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Some mathematical properties of order book models

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Some empirical and mathematical properties of limit order books

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Limit order books

- **Joint works (some in progress) with I. Muni Toke, A. Jedidi**
- **References**
  - **I. Muni Toke, *Market making behaviour and its impact on the Bid-Ask spread,* in *Econophysics of Order-driven Markets,* Abergel, F.; Chakrabarti, B.K.; Chakraborti, A.; Mitra, M. (Eds.), Springer, 2011**
  - **F. Abergel, A. Jedidi, *A mathematical approach to order book modelling*, http://papers.ssrn.com/sol3/papers.cfm?abstract_id=1740889**
  - **F. Abergel, A. Chakraborti, I. Muni Toke, M. Patriarca, *Econophysics I: empirical facts* and *Econophysics II: agent-based models*, to appear in *Quantitative Finance***

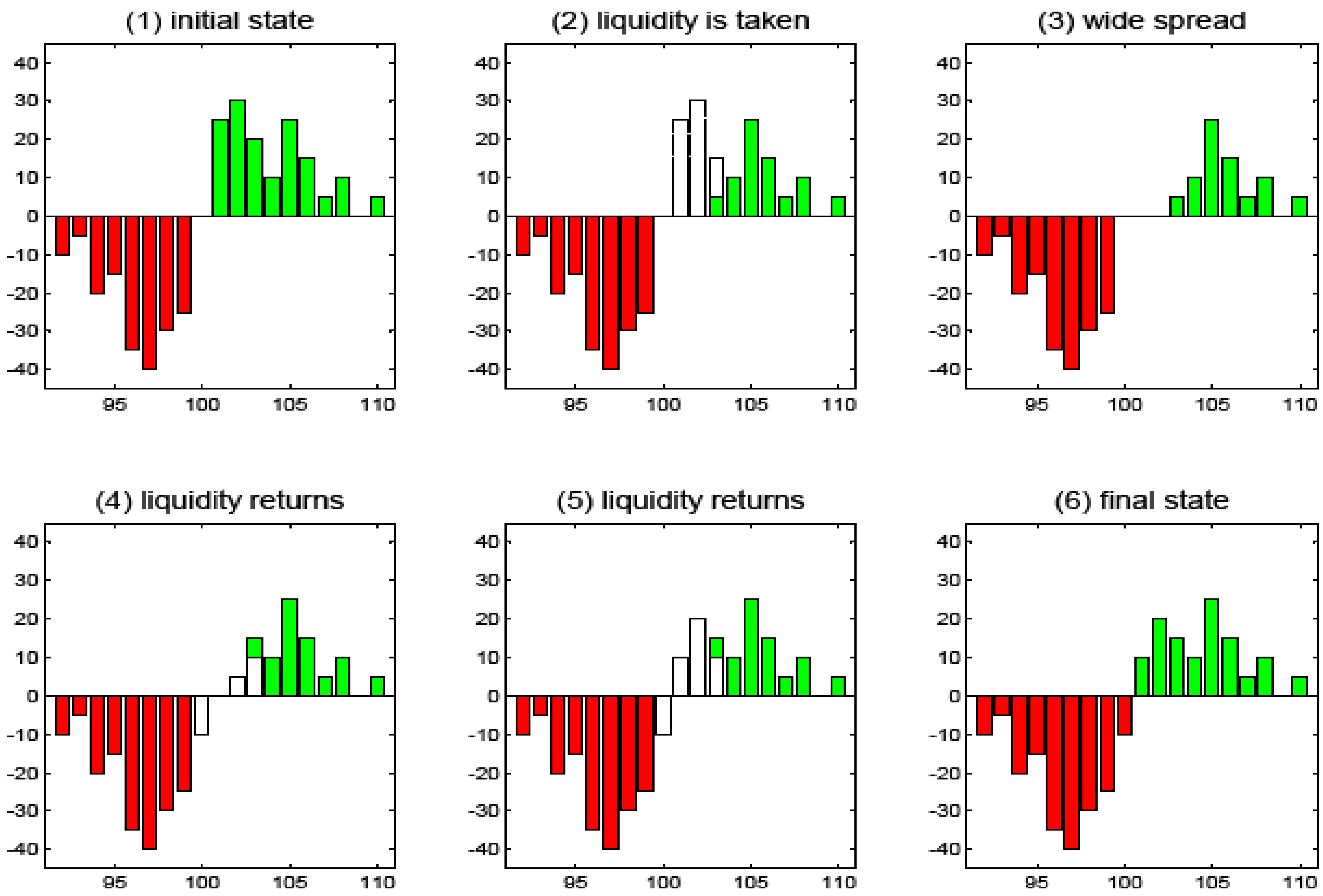
Limit order books

Summary

- **Empirical properties of the order book**
  - **Stationary statistical properties**
  - **Dynamical statistical properties**

- **Mathematical models**
  - **Mathematical framework**
  - **Price dynamics**

Limit order book



Statistical properties I

- **A host of empirical studies going back to ~20 years, addressing the two following questions:**
  - **When will the next event take place ?**
  - **Where will the next event take place ?**
- **Under independence assumptions:**

