# One Penny Arbitrages
# or: Free Snacks without Free Lunches

Erio Castagnoli, Gino Favero, Claudio Tebaldi
Università Bocconi e Università di Parma

ICTP Trieste

# Introduction

- Rational valuation in financial markets requires a more realistic description of liquidity effects.
- The effects on prices of traders' competition for liquidity needs has been widely analyzed in microstructural market models (Micro approach);
- The efficient market hypothesis and in particular the No Arbitrage Valuation approach abstracts from specific micromodels and considers a Dutch Book argument to verify consistency properties on aggregate prices (Macro approach).
- In complete markets, FTAP is in apparent contradiction with a statistical theory of fluctuations: the unique pricing functional completely describes information uncertainty faced by investors, all trades occur at equilibrium prices which instantaneously aggregate all available information. This is essentially the reason of the inability of traditional NA models to describe liquidity effects.

# Motivation and research agenda

- Statistical Mechanics point of view: a proper definition of liquidity requires a statistical theory of price formation which bridges microstates with macrostates. No "noise traders" solutions are considered.
- The first step of this research program: apply a "dutch book consistency argument" to a stylized static version of a limit order book and derive the basic properties of the corresponding valuation operator.
- First conclusion, the stylized properties of market valuation operator has many "realistic" features with a mild increase of computational complexity. In particular the "uncertainty" faced by investors on the liquidation procedure generates a valuation model where the necessity of being robust with respect to disequilibrium pricing functionals (statistical fluctuations) arises endogenously.

# The simplest model for uncertainty and information

- Consider a world with two dates. In date 0 (the present) a set of agents, trade in a finite set of assets.
- Between date 0 and date 1 (the future) the state of nature is revealed.
- The state space is $\Omega = \{\omega_1, ..., \omega_S\}$ and $\mathcal{X}$ the vector space of bounded real random variables $X(\omega)$ defined on $\Omega$ is the space of tradable contracts in the market. There are $N$ fundamental traded assets $\{Y_n(\omega)\}_{n=1,...,N}$,
- No information asymmetries among agents about the risk profile of the traded contracts
- We leave the possibility that each agent submits a limit order or a market order as in a pure limit order market.

# Description of the limit order books

Non-executed order prices can be ordered as

$$C_{bid}^{l_{\min}} < \ldots < C_{bid}^{1} < C_{ask}^{1} < \ldots < C_{ask}^{l_{\max}} \tag{1}$$

and by definition a market order of size $q$ on stock $Y_n$ will pay:

$$C_{ask}(qY_n) = \sum_{0 \le i \le j-1} n_{ask}^{i} C_{ask}^{i} + (q - \sum_{0 \le i \le j-1} n_{ask}^{i}) C_{ask}^{j}$$

A unique curve can be defined as follows:

$$\begin{aligned} C_{ask}(qY_n) &= C(qY_n) \\ C_{bid}(qY_n) &= -C(-qY_n) \end{aligned}$$

where ask prices are represented by the positive buy-side of the curve, while the negative sell-side of the curve represents the opposite of bid prices.

## Definition

A total cost function, $C(qY_n)$, is an extended real-valued function which is nondecreasing, convex, lower semicontinuous and vanishing at 0.

## Lemma

*The limit order book can be identified with the graph of the subdifferential:*

$$\partial C(qY_n) = \left\{c \in \mathbb{R} \mid C(qY_n) \geq C\left(q' Y_n\right) + c\left(q - q'\right) \quad \forall q' \in \mathbb{R}\right\}$$

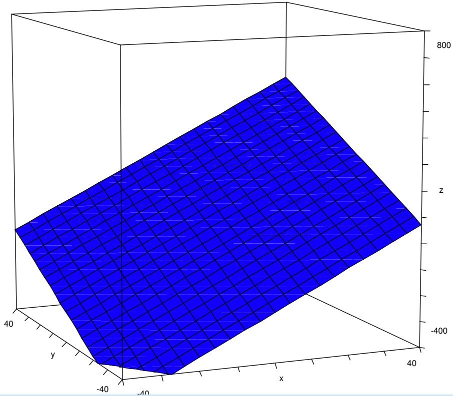
*total cost function* $C(qY_n)$. *The marginal cost function* $c(qY_n)$ *is equal to* $\nabla_q C(qY_n)$ *when the function is differentiable and to the closed interval between the right and the left limits* $\left[\nabla_q^- C(qY_n), \nabla_q^+ C(qY_n)\right]$.

