The Abdus Salam International Centre for Theoretical Physics

United Nations Educational, Scientific and Cultural Organization

IAEA International Atomic Energy Agency

2229-10

School and Workshop on Market Microstructure: Design, Efficiency and Statistical Regularities

21 - 25 March 2011

How to model asset dynamics on the basis of dependence, anomalous scaling and non-stationarirty

Attilio STELLA
Universita' di Padova
Dipt. di Fisica "G. Galilei'
Via F. Marzolo 8, 35131
Padova
ITALY

How to model asset dynamics on the basis of dependence, anomalous scaling and non-stationarity

Attilio L. Stella

Dipartimento di Fisica e Sezione INFN, Università di Padova, Italy.

F. Baldovin, D. Bovina, F. Camana, M. Zamparo

Dip. Fisica e Sezione INFN, Università di Padova

- Independence and normal scaling (stability) of Gaussian distributions have been at the basis of key developments in quantitative finance (Brownian motion, geometric Brownian motion, . . .) since the pioneering work of Bachelier.

- Main message: a generalization of Gaussian stability to cases of strongly dependent random variables whose sum obeys anomalous scaling opens novel perspectives in financial modeling.

Summary

- Construction of non-Markovian, self-similar, time-inhomogeneous stochastic processes with anomalous scaling properties.
- Consistency of the processes with statistics of ensemble of histories extracted from high frequency financial data (S&P).
- Generation of long time-series to be compared with the historical ones.

$X_1, X_2, \ldots, X_t, \ldots$: Sequence of independent random returns

$$G_\sigma\left(x\right) \equiv \frac{\exp\left(-\frac{x^2}{2\sigma^2}\right)}{\sqrt{2\pi\sigma^2}}$$

Simple process

$$p^{(T)}(x_1, x_2, \ldots, x_T) = \prod_{t=1}^{T} G_\sigma(x_t)$$

$$Y_T = X_1 + X_2 + \cdots + X_T$$

Stability:

$$\int dy_2 \, p^{(2)}(y_3 - y_2, y_2 - y_1) = \frac{1}{2^{1/2}} \, p^{(1)} \left(\frac{y_3 - y_1}{2^{1/2}} \right)$$

Normal scaling:

$$T^{1/2} p_{Y_T}(T^{1/2} y) = p_{Y_1}(y) = G_\sigma(y)$$

Anomalous scaling

$$T^D \, p_{Y_T}(T^D \, y) = g(y)$$

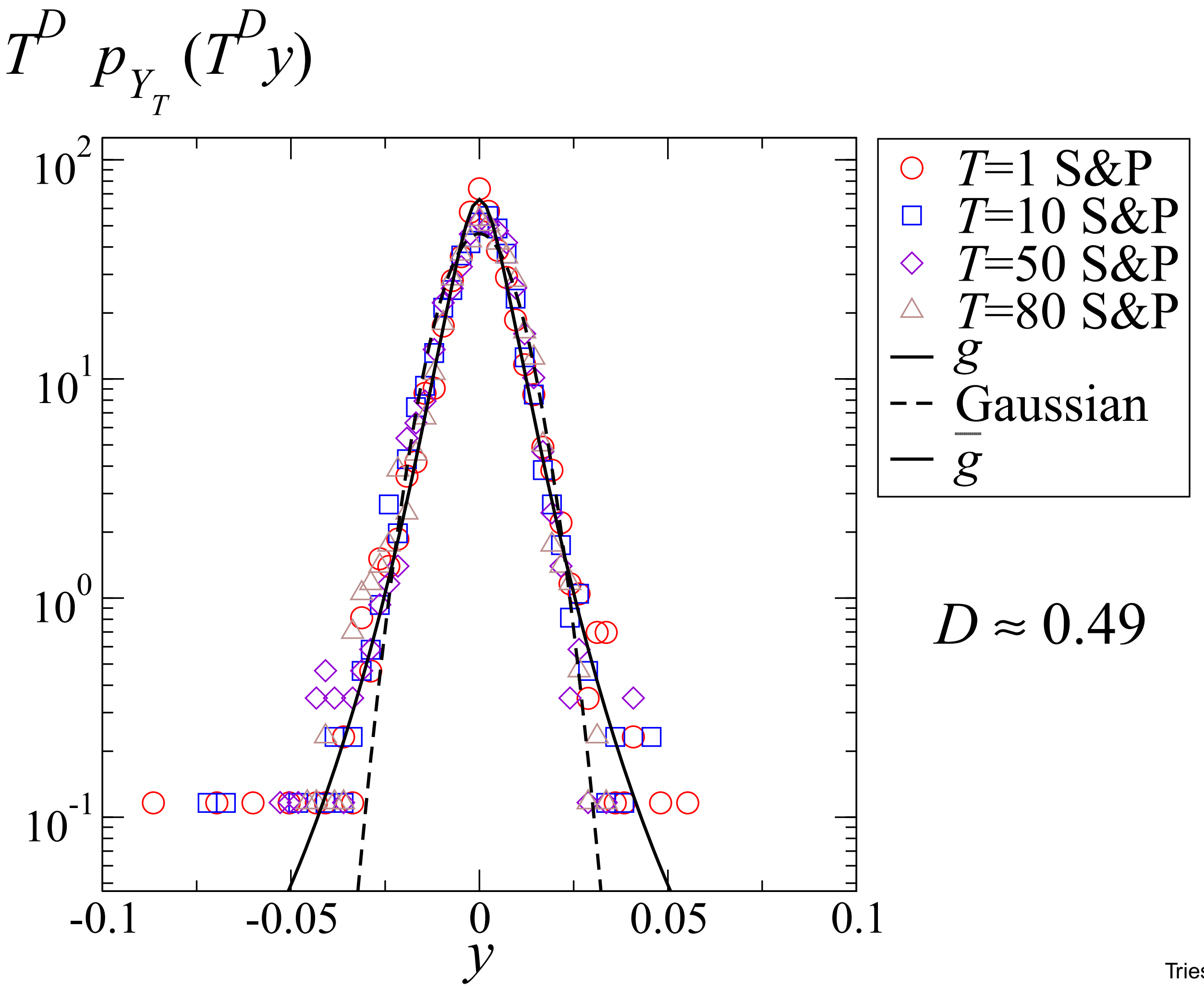
g non-Gaussian and/or $D \neq 1/2$.

Ubiquitous in natural and social phenomena and most often due to strong dependence of the X_i's.

E.g., in finance anomalous scaling holds to good approximation for the empirical PDF of the aggregated return Y_T in a limited range of T's

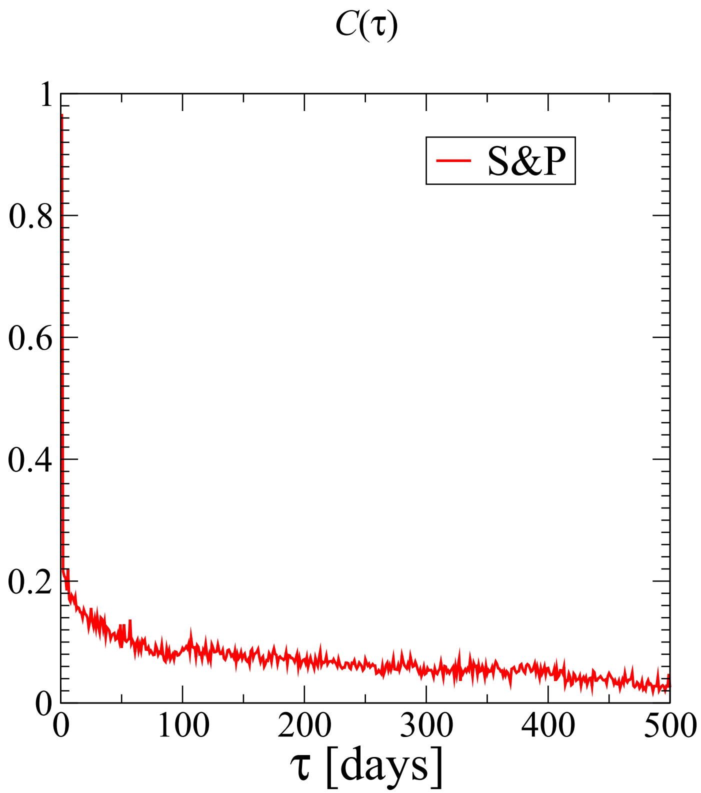
S&P 500 $S(t)$

$Y_T \equiv \ln S(t+T) - \ln S(t)$ (detrended),
sliding interval along t ($\simeq 15386$ [days])



