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Liquidity and coherent risk measures

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Introduction - 1

- What is *liquidity risk*? There are several answers
  - **Treasurer**'s answer: the risk of running short of cash
  - **Trader**'s answer: the risk of trading in *illiquid* markets, i.e. markets where exchanging assets for cash may be difficult or even impossible
  - **Central Bank**'s answer: the risk of concentration of cash among few economic agents and related systemic effects
- The three facets of liquidity risk are interconnected
- Setting a precise mathematical framework is not easy

Introduction - 2

- We do **not** propose a measure for liquidity risk
- Instead, we propose an approach for valuing a portfolio under liquidity risk
- The value depends on
  - external factors (**market liquidity**)
  - internal factors (**liquidity constraints**)
- The same portfolio is valued differently by different owners
- The liquidity-adjusted value affects the portfolio risk (as measured by a coherent risk measure)

References

- *Liquidity risk theory and coherent risk measures*
  C.Acerbi, G.Scandolo. Quantitative Finance 8-7 (2008)

- *The value of liquidity*
  C.Acerbi, C.Finger. RiskMetrics Technical paper (2010)

- *What my friend means to say is...*
  C.Finger. RiskMetrics Research Monthly (2009)

Outline

- A critique to coherent risk measures
- The theoretical framework
  - Portfolios and Marginal Supply-Demand Curves
  - Liquidation value vs. usual mark-to-market value
  - Liquidity policies and general mark-to-market values
  - Numerical examples
- Back to coherent risk measures

Coherent risk measures - 1

- A **coherent** risk measure $\rho = \rho(X)$ satisfies
  1. *Cash equivariance*
     $$\rho(X + c) = \rho(X) - c \quad \forall c$$
  2. *Monotonicity*
     $$\rho(X) \leqslant \rho(Y) \quad \text{if } X \geqslant Y$$
  3. *Positive homogeneity* (PH)
     $$\rho(\lambda X) = \lambda \rho(X) \quad \forall \lambda > 0$$
  4. *Subadditivity* (Sub)
     $$\rho(X + Y) \leqslant \rho(X) + \rho(Y)$$

  (see Artzner et al., Math Fin 1998)
- Remind: Value-at-risk misses (Sub), while Expected Shortfall (aka CVaR) is coherent

Coherent risk measures - 2

- Critique to properties (PH) and (Sub):
  - *doubling the portfolio, its risk should more than double in many cases*
- Instead

$$\begin{aligned} \text{(PH)} \quad &\Longrightarrow \quad \rho(2X) = 2\rho(X) \\ \text{(Sub)} \quad &\Longrightarrow \quad \rho(2X) \leqslant 2\rho(X) \end{aligned}$$

  In other words, coherent risk measures do not seem to take into account liquidity risk

- (PH) and (Sub) were replaced by the weaker property of convexity. For a **convex** risk measure it may well happen that

$$\rho(2X) > 2\rho(X)$$

  (see Follmer/Schied, 02, Frittelli/Rosazza, 02)

Convex risk measures - 1

- Have convex risk measures been a remedy? **No**
- First of all, one of the simplest examples of convex r.m. which is not coherent is

$$\rho(X) = -\log \mathbb{E}\left[e^{-X}\right]$$

  It is called an **entropic** risk measure (by mathematicians at least...)

  - Would you ever use such a risk measure for day-to-day risk management?
  - How could you explain in plain words: *the entropic risk of my portfolio is 2000 Euro*?
  - Convex risk measures have had almost no impact in practice. They fail to satisfy property 0 for a risk measure:

    The statement $\rho(X) = 2000$ *can be explained in plain words*

Convex risk measures - 2

- Our point against convex r.m. is different: they are not needed to account for liquidity risk
- The key question is: what is X, the argument of ρ ? X is the PL, or alternatively, the market value $V(\bar{\mathbf{p}})$ of the portfolio $\bar{\mathbf{p}}$
- With a slight abuse we write

$$\rho(\bar{\mathbf{p}}) = \rho(V(\bar{\mathbf{p}}))$$

  so the risk measure is at **portfolio-level**
- If we do not take into account liquidity issues, V is **linear** in $\bar{\mathbf{p}}$ and therefore

$$\rho(\lambda\bar{\mathbf{p}}) = \rho(V(\lambda\bar{\mathbf{p}})) = \rho(\lambda V(\bar{\mathbf{p}})) = \lambda\rho(\bar{\mathbf{p}})$$
$$\rho(\bar{\mathbf{p}} + \bar{\mathbf{q}}) = \rho(V(\bar{\mathbf{p}}) + V(\bar{\mathbf{q}})) \leqslant \rho(V(\bar{\mathbf{p}})) + \rho(V(\bar{\mathbf{q}})) = \rho(\bar{\mathbf{p}}) + \rho(\bar{\mathbf{q}})$$

Convex risk measures - 3

- However, in the presence of illiquidity we may well have that

$$V(2\bar{\mathbf{p}}) \neq 2V(\bar{\mathbf{p}})$$

  and/or

$$V(\bar{\mathbf{p}} + \bar{\mathbf{q}}) \neq V(\bar{\mathbf{p}}) + V(\bar{\mathbf{q}})$$

  that is, V need **not** be linear anymore
- Therefore, even for a coherent r.m. $\rho = \rho(X)$, the corresponding r.m. on portfolios may fail to satisfy (PH) and (Sub), so we may well have

$$\rho(2\bar{\mathbf{p}}) > 2\rho(\bar{\mathbf{p}})$$

Basic notation

- It is possible to trade in
  - N **illiquid** assets (equity for simplicity)
  - **cash**, which is by definition the only liquidity risk-free asset
- A **portfolio** is a vector $\bar{\mathbf{p}} = (p_0, \mathbf{p}) \in \mathbb{R}^{N+1}$
  - p_0 is the amount of cash
  - $\mathbf{p} = (p_1, \ldots, p_N)$ is the vector of positions in assets
  - p_n is the number of assets of type n
- For instance $\bar{\mathbf{p}} = (5000, 100, 200, -50)$
  - invests $p_0 = 5000$ in cash
  - takes long positions in the first two assets
  - takes a short position in the third asset

Our approach in a nutshell

- We observe the external factors (**market liquidity**)
- We define, in a natural way, an **upper** and a **lower value** for a portfolio (that is, two ways of marking-to-market)
- We specify the internal factor (**liquidity constraint**)
- Given a portfolio $\bar{\mathbf{p}}$ we **disinvest** part of it, obtaining its lower value in cash, in order for the remaining portfolio to satisfy the liquidity constraint
- We do this in an **optimal** way, i.e. we maximize the upper value of the remaining portfolio
- Not only we end up with the value of our portfolio, but we also have a practical recipe to meet the liquidity constraint

Market liquidity - 1

- In a perfectly liquid market
  - S_n is the unique price, for selling/buying a unit of asset n; this price does **not** depend on the size of the trade
  - The value of a portfolio

$$V(\bar{\mathbf{p}}) = p_0 + \sum_{n=1}^{N} p_n S_n$$

    is linear in $\bar{\mathbf{p}}$
- In illiquid markets
  - the unit price $S_n = S_n(x)$ will depend on the size $x \in \mathbb{R}$ ($x > 0$ is a sale, $x < 0$ is a purchase) of the trade
  - The value of a portfolio

$$V(\bar{\mathbf{p}}) = p_0 + \sum_{n=1}^{N} p_n S_n(p_n)$$

    is not linear anymore
- In both cases $S_0 = 1$

Market liquidity - 2

- In order-driven markets (Borsa Italiana for instance), at any moment and for any asset there is a **Marginal Supply-Demand Curve** (**msdc**), *m*, giving the marginal price for any trade
- If $x > 0$ (x integer), $m(x)$ is the price of the x-th share when selling $p \geqslant x$ shares. The total income when selling p assets is

$$P(p) = \sum_{1 \leqslant x \leqslant p} m(x)$$

  (P for proceeds)
- If $x < 0$, $m(x)$ is the price of the $|x|$-th share when buying $|p| \geqslant |x|$ shares. The total outcome when buying $|p|$ assets is ($p < 0$)

$$P(p) = -\sum_{1 \leqslant x \leqslant |p|} m(-x)$$

- m is **decreasing** (actually constant on long intervals); $m(-1) > m(1)$ are the **best ask** and **bid** prices