The GE assumption of constant returns to scale is removed, the price of an asset is not an homogeneous function This has important effects on the valuation functional.

# A visual example (1)

$$y^1 = \begin{bmatrix} 5 \\ 10 \end{bmatrix} \qquad \widetilde{\pi}(y^1) = 7$$

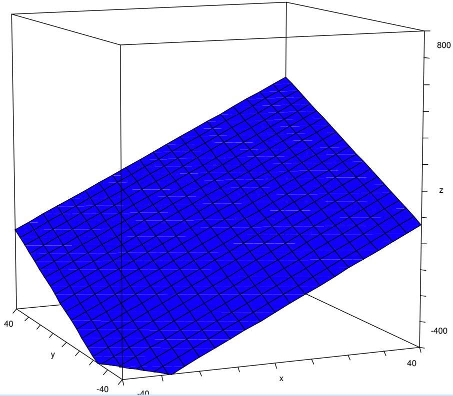
$$y^2 = \begin{bmatrix} 8 \\ 4 \end{bmatrix} \qquad \widetilde{\pi}(y^2) = 5.2$$



# A visual example (2)

$$y^1 = \begin{bmatrix} 5 \\ 10 \end{bmatrix} \qquad \begin{matrix} \widetilde{\pi}_b(y^1) = 6.5 \\ \widetilde{\pi}_a(y^1) = 7.5 \end{matrix}$$

$$y^2 = \begin{bmatrix} 8 \\ 4 \end{bmatrix} \qquad \begin{matrix} \widetilde{\pi}_b(y^2) = 5 \\ \widetilde{\pi}_a(y^2) = 6 \end{matrix}$$



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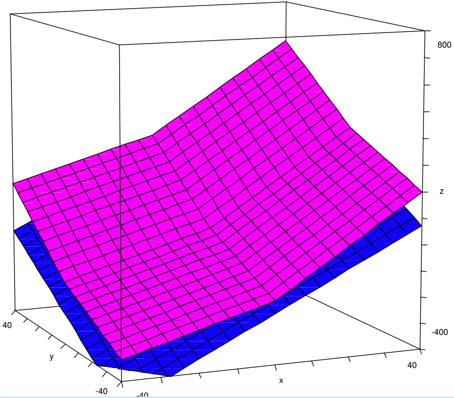

# A visual example (3)

$$y^1 = \begin{bmatrix} 5 \\ 10 \end{bmatrix} \qquad \begin{array}{c} \widetilde{\pi}_b(y^1) = 6.5 \\ \widetilde{\pi}_a(y^1) = 7.5, \\ 8\ (> 20) \end{array}$$

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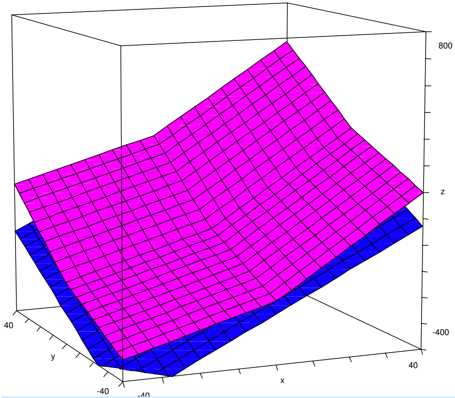

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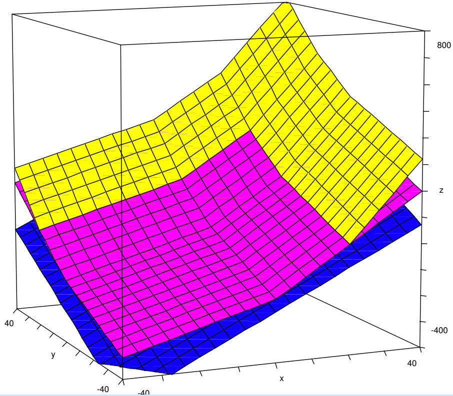

# The granular price functional: an example

$$y^1 = \begin{bmatrix} 5 \\ 10 \end{bmatrix} \quad \begin{matrix} \widetilde{\pi}_b(y^1) = 6.5 \\ \widetilde{\pi}_a(y^1) = 7.5, \\ 8 \ (> 20) \end{matrix}$$

$$\widetilde{\pi}(ay^1) = \begin{cases} 6.5a & a < 0 \\ 7.5a & 0 \leqslant a < 20 \\ 7.5 \cdot 20 + 8(a - 20) & a \geqslant 20 \\ (= 8a - 10) & \end{cases}$$