Zero-intelligence models

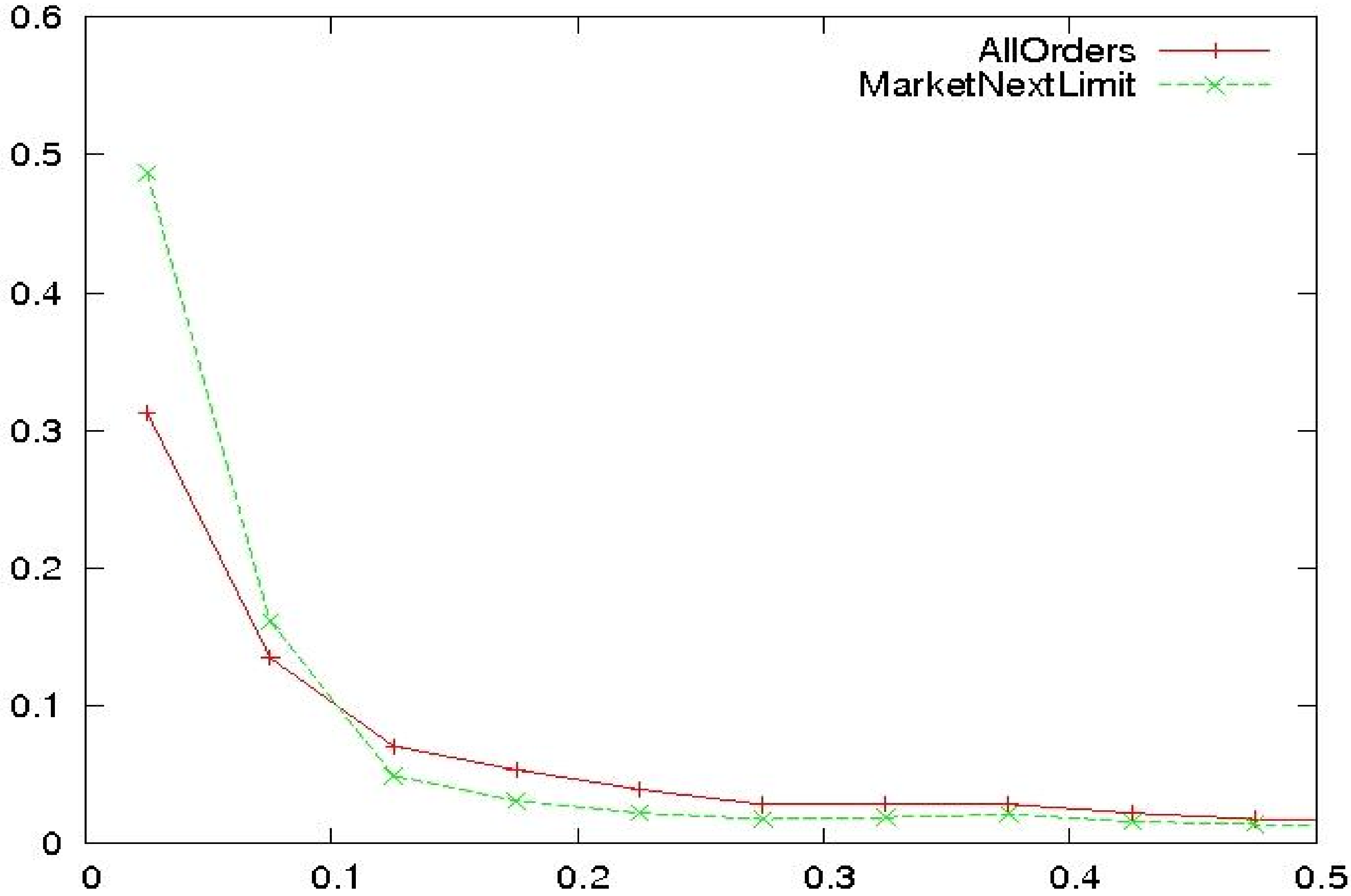
Statistical properties I

- Such unconditional statistics do not fully reflect the dynamics of a limit order book.
- Many interesting phenomena are not described this way
  - Volatility clustering
  - Leverage
  - Autocorrelation of the order flow
- In real markets, agents observe the state of the market and adapt to it
- An example (Muni-Toke): empirical evidence of market making and market taking

Statistical properties II
Market making

- **Following a market order**
  - **New limit orders arrive more rapidly than unconditional limit orders**
  - **No significant correlation between the respective signs of the market and limit orders**