Market liquidity - 3

- A snapshot of (part of) the *Poltrone Frau S.p.A.* order book

			best ask	best bid		
price	0.975	0.964	**0.955**	**0.925**	0.92	0.90
q.ty	20	18	50	33	10	50

(real quantities are obtained $\times 100$)

- The bid-ask spread is 0.03 or 0.8% as percentage of the mid price

- The corresponding msdc is

$m(x)$	0.975	0.964	0.955	0.925	0.92	0.90
x	[-88,-69]	$[-68, -51]$	$[-50, -1]$	$[1, 33]$	$[34, 43]$	[44,93]

Market liquidity - 4

- If we sell $p = 50$ shares, the income is

$$P(50) = \sum_{1 \leqslant x \leqslant 50} m(x) = 33 \cdot 0.925 + 10 \cdot 0.92 + 7 \cdot 0.90 = 46.02$$

  We are fully *marking-to-market*. We call this quantity the liquidation value or the **lower value** of 50 shares and denote it $L(50)$.

- If we instead mark to the best bid price we have

$$U(50) = 50 \cdot 0.925 = 46.25 > L(50),$$

  that we call the **upper value** of 50 shares. The difference

$$C(50) = U(50) - L(50) = 0.23$$

  is the **liquidation cost**

Market liquidity - 5

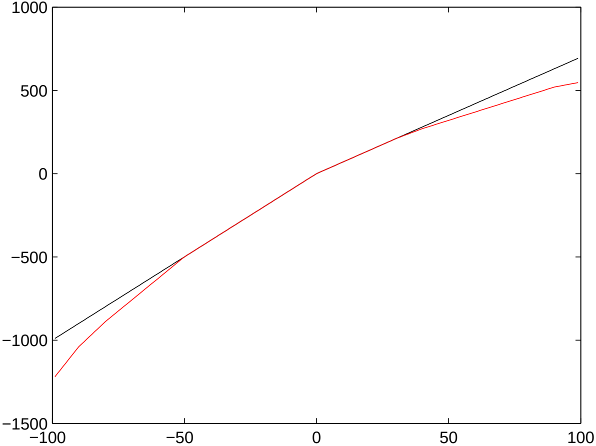


Figura: Upper value (black) and lower value (red) for different trades

Upper and lower value - 1

- Given msdc m_n for any asset and a portfolio $\bar{\mathbf{p}} = (p_0, \mathbf{p})$
- The **Lower Value** (or Liquidation value) of $\bar{\mathbf{p}}$ is

$$\begin{aligned} L(\bar{\mathbf{p}}) &= p_0 + \sum_n L(p_n) \\ &= p_0 + \sum_{p_n>0} \sum_{x=1}^{p_n} m_n(x) + \sum_{p_n<0} \sum_{x=1}^{|p_n|} m_n(-x) \end{aligned}$$

- The **Upper Value** of $\bar{\mathbf{p}}$ is

$$\begin{aligned} U(\bar{\mathbf{p}}) &= p_0 + \sum_n U(p_n) \\ &= p_0 + \sum_{p_n>0} p_n m_n(1) - \sum_{p_n<0} p_n m_n(-1) \end{aligned}$$