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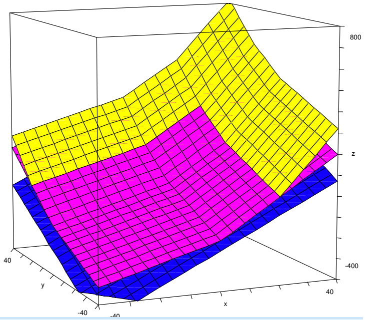
$$\widetilde{\pi}(ay^2) = \begin{cases} 5a & a < 0 \\ 6a & 0 \leqslant a < 2 \\ 6 \cdot 20 + 6.5(a - 20) & a \geqslant 20 \\ (= 6.5a - 10) & \end{cases}$$



# Market ineffectiveness

A (strong) **arbitrage** (**A**) is a traded asset (or a portfolio) with a positive payoff and a negative price: $x \geqq 0$, $\widetilde{\pi}(x) < 0$. A **weak arbitrage** has a nonnegative payoff at a nonpositive price: $y \geq 0$, $\widetilde{\pi}(y) \leqslant 0$ (suitable meaning of inequalities).

A (strong) **convenient super-hedging opportunity** (**SH**) is a pair of assets (or portfolios) such that $x \geqq y$ and $\widetilde{\pi}(x) < \widetilde{\pi}(y)$; a **weak** one takes place when $x \geq y$ and $\widetilde{\pi}(x) \leqslant \widetilde{\pi}(y)$.

Note that a (strong) A is just a (strong) SH of the **null payoff**.

Classically, the concern is to **remove As** from the market.

Our point is: several facts point out that **absence of SHs** should be the primary concern, not just a complementary one.

# SH in the linear world

If $\widetilde{\pi}$ is linear, then $x - y$ is an **A** whenever $x$ is a **SH** of $y$. In other words, they are the same thing!

It is peaceful that no market at equilibrium allows for arbitrages.

Moreover, taking arbitrages away takes us in a heavenly place:

- there exists a pair $(Q, B)$ such that $\widetilde{\pi}(x) = B \cdot \mathrm{E}^Q x$ for all $x$;
- such a pair is unique if the market is **complete**.

If the market is **incomplete**, there are infinitely many pairs $(\mu, B_\mu)$ such that $\widetilde{\pi}(x) = B_\mu \cdot \langle \mu, x \rangle$. In such a case, the **super-hedging** price of a non-attainable claim $z$ (*i.e.*, $\min\{\widetilde{\pi}(x) \colon x \geqq z\}$) obtains as the **max** of $B_Q \cdot \mathrm{E}^Q x$ over the pairs with $\mu = Q$ a (probability, *i.e.*, a) **positive measure**.

Thus, arbitrages are the natural concern in perfect markets.

The price of the portfolio with composition $\mathbf{a} = \{a_n\}_{n\in\mathbb{N}}$ is given by:

$$\pi(\mathbf{Ya}) \triangleq \sum_{n=1}^{N} C(Y_n a_n)$$

**Proposition**

*Assume a market with convex price quantity curves where NCSR and the Positivity Axiom hold, then the (supehedging) valuation functional* $\pi\colon \mathcal{X} \longrightarrow \mathbb{R}$ *has the following properties:*

a) $\pi$ *is grounded* $\pi(0) = 0$;

b) $\pi$ *is monotonic:* $X \leq Y \Rightarrow \pi(X) \leq \pi(Y)$;

c) $\pi$ *is convex:*

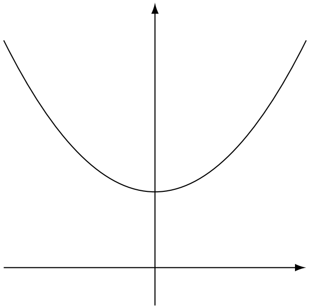
$\forall\lambda \in [0,1], \quad \pi(\lambda X + (1-\lambda)Y) \leq \lambda\pi(X) + (1-\lambda)\pi(Y)$;

# What if the functional is not increasing?

The family of affine functions dominated by a convex one can be described as

$$\mathscr{A} = \{\varphi(\cdot) + b \colon \varphi(\cdot) \in \mathscr{C}, b \in (-\infty, b_\varphi)\},$$

with $\mathscr{C}$ a convex set of linear functions and $b_\varphi \in \mathbb{R}$