$D \approx 0.49$

Anomalous scaling due to dependence:



$$C(\tau) \equiv \frac{\overline{|x_{t+\tau}|\,|x_t|} - \overline{|x_{t+\tau}|}\;\overline{|x_t|}}{\overline{|x_t|^2} - \overline{|x_t|}^2}$$

Anomalous scaling is an important symmetry, which imposes strong constraints on the evolution of financial processes.

Self-similar process with dependent increments and anomalous scaling

$$p^{(T)}(x_1,\ldots,x_T) \equiv \int_0^{+\infty} d\sigma\ \rho(\sigma) \prod_{t=1}^{T} G_\sigma\left(x_t\right)$$

$$\rho(\sigma) \geq 0, \qquad \int_0^{+\infty} d\sigma\ \rho(\sigma) = 1$$

$$\int dx_T\ p^{(T)}(x_1,\ldots,x_T) = p^{(T-1)}(x_1,\ldots,x_{T-1})$$

Correlated stability holds

$$T^{1/2}\, p_{Y_T}(T^{1/2}\, y) = p_{Y_1}(y) = \int_0^{+\infty} d\sigma\ \rho(\sigma)\ G_\sigma\,(y) \equiv g(y)$$

g: Scaling function

Time-inhomogeneity allows more general form of anomalous scaling:

$$p^{(T)}(x_1, x_2, \ldots, x_T) = \int_0^{+\infty} d\sigma \rho(\sigma) \prod_{t=1}^{T} \left[\frac{1}{a_t} G_\sigma \left(\frac{x_t}{a_t} \right) \right]$$

$$\int dy_2 \, p^{(2)}(y_3 - y_2, y_2 - y_1) = \frac{a_1}{\sqrt{a_1^2 + a_2^2}} \; p^{(1)} \left(\frac{(y_3 - y_1) \, a_1}{\sqrt{a_1^2 + a_2^2}} \right)$$

$$a_t = \left[t^{2D} - (t-1)^{2D} \right]^{1/2} \quad \Rightarrow \quad \sum_{t=1}^{T} a_t^2 = T^{2D}$$

Anomalous scaling for p_{Y_T}:

$$T^D \, p_{Y_T}(T^D \, y) = p_{Y_1}(y) = g(y)$$

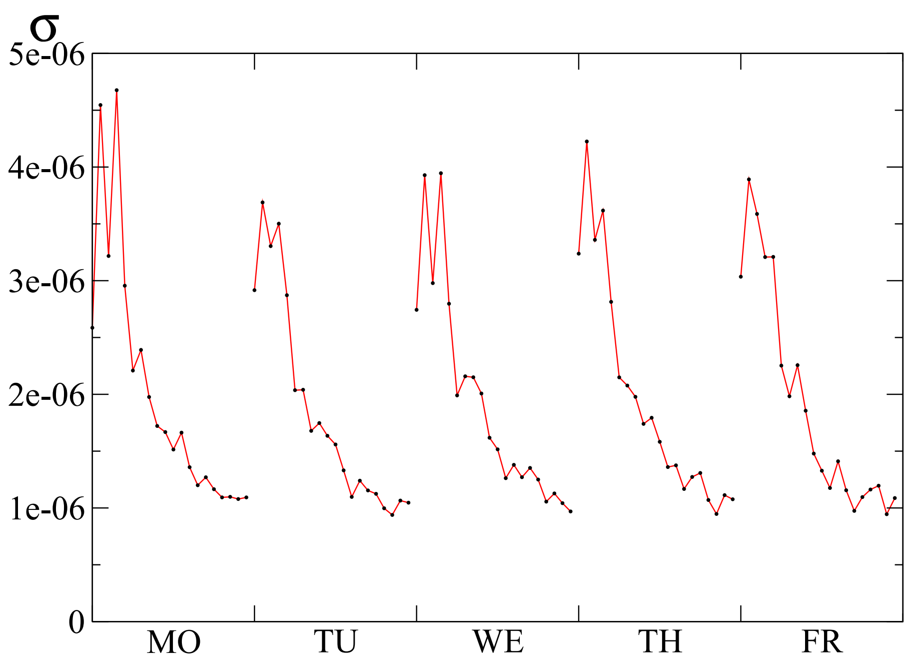
- Convex combinations of Gaussians with different widths allow very general g's and are often considered in phenomenological analyses.