- Notice that $U(\bar{\mathbf{p}}) \geqslant L(\bar{\mathbf{p}})$: the **liquidation cost** is

$$C(\bar{\mathbf{p}}) = U(\bar{\mathbf{p}}) - L(\bar{\mathbf{p}})$$

Upper and lower value - 2

- Consider the portfolio $\bar{\mathbf{p}} = (p_0, p_1, p_2) = (100, 40, -30)$
- The two order books are

			best ask	best bid		
price	0.975	0.964	**0.955**	**0.925**	0.92	0.90
q.ty	20	18	50	33	10	50
price	3.125	3.11	**3.09**	**3.075**	3.06	3.04
q.ty	35	10	23	15	40	20

- Then we have

$$\begin{aligned} L(\bar{\mathbf{p}}) &= 100 + (33 \cdot 0.925 + 7 \cdot 0.92) - (23 \cdot 3.09 + 7 \cdot 3.11) \\ &= 100 + 36.965 - 92.84 = 44.125 \end{aligned}$$

and

$$\begin{aligned} U(\bar{\mathbf{p}}) &= 100 + 40 \cdot 0.925 - 30 \cdot 3.09 \\ &= 100 + 37 - 92.70 = 44.30 \end{aligned}$$

Upper and lower value - 3

- Here is an example on real data (provided by Carlo)
- A portfolio $\bar{\mathbf{p}}$ invests in $N = 36$ US equities ($p_0 = 0$), from very liquid to **very illiquid** stocks
- Weights are fixed (they range from 1.5% to 7.5%), and we change the amount invested
- Using order books, on some day he obtained (amounts in USD)

mid price MtM	1mln	10mln	100mln	1bln
U	998.483	9.985.000	99.848.000	998.483.000
L	996.462	9.810.000	88.617.000	850.649.000
C/mid	20bp	1.74%	11.23%	14.78%

- **Most** of the liquidation cost at 100mln is carried by the 4 most illiquid stocks

Upper and lower value - 4

- It is mathematically convenient to define $m = m(x)$ for arbitrary real x, so that, for instance

$$P(p) = \int_0^p m(x)\, dx$$

and the best bid and ask prices are $m(0^+)$ and $m(0^-)$

- Therefore

$$L(\bar{\mathbf{p}}) = p_0 + \sum_n \int_0^{p_n} m_n(x)\, dx$$

$$U(\bar{\mathbf{p}}) = p_0 + \sum_n ((p_n)^+ m_n^+ - (p_n)^- m_n^-)$$

Models for m

- A **simple** model is

$$m(x) = ae^{-kx}, \qquad k > 0$$

  a is the mid price, k is a measure of the **slippage**

- A bid-ask spread is easily incorporated

- Another, more sophisticated model is due to Almgren et al. (Risk, 2005)

$$m(x) = m_0(1 - c_1 x - c_2|x|^{5/8} sgn(x)),$$

  where m_0 is the best bid or ask according to $x > 0$ or $x < 0$ and c_1 and c_2 are two positive constants that depend on **market liquidity indicators** (outstanding number of stocks, average daily volume, 1-day volatility)

Upper and lower value - 5

- As a consequence, we can considers portfolios $\bar{\mathbf{p}}$ in $\mathbb{R}^{N+1}$
- U, L and C have some interesting properties
- U and L are **concave**, while C is convex
- if $p_n \cdot q_n \geqslant 0$ for any n, then

$$L(\bar{\mathbf{p}} + \bar{\mathbf{q}}) \leqslant L(\bar{\mathbf{p}}) + L(\bar{\mathbf{q}})$$

  and similarly for U

- if $\lambda \geqslant 1$, then

$$L(\lambda\bar{\mathbf{p}}) \leqslant \lambda L(\bar{\mathbf{p}}) \quad \text{and} \quad U(\lambda\bar{\mathbf{p}}) = \lambda U(\bar{\mathbf{p}})$$