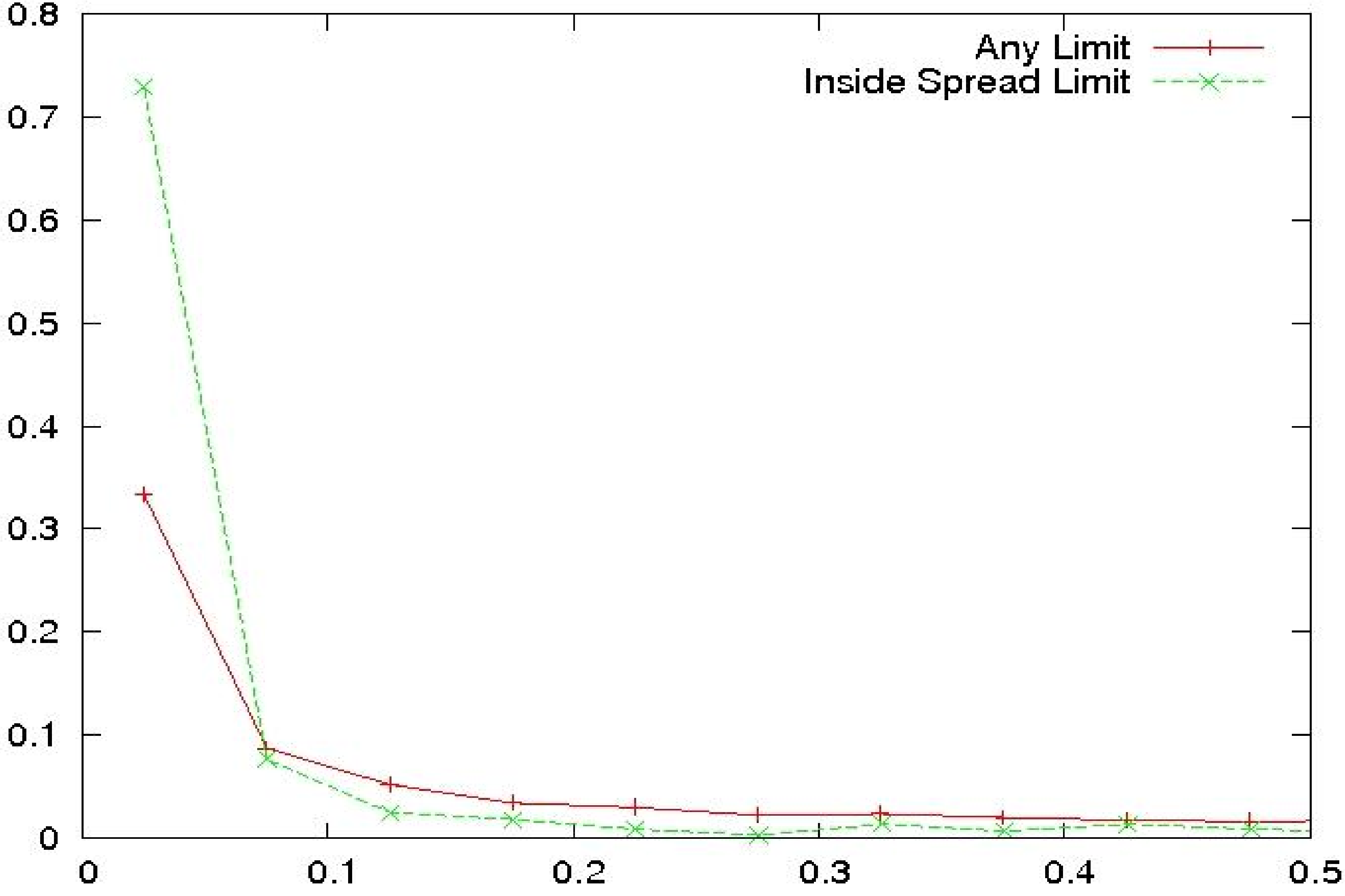
Statistical properties II
Market making



Statistical properties II
Market taking

- **Following a limit order**
  - **New market orders do not arrive more rapidly…**
  - **… except when the limit order fell within the spread**

Statistical properties II
Market taking



Statistical properties II

- **Several recent studies accounting for dependencies (Large 2007, Muni Toke 2010, Eisler 2010)…**
  - **Conditional inter-event duration**
  - **Lead and lag relationship**
  - **Conditional price and volume distributions**

- **… lead to models involving**

  - ***State-dependent intensities and placement***
    - ***Mutually excited processes***

Mathematical framework

- **The limit order book: a vector valued point process**

- **Main questions to be addressed**
  - **Stationarity**
  - **Price and spread dynamics**
  - **Scaling and long time asymptotics**

Mathematical framework

