaptive agents