- U, L and C are continuous

Liquidity constraint - 1

- When using U we are not considering the msdc (only the bis-ask spread). It is like we do **not** have to liquidate any part of the portfolio
- When using L, instead, we are considering the whole msdc. It is like we have to liquidate the **entire** portfolio
- Between the two extreme cases, there are infinite attitude towards liquidity risk. We describe them through a set of portfolios, to be called a **liquidity policy**.
- A liquidity policy collects all portfolios that are acceptable (by the owner) because:
  - they have sufficiently **good liquidity features** and/or
  - they allow the owner to meet **funding requirements**

Liquidity constraint - 2

- A **liquidity policy** is a set $\mathcal{L}$ of portfolios, that is $\mathcal{L} \subset \mathbb{R}^{N+1}$ such that
  - if $\bar{\mathbf{p}} = (p_0, \mathbf{p}) \in \mathcal{L}$ then $\bar{\mathbf{p}} + a = (p_0 + a, \mathbf{p}) \in \mathcal{L}$ for any $a > 0$
  - if $(p_0, \mathbf{p}) \in \mathcal{L}$, then $(p_0, \mathbf{0}) \in \mathcal{L}$
- In plain words, adding liquidity or considering the cash part only, does not worsen the liquidity properties of the portfolio
- For technical reasons, we also assume $\mathcal{L}$ to be convex (the blend of two acceptable portfolios is acceptable as well) and closed (for instance, defined through equalities and loose inequalities)

Liquidity constraint - 3

- A **cash-type** liquidity policy is in the form

$$\mathcal{L} = \{\bar{\mathbf{p}} : p_0 \geqslant a_0\}$$

  for some fixed amount $a_0 > 0$

- A mutual fund with 1bln USD assets (mark-to-mid) by prospectus must be prepared to liquidate 20% of its assets. In this case $a_0 = 0.2$bln USD

- Important: the mutual fund does not have to meet the liquidity policy all the time (in fact this liquidity policy is seldom met). However, the fund must **be prepared** to satisfy it and value its assets in accordance.

Liquidity constraint - 4

- We have examples of liquidity policies in everyday finance:
  - ALM constraints
  - Risk managements limits, Basel II
  - Investment policies
  - Margin limits
- For a portfolio of bonds ($n = 1, \ldots, K$) and swaps ($n = K + 1, \ldots, N$)

$$\mathcal{L} = \{\bar{\mathbf{p}} \,:\, -S \leqslant p_n \leqslant M \; \forall n \leqslant K, \; \left| \sum_n p_n D_n \right| \leqslant D\}$$

imposes limits on the **notional** of the bonds (M for long positions, S for short ones) and a limit on the **total sensitivity** of the portfolio to a 1bp shift for the rates

Liquidity constraint - 5

- For a portfolio of options (on a single equity index)

$$\mathcal{L} = \left\{ \bar{\mathbf{p}} \,:\, p_0 \geqslant \max_k \left(\sum_n p_n C_n^{(k)} \right)^- \right\}$$

imposes a **minimum cash requirement** based on a stress testing of the portfolio with K scenarios. Here $C_n^{(k)}$ is the price of the option n under the scenario k

Optimal disinvestment - 1

- We start with a portfolio $\bar{\mathbf{p}}$ (which is not in $\mathcal{L}$)
- We **disinvest** the assets portfolio $\mathbf{r} \in \mathbb{R}^N$ ending up with

$$\bar{\mathbf{p}} \quad \rightarrow \quad \bar{\mathbf{q}} = (p_0 + L(\mathbf{r}), \mathbf{p} - \mathbf{r})$$

  in order for the resulting portfolio $\bar{\mathbf{q}}$ to be in $\mathcal{L}$
- We evaluate the portfolio $\bar{\mathbf{q}}$ with the **upper value**

$$U(\bar{\mathbf{q}}) = p_0 + L(\mathbf{r}) + U(\mathbf{p} - \mathbf{r})$$

  where $L(\mathbf{r}) = L(0, \mathbf{r})$ and $U(\mathbf{p} - \mathbf{r}) = U(0, \mathbf{p} - \mathbf{r})$
- We choose $\mathbf{r}$ in order for $U(\bar{\mathbf{q}})$ to be **maximized**