- **Back to the simplest example: zero-intelligence model with limit orders, market orders and cancellations (Farmer, Smith, Guillemot, Krishnamurthy, 2003)**

$$dL_t^{i\pm} \qquad \lambda_L^{i\pm} \frac{\Delta P}{\tau}$$

$$dM_t^{\pm} \qquad \lambda_M^{\pm} \frac{1}{\tau}$$

$$dC_t^{i\pm} \qquad \lambda_C^{i+} \frac{a_i}{\tau}, \ \lambda_C^{i-} \frac{b_i}{\tau}$$

Mathematical framework

- **Two sets of variables**

$$a_1, \ldots, a_N ; b_1, \ldots, b_N$$

$$a_i \equiv a(i\Delta P) \qquad b_i \equiv b(i\Delta P)$$

- **Coupled dynamics**

$$A_i = \sum_{k=1}^{i} a_i \qquad B_i = \sum_{k=1}^{i} b_i$$

- **Two basic types of events**
  - **Jump: a change in the quantities**
  - **Shift : renumbering after a change of one of the best quotes**

Stationarity

- In this simple model, there exists a Lyapunov function (the total available volume), thanks to the exponential damping effect of cancellations
- Therefore, there exists a stationary distribution with exponential convergence
- This result can be generalized to state-dependent intensities