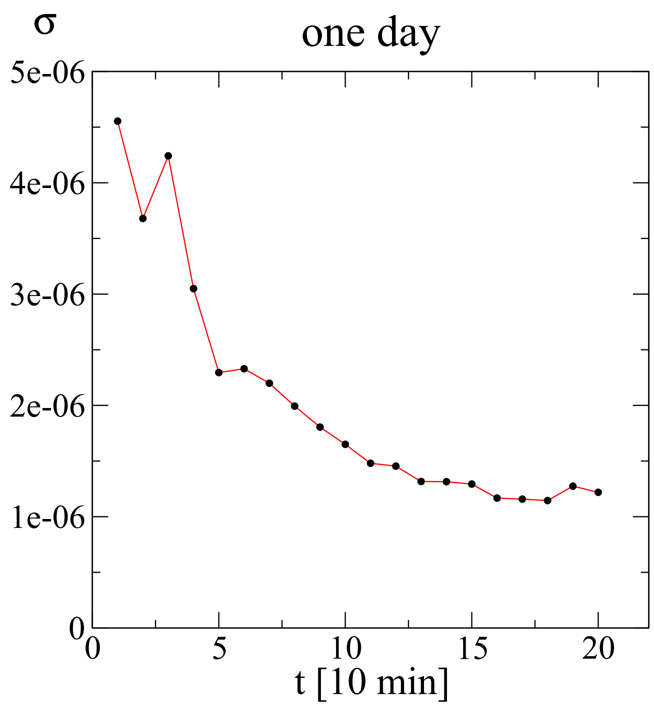
- The generalization of the Gaussian stability allows to establish limit theorems for sums of strongly dependent random variables.

- The detection of time inhomogeneity is very problematic if a single time series is available. To build a statistics along a single series one assumes stationarity of the sampled quantities

- There are high frequency financial data from which one can extract whole ensembles of histories, supposedly all governed by the same, non-stationary stochastic rules

Standard Poor's index (1985-2010, every 10 minutes) appears to produce every day a realization of an underlying non-stationary process in window from 8.40 a.m. to 12 a.m. NY-time.





σ: volatility over 10-minutes interval

- Unique opportunity of studying ensemble of 6283 histories generated by same financial process.
- Each history given by sequence of log-returns X_i ($i = 1, 2, \ldots, 20$) on intervals of 10 $[\mathrm{min}]$.
- Ensemble averages:

$$\langle X_i \rangle \simeq 0$$

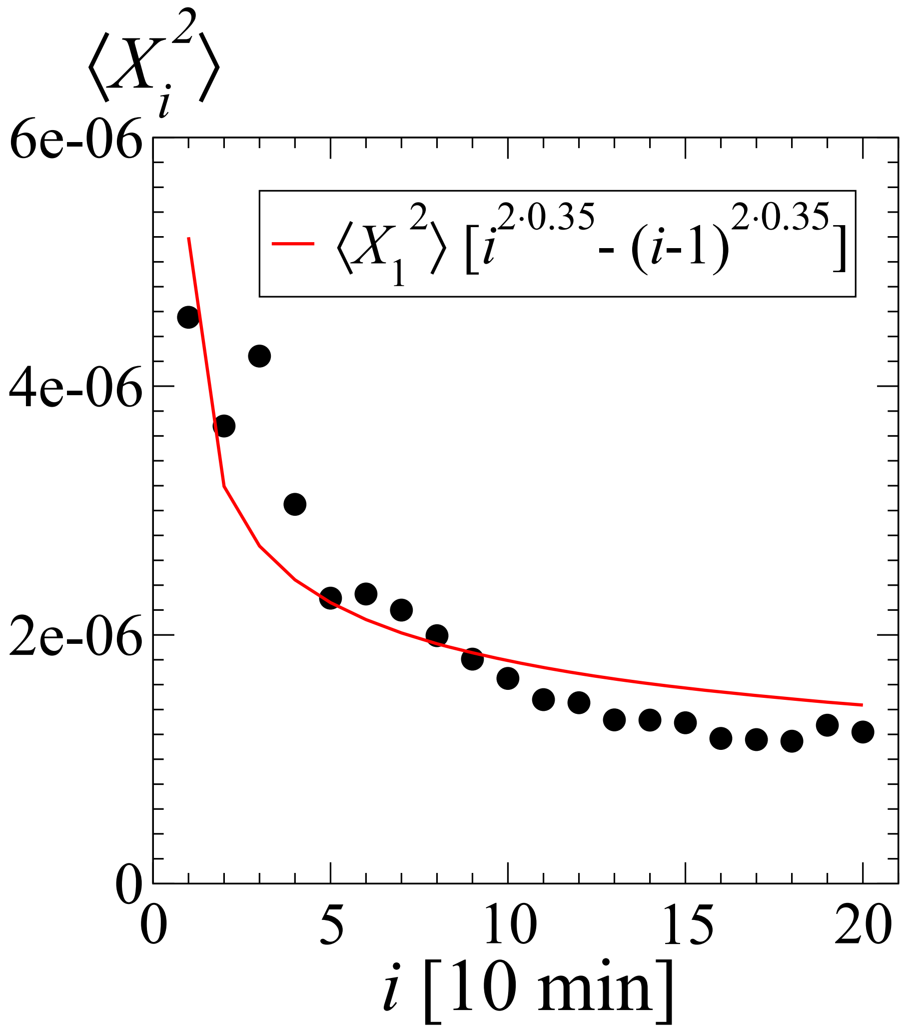
$$\langle X_i X_j \rangle \simeq \langle X_i^2 \rangle \, \delta_{ij}$$

$$\ldots$$

Validation of our model?

Non-stationarity

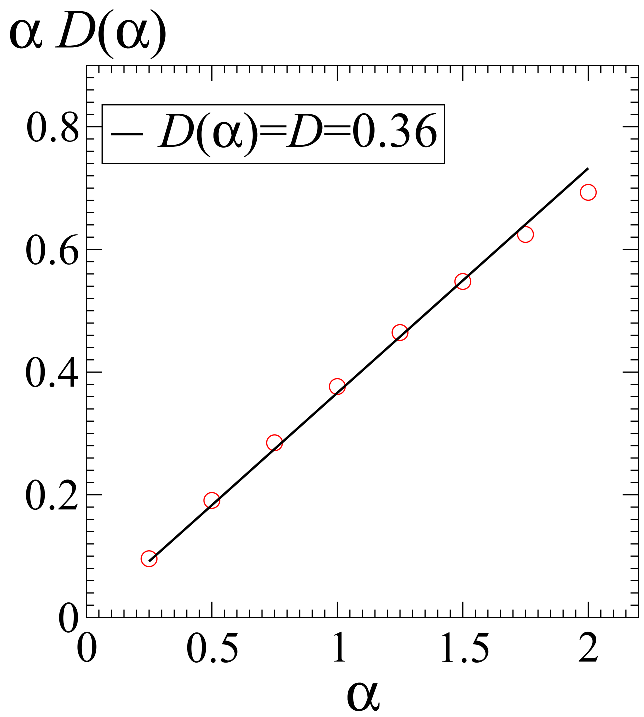
Model: $\langle X_i^2 \rangle = \langle \sigma^2 \rangle_\rho \, a_i^2 = \langle X_1^2 \rangle \, a_i^2 = \langle X_1^2 \rangle \; \left[i^{2D} - (i-1)^{2D} \right]$