Optimal disinvestment - 2

- The $\mathcal{L}$-**value** of the portfolio $\bar{\mathbf{p}}$ is then

$$V_{\mathcal{L}}(\bar{\mathbf{p}}) = p_0 + \max\{L(\mathbf{r}) + U(\mathbf{p} - \mathbf{r}) \,:\, (p_0 + L(\mathbf{r}), \mathbf{p} - \mathbf{r}) \in \mathcal{L}\}$$

- The previous expression is not handy at first sight, but it is immediate to prove that

it is a **concave maximization program** *(in finite dimension)*

- Indeed, the map $\mathbf{r} \mapsto L(\mathbf{r}) + U(\mathbf{p} - \mathbf{r})$ is concave and continuous and the set of constraints for $\mathbf{r}$ is convex and closed
- This is crucial for a practical implementation (in particular when N is large)

Optimal disinvestment - 3

- For a cash type liquidity policy we have

$$V_{\mathcal{L}}(\bar{\mathbf{p}}) = p_0 + \max\{L(\mathbf{r}) + U(\mathbf{p} - \mathbf{r}) \, : \, L(\mathbf{r}) \geqslant a_0 - p_0\}$$

- If $\mathcal{L}$ is the set of all portfolios, we have $V_{\mathcal{L}} = U$, while if $\mathcal{L} = \{(p_0, \mathbf{0}) \, : \, p_0 \in \mathbb{R}\}$ then $V_{\mathcal{L}} = L$
- In general, if $\mathcal{L} \subset \mathcal{L}'$, then $V_{\mathcal{L}} \leqslant V_{\mathcal{L}'}$
- In particular, $V_{\mathcal{L}} \leqslant U$ for any liquidity policy $\mathcal{L}$

Implementation

- The $\mathcal{L}$-value

$$V_{\mathcal{L}}(\bar{\mathbf{p}}) = p_0 + \max\{L(\mathbf{r}) + U(\mathbf{p} - \mathbf{r}) \, : \, (p_0 + L(\mathbf{r}), \mathbf{p} - \mathbf{r}) \in \mathcal{L}\}$$

  may sometime be computed analytically: if $\mathcal{L}$ is defined through equalities we use the **Lagrange method**

- Even if this is not feasible, a numerical implementation is almost always quite easy

- Just carefully translate the liquidity policy as a set of linear inequalities and use **convex optimization routines** (in C, Fortran, Matlab) for numerically solving the problem

An analytical example - 1

- Assume the msdc of 1 USD of bond n is $m_n(x) = e^{-k_n x}$, for some $k_n > 0$
- The portfolio is $\bar{\mathbf{p}} = (0, p_1, \dots, p_N)$, where p_n is the number of USD invested in bond n. Let $\sum_n p_n = 1$mln
- The liquidity policy is

$$\mathcal{L} = \{\bar{\mathbf{q}} \,:\, q_0 \geqslant 0.5\text{mln}\}$$

  that is, we have to be prepared to liquidate half of our portfolio.
- We easily compute

$$L(\mathbf{r}) = \sum_n \frac{1 - e^{k_n r_n}}{k_n}$$

  and of course $U(\mathbf{r}) = \sum_n r_n$

An analytical example - 2

- We have to solve ($p_0 = 0$)

$$\max_{\mathbf{r}} L(\mathbf{r}) + U(\mathbf{p} - \mathbf{r})$$
$$\text{sub: } L(\mathbf{r}) \geqslant 0.5\text{bln}$$

that is

$$\max_{\mathbf{r}} \sum_n \left(\frac{1 - e^{k_n r_n}}{k_n} + p_n - r_n \right)$$
$$\text{sub: } \sum_n \left(\frac{1 - e^{k_n r_n}}{k_n} \right) \geqslant 0.5\text{bln}$$