Extension to Hawkes processes

- **Hawkes processes:**
  - **a point process with stochastic intensity**
  - **The intensity is excited by the previous jumps (autoregressive process)**

$$\lambda_t^j = \lambda_0^j + \sum_{p=1}^{N} \int_{-\infty}^{t} \varphi_{jp}(t-s)\, dN_s^P$$

  - **Typical choice: exponential kernels**

$$\lambda_t^j = \lambda_0^j + \sum_{p=1}^{N} \int_{-\infty}^{t} \alpha_{jp} e^{-\beta_{jp}(t-s)} dN_s^P$$

  - **Becomes a Markov process in 1D (or higher with equal decay rates)**

Extension to Hawkes processes

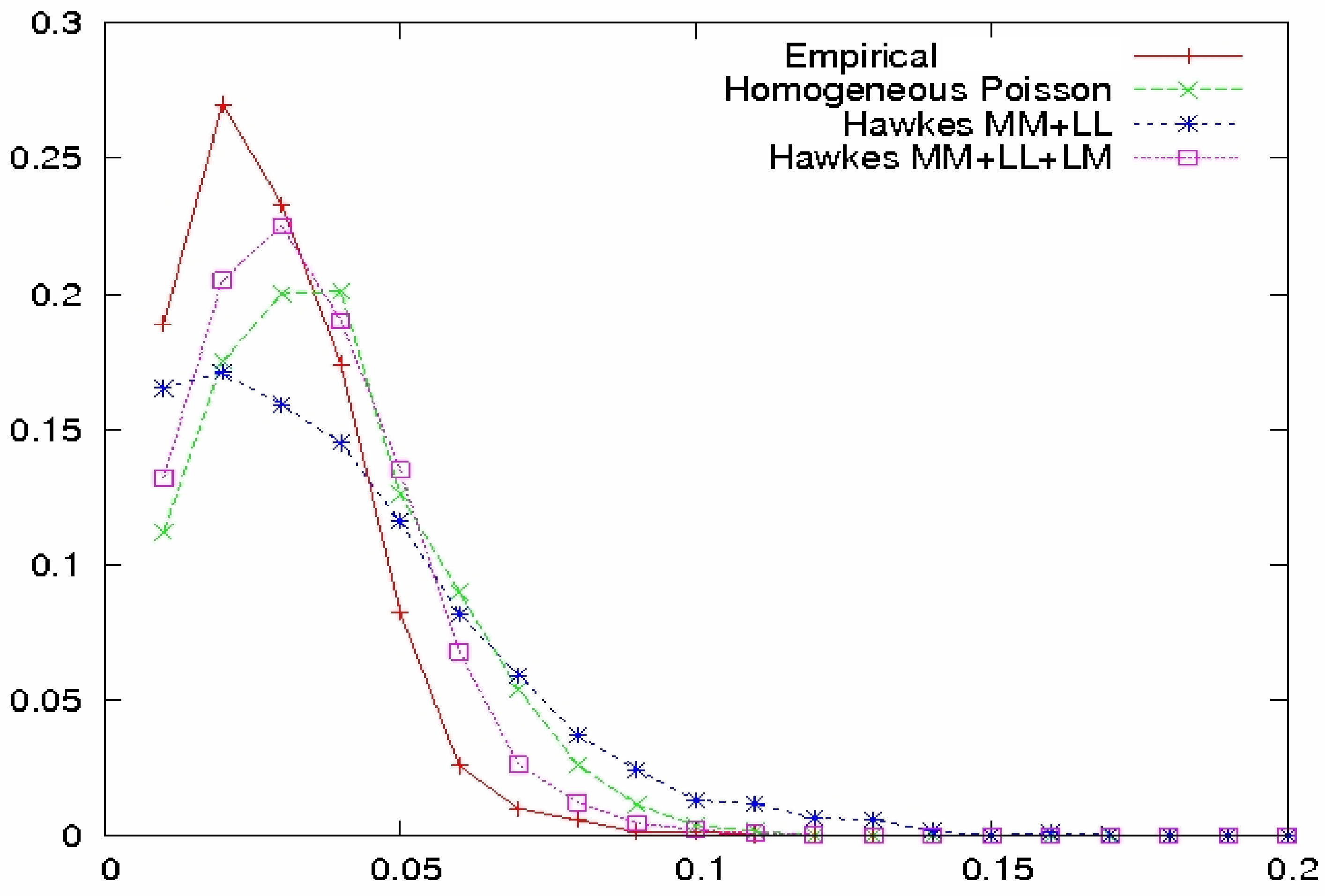
- Clustering of orders easily described
- Leverage modelled thanks to asymetric kernels
- Stationarity conditions related to the values of the Hawkes parameters