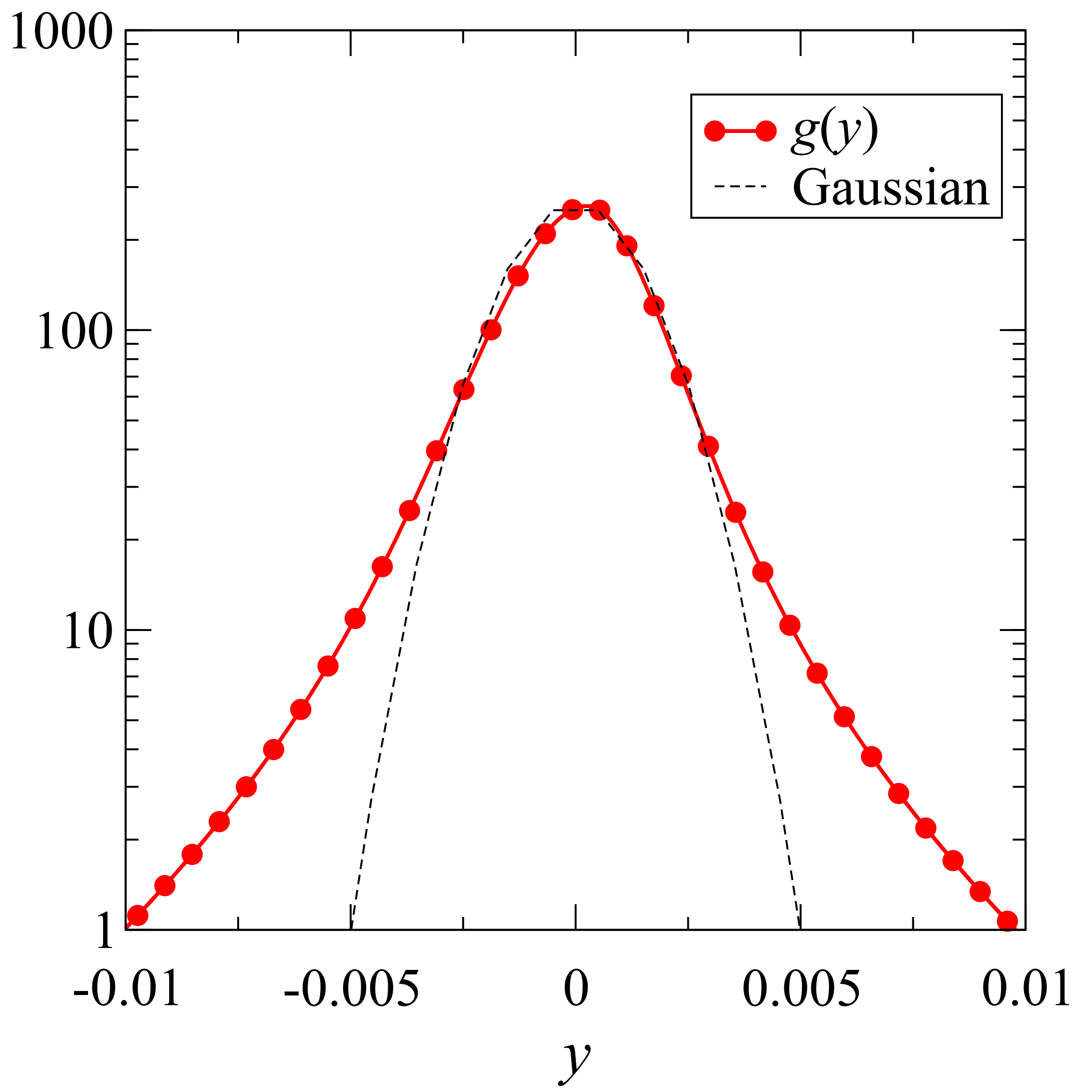
Value of $D \simeq 0.36$ confirmed by scaling of moments of p_{Y_T}:

$$T^D\, p_{Y_T}\left(T^D\, y\right) = g(y) \quad \Rightarrow \quad \langle |Y_T|^\alpha \rangle \sim T^{\alpha D(\alpha)},$$

with $D(\alpha) = const. = D$



Non-Gaussianity



Stability generalized to take skewness and leverage effect into account

$$p^{(T)}(x_1, x_2, \ldots, x_T) = \int_{-\infty}^{+\infty} d\mu \int_0^{+\infty} d\sigma \; \psi(\mu, \sigma) \prod_{t=1}^{T} G_{\mu b_t, \, \sigma a_t}(x_t)$$

where

- $G_{\mu,\sigma}(x) \equiv \frac{1}{\sqrt{2\pi \; \sigma^2}} e^{(x-\mu)^2/2\sigma^2}$,
- $a_t \equiv \sqrt{t^{2D} - (t-1)^{2D}}$,
- $b_t \equiv t^D - (t-1)^D$.