An analytical example - 3

- Using the Lagrangian multiplier method we obtain the solution

$$r_n = k_n^{-1} \log(1 + \theta)$$

$$\theta = 0.5\text{mln} \left(\sum_n k_n^{-1} - 0.5\text{mln} \right)$$

- The resulting value is

$$V_{\mathcal{L}}(\bar{\mathbf{p}}) = 1.5\text{mln} - \log \left(\frac{\sum_n k_n^{-1}}{\sum_n k_n^{-1} - 0.5\text{mln}} \right) \sum_n k_n^{-1}$$

An analytical example - 4

- If $k_n = k = 10^{-5}$ and $p_n = 1\text{mln}/N$ (equally weighted portfolio), then
$$V_{\mathcal{L}}(\bar{\mathbf{p}}) = 1.5\text{mln} - \frac{N}{k}\log\left(\frac{N/k}{N/k - 0.5\text{mln}}\right)$$
- The function $V_{\mathcal{L}}$ is increasing in N

An analytical example - 5

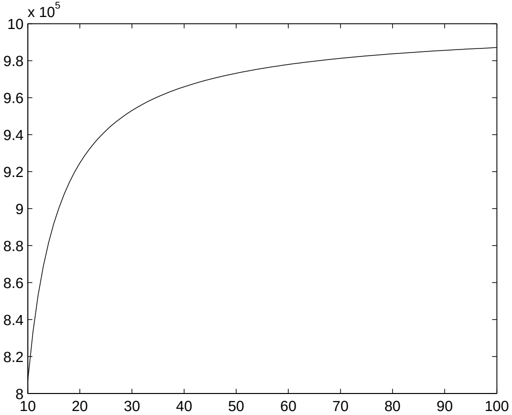


Figura: $\mathcal{L}$-Value of the equally weighted portfolio as N increases

An analytical example - 6

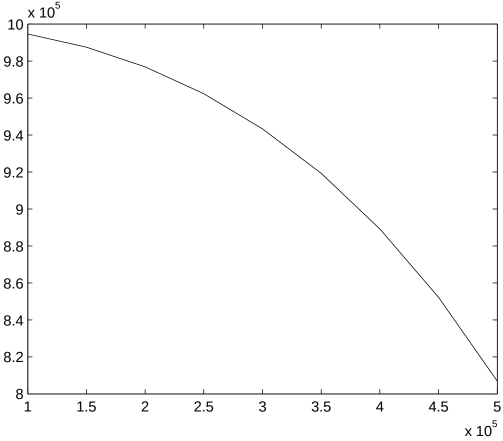


Figura: $\mathcal{L}$-Value of the equally weighted portfolio with $N = 10$ as c in the liquidity policy increases from 0.1mln to 0.5mln

Properties of the $\mathcal{L}$-value - 1

- **Concavity**

$$V_{\mathcal{L}}(\lambda\bar{\mathbf{p}} + (1-\lambda)\bar{\mathbf{q}}) \geqslant \lambda V_{\mathcal{L}}(\bar{\mathbf{p}}) + (1-\lambda)V_{\mathcal{L}}(\bar{\mathbf{q}}) \qquad \lambda \in (0,1)$$

- That is, the value of a blend is no less than the blend of the values
- A sort of diversification effect, triggered by illiquidity, not by correlation
- It can be called a **granularity effect**

Properties of the $\mathcal{L}$-value - 2

- **Cash supervariance**

$$V_{\mathcal{L}}(\bar{\mathbf{p}} + a) \geqslant V_{\mathcal{L}}(\bar{\mathbf{p}}) + a \qquad a > 0$$

- That is, liquidity injection has an added value
- Indeed, adding 1 USD increases the cash position of 1 USD **and improves** the liquidity-adjusted value of the assets portfolio