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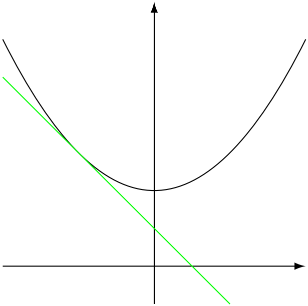

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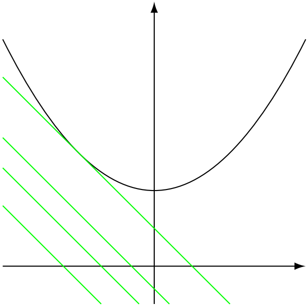

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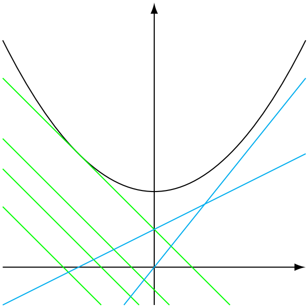

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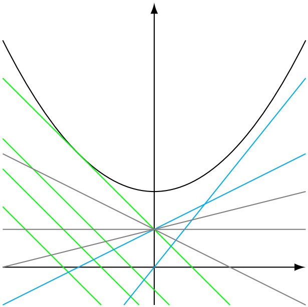

The maximal **increasing** convex function dominated by the original one obtains via

$$\mathscr{A}_+ = \{\varphi(\cdot) + b\colon \varphi(\cdot) \in \mathscr{C}_+, b \in (-\infty, b_\varphi)\},$$

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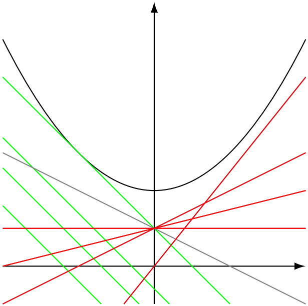

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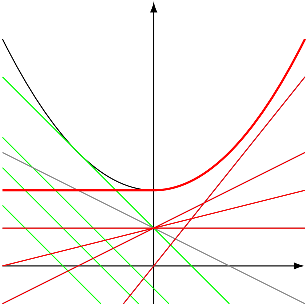

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This allows to determine the best price for a given payoff, after exploiting possible SHs. (Incidentally, taking just arbitrages away is not that simple or meaningful.)

# Local vs global consistency properties

- Efficiency statement, like EMH assume that prices are already formed, this implies that we should expect that conditions of market efficiency are "robust" and not unstable with respect to small violations.
- In the next example it is proved that the standard NA condition can be removed without generating an unbounded profit, but only "a free snack"
- The weakest definition of no arbitrage, the no scalable arbitrage condition of Pennanen, is an asymptotic condition:

$$(\cap_{\alpha>0}\alpha C) \cap \mathcal{X}_{+} = \{0\}$$

where $\mathcal{X}_{+}$ is the set of nonnegative bounded contingent claims and $C$ is the set of bounded contingent claims that can be hedged (super-replicated) with zero initial investment

# A free snack

Take $\Omega = \{\omega_1, \omega_2\}$. Two assets are traded:

- $y^1(\omega_1) = 8$, $y^1(\omega_2) = 10$; bid price is $7$, ask price is $8$;
- $y^2(\omega_1) = 6$, $y^2(\omega_2) = 9$; the bid price is $5.1$; the ask price is $5.2$ for $\leqslant 4$ units, and $6.2$ for every unit beyond the 5th.

An **arbitrage** is possible: short-selling $3$ units of $y^1$ and buying $4$ units of $y^2$ yields the payoff $-3 \cdot \left[\begin{smallmatrix} 8 \\ 10 \end{smallmatrix}\right] + 4 \cdot \left[\begin{smallmatrix} 6 \\ 9 \end{smallmatrix}\right] = \left[\begin{smallmatrix} 0 \\ 6 \end{smallmatrix}\right] \geq 0$ at the "price" $-3 \cdot 7 + 4 \cdot 5.2 = -0.2$.

Yet, this is not an opportunity to grow unlimitedly rich. Doubling the above position yields the payoff $\left[\begin{smallmatrix} 0 \\ 12 \end{smallmatrix}\right]$, but costs $-6 \cdot 7 + 4 \cdot 5.2 + 4 \cdot 6.2 = 3.6$.