$$\left(E\left(\lambda^{j}\right)\right)=\left(Id-\left[\frac{\alpha_{jp}}{\beta_{jp}}\right]\right)^{-1}\left(\lambda_0^{j}\right)$$

- Stationarity conditions are found satisfied in empirical studies (Muni Toke, Hewlett, Large…)

Hawkes processes
Spread distribution

- **A consequence of better modelling: spread distribution**



Price dynamics

- **Price dynamics depend on**
  - **Events affecting the best limits**
  - **The “first gap” process**
- **A useful representation for the best Ask and Bid prices:**

$$dP_t^A = \Delta P \left\{ \left(\left(A_t^{-1}(\tau) - A_t^{-1}(0) \right) \left(dM_t^+ + dC_t^{i_A+} \right) \right) - \sum_{i < B_t^{-1}(0)} \left(B_t^{-1}(0) - i \right)^+ dL_t^{i+} \right\}$$

$$dP_t^B = -\Delta P \left\{ \left(\left(B_t^{-1}(\tau) - B_t^{-1}(0) \right) \left(dM_t^- + dC_t^{i_B-} \right) \right) + \sum_{i < A_t^{-1}(0)} \left(A_t^{-1}(0) - i \right)^+ dL_t^{i-} \right\}$$

Price dynamics

- The expressions above provide a natural interpretation of the price changes: they are due to
  - New limit orders that fall within the spread, for which one can safely assume some independence assumptions
  - Events that modify the best quotes (either cancellations or market orders), for which the price changes depends on the first gaps $\left(A_t^{-1}(\tau)-A_t^{-1}(0)\right)$ and $\left(B_t^{-1}(\tau)-B_t^{-1}(0)\right)$
- The price process has the following representation $$dP_t = \sum_i X_i^t dN_t^i$$
  - A Bachelier market has a similar representation with i.i.d. marks
  - The marks may be assumed to be identically distributed (under stationarity), but not independent.
  - The long time dynamics is sensitive to the dependence structure of these processes

Price dynamics

- **A mathematical result**
  - **The centered price process in a zero-intelligence model with proportional cancellation rate scales to a brownian motion in the long time limit**
    - **Not a surprise from the physicist's point of view…**
    - **A first general result relating order book models and classical price models**
    - **The "spurrious" randomness of the volatility due to the memory of the order book vanishes exponentially fast in this simple case**
    - **Extensions to state dependent intensities, Hawkes processes**

Price dynamics

- The case of local (endogenous) or stochastic (exogenous) intensities allows one to mimick some classical local and stochastic volatility models
- The “leaner” the order book, the closer the dynamics is to standard diffusion models
- Long memory may appear in the case of slow cancellation rates, slow decay kernel…

Order book modelling

Conclusion

- **Empirical studies of the order book**
  - **A large body of empirical results**
  - **Conditional quantities contain a lot of relevant information**
  - **The behaviour of market participants at the best limits tends to "control" the dynamics of price and spread**

- **Mathematical modelling**
  - **A general framework suitable for many extensions**
  - **An approach bridging the gap between order book dynamics and price process**