Back to coherent risk measures - 1

- Consider a coherent r.m. ρ
- Take a stochastic model for the evolution of the msdc's at the horizon T
- Fix a liquidity policy $\mathcal{L}$
- Then

$$\rho_{\mathcal{L}}(\bar{\mathbf{p}}) = \rho(V_{\mathcal{L},T}(\bar{\mathbf{p}}))$$

defines a risk measure at **portfolio-level**

Back to coherent risk measures - 2

- As $V_{\mathcal{L}}$ is concave, $\rho = \rho_{\mathcal{L}}$ is **convex**

$$\rho(\lambda \bar{\mathbf{p}} + (1-\lambda)\bar{\mathbf{q}}) \leqslant \lambda \rho(\bar{\mathbf{p}}) + (1-\lambda)\rho(\bar{\mathbf{q}}) \qquad \lambda \in (0,1)$$

- Convexity is a consequence of illiquidity of the markets, not a property to be imposed in general

- As $V_{\mathcal{L}}$ is cash-supervariant, ρ is **cash-subvariant**

$$\rho(\bar{\mathbf{p}} + a) \leqslant \rho(\bar{\mathbf{p}}) - a \qquad a > 0$$

- The injection of 1 USD lowers the portfolio risk of more than 1 USD

Back to coherent risk measures - 3

- In the limit of **no illiquidity** we recover subadditivity and cash equivariance
- Indeed, if $V_{\mathcal{L}} = U$, then $\rho(\bar{\mathbf{p}}) = \rho(U(\bar{\mathbf{p}}))$ is (easy check)
  - positively homogeneous
  - subadditive
  - cash-equivariant
- Notice that **we are not ruling out bid-ask spreads**

A numerical example - 1

Consider (T is fixed)

$$m_n(x) = A_n \exp\{-K_n x\},$$

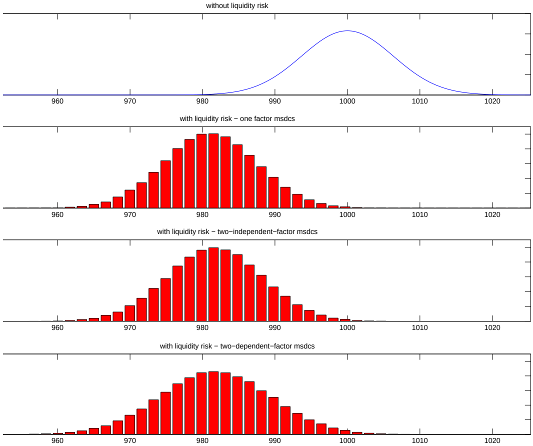
where, $A_n > 0$ and $K_n \geqslant 0$ are r.v. There can be

- *Market risk only*: A_n jointly lognormal, $K_n = 0$
- *Market and non-random liquidity risk*: A_n jointly lognormal, $K_n > 0$ constant
- *Market and independent random liquidity risk*: (A_n, K_n) jointly lognormal, with $A_n \perp K_n$
- *Market and correlated random liquidity risk*: (A_n, K_n) jointly lognormal, with A_n and K_n negatively correlated

A numerical example - 2

- For a given portfolio $\mathbf{p}$ and $\mathcal{L} = \{\mathbf{q} : q_0 \geqslant a\}$, in any of the 5 previous situations we:
  - set $N = 10$, A_n and K_n identically distributed for different n
  - we perform 100k simulations of $(m_n(x))_n$
  - for any outcome of the simulation we compute $V_{\mathcal{L}}(\mathbf{p})$
  - we repeat for different inputs ($\mathbf{p}$, a, mean, variances and correlations of A_n and K_n)

A numerical example - 3



Conclusion

Messages:

- Liquidity risk arises when **msdc are ignored** (it does not only depend on the bid-ask spread)
- Liquidity risk can be captured by a redefinition of the **concept of value**, which depends on a liquidity policy
- The same portfolio is valued **differently** according to the liquidity needs of the owner
- **Coherent** risk measures are perfectly **adequate** to deal with liquidity risk

To do:

- study possible realistic (yet analytically tractable) stochastic models for a msdc
  (many studies of the components of bid-ask spread in the literature)
- portfolio optimization with liquidity risk (three dimensional problem: return-volatility-illiquidity)