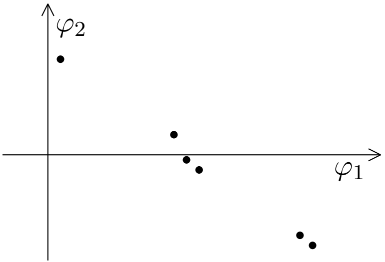
Is such a situation impossible to see in a financial market? Should this model be thrown away? (Pennanen, to appear)

# So, how does this example work, anyway?

The possible prices are $7$ and $8$ for $y^1 = [8 \;\; 10]^{\mathrm{T}}$, and $5.1$, $5.2$ and $6.2$ for $y^2 = [6 \;\; 9]^{\mathrm{T}}$. This induces $3 \times 2 = 6$ possible price pairs, and each identifies a risk neutral measure $[\varphi_1 \;\; \varphi_2]$ such that

$$\begin{cases} 8\varphi_1 + 10\varphi_2 = \pi_1 \\ 6\varphi_1 + \;\; 9\varphi_2 = \pi_2 \end{cases}$$

These are the extreme points of $\mathscr{C}$.



| Region | $a$ | $\pi$ |
|---|---|---|
| $\mathcal{B}_1$ | $a_1, a_2 \leqq 0$ | $\begin{bmatrix} 7 & 5.1 \end{bmatrix}$ |
| $\mathcal{B}_2$ | $a_1 > 0, a_2 \leqq 0$ | $\begin{bmatrix} 8 & 5.1 \end{bmatrix}$ |
| $\mathcal{B}_3$ | $a_1 \leqq 0, 0 < a_2 \leqq 4$ | $\begin{bmatrix} 7 & 5.2 \end{bmatrix}$ |
| $\mathcal{B}_4$ | $a_1 > 0, 0 < a_2 \leqq 4$ | $\begin{bmatrix} 8 & 5.2 \end{bmatrix}$ |
| $\mathcal{B}_5$ | $a_1 \leqq 0, a_2 > 4$ | $\begin{bmatrix} 7 & 6.2 \end{bmatrix}$ |
| $\mathcal{B}_6$ | $a_1 > 0, a_2 > 4$ | $\begin{bmatrix} 8 & 6.2 \end{bmatrix}$ |

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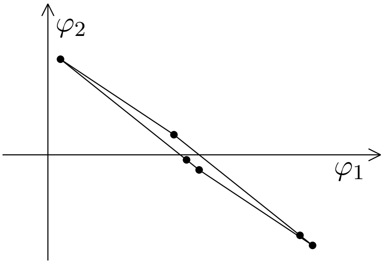


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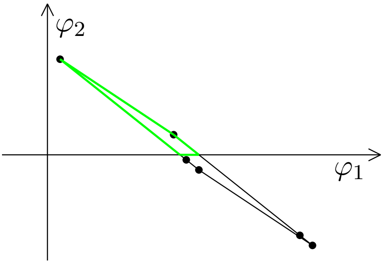

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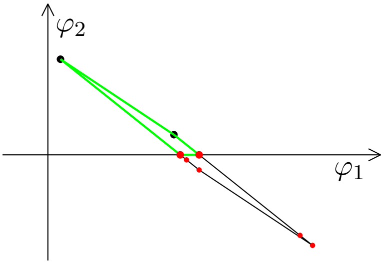


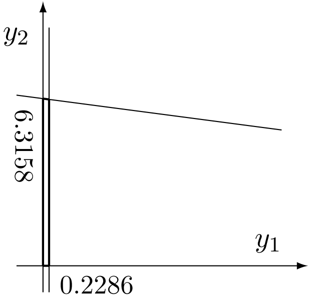

Then, the $b_\varphi$ are determined for the four extreme points in $\mathscr{C}_+$, by determining the best hedging price for a single payoff.

It turns out, therefore, that

$$\widetilde{\pi}\left(\begin{bmatrix} y_1 & y_2 \end{bmatrix}^{\mathrm{T}}\right) = \max\left\{\begin{array}{l} 0.875y_1 - 0.2 \\ y_1 - 3.2 \\ \frac{1}{12}y_1 + \frac{19}{30}y_2 - 4 \\ \frac{5}{6}y_1 + \frac{2}{15}y_2 - 4 \end{array}\right\}.$$

The arbitrages allowed by such a functional are given by

$$\begin{cases} \widetilde{\pi}\left(\begin{bmatrix} y_1 & y_2 \end{bmatrix}^{\mathrm{T}}\right) \leqslant 0 \\ y_1, y_2 \geqslant 0 \end{cases}$$




Not a big deal, after all. Moreover, thinking about a “book of supply and demand”, once such arbitrages are exploited, only the higher prices remain in the market, so arbitrages disappear.

# A representation for the valuation functional

Convexity of the functional implies that

$$\pi(y) = \max_{\varphi \in \Phi} \left[ \varphi(y) + c_\varphi \right],$$

where $\Phi$ is compact convex set of linear functionals and $c_\varphi : \Phi \longrightarrow [-\infty, +\infty]$ is the Fenchel conjugate of $\pi$:

$$c_\varphi = \inf_y \left[ \varphi(y) - \pi(y) \right].$$

the value of $c_\varphi$ is the gain obtained by having used the best prices compared to the marginal ones.

| Extremal points $L_+$ | | $(\varphi^r, c_{\varphi^r})$ |
|---|---|---|
| $\begin{bmatrix} 0.875 & 0 \end{bmatrix}$ | $\varphi^1 = \begin{bmatrix} 1 & -0.1 \end{bmatrix}$ | $c_{\varphi^1} \simeq -0.545455$ |
| $\begin{bmatrix} 1 & 0 \end{bmatrix}$ | $\varphi^2 = \begin{bmatrix} 1.75 & -0.6 \end{bmatrix}$ | $c_{\varphi^2} \simeq -3.272727$ |
| $\begin{bmatrix} 0.875 & 0 \end{bmatrix}$ | $\varphi^3 = \begin{bmatrix} \frac{11}{12} & -\frac{0.4}{12} \end{bmatrix}$ | $c_{\varphi^3} = -0.2$ |
| $\begin{bmatrix} 1 & 0 \end{bmatrix}$ | $\varphi^4 = \begin{bmatrix} \frac{20}{12} & -\frac{6.4}{12} \end{bmatrix}$ | $c_{\varphi^4} = -3.2$ |
| $\begin{bmatrix} \frac{1}{12} & \frac{7.6}{12} \end{bmatrix}$ | $\varphi^5 = \begin{bmatrix} \frac{1}{12} & \frac{7.6}{12} \end{bmatrix}$ | $c_{\varphi^5} = -4$ |
| $\begin{bmatrix} \frac{10}{12} & \frac{1.6}{12} \end{bmatrix}$ | $\varphi^6 = \begin{bmatrix} \frac{10}{12} & \frac{1.6}{12} \end{bmatrix}$ | $c_{\varphi^6} = -4$ |

# The set of supporting state prices

Assume a convex set $A \subset \mathbb{R}^m$:

$$\pi_A(y) = \pi|_A(y) = \max_{\varphi \in \Phi} \left[\varphi(y) - c_\varphi\right] \qquad (y \in A).$$

This is equivalent to restrict $\Phi$ to a $\Phi_A$ a smaller set which includes all the linear functionals $\varphi$ which determine $\pi$ for $y \notin A$:

$$\pi_A(y) = \operatorname*{argmax}_{\varphi \in \Phi_A} \left[\varphi(y) - c_\varphi\right] \qquad (y \in A).$$

Obviously

$$A \subseteq B \iff \Phi_A \subseteq \Phi_B$$

$\Phi_A$ is the set of state prices used in the valuation.
In this valuation approach there are many consistent pricing rules. Robust choice of the pricing measure to be used is an issue here. Uncertainty is endogenously generated.

# Ambiguity Aversion and Limit Order Books

- Define the degree of ambiguity on the random payment $y$:

$$\Delta(y) = \pi_a(y) - \pi_b(y) = \pi(y) + \pi(-y)$$

  the multiplicity of supporting state prices $\varphi \in \Phi$ determines an uncertainty on the proper price of the security $y$.

- It is possible to decompose into discount rate uncertainty and probabilistic uncertainty:

$$\Delta(y) = \Delta^{\ell}(y) + \Delta_a^q(y) + \Delta_b^q(y)$$

- This preliminary result shows that the more illiquid the market, the larger the uncertainty on the "true price" of the security which can be inferred by prices. When illiquidity makes markets less informationally efficient then an ambiguity averse agent will reduce trading, increasing market illiquidity and creating a "liquidity spiral".

# Conclusions

- A stylized limit order book market model is still an interesting place where consistency arguments can be used
- The partial equilibrium valuation functional which is obtained seems to provide a satisfactory description of markets where illiquidity is an issue
- There are promising developments toward a full equilibrium story where uncertainty and statistical fluctuations are endogenously generated and is taken into account by the agents' valuation models.