$\psi(\mu, \sigma)$ chosen such that

$$\langle \psi(\mu, \sigma) \rangle = 0 \quad \forall t$$

$$c_{\alpha\beta}(j) \equiv \frac{\langle |X_1|^\alpha \, |X_j|^\beta \rangle}{\langle |X_1|^\alpha \rangle \langle |X_j|^\beta \rangle}$$

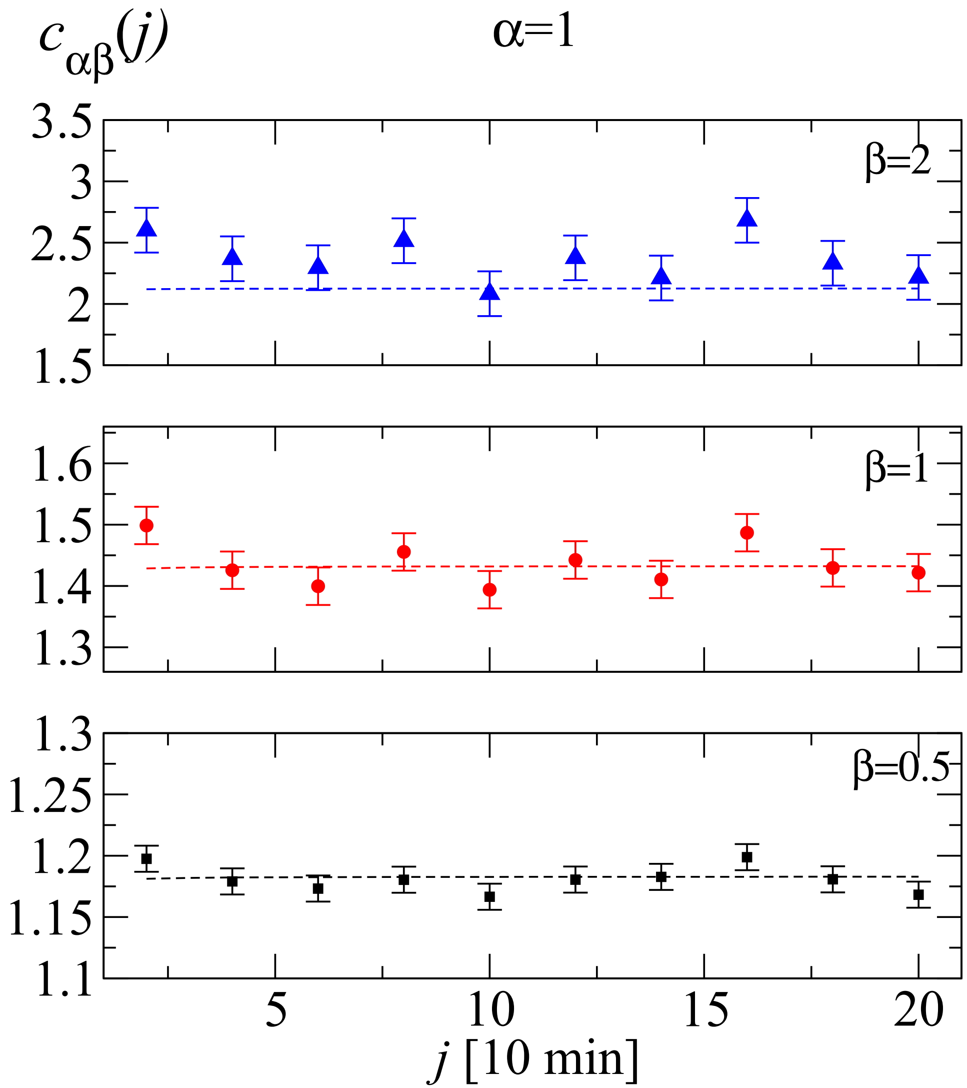
Our model:

$$c_{\alpha\beta}(j) = \frac{B_\alpha B_\beta}{B_{\alpha+\beta}} \frac{\langle |X_1|^{\alpha+\beta} \rangle}{\langle |X_1|^\alpha \rangle \, \langle |X_1|^\beta \rangle} = const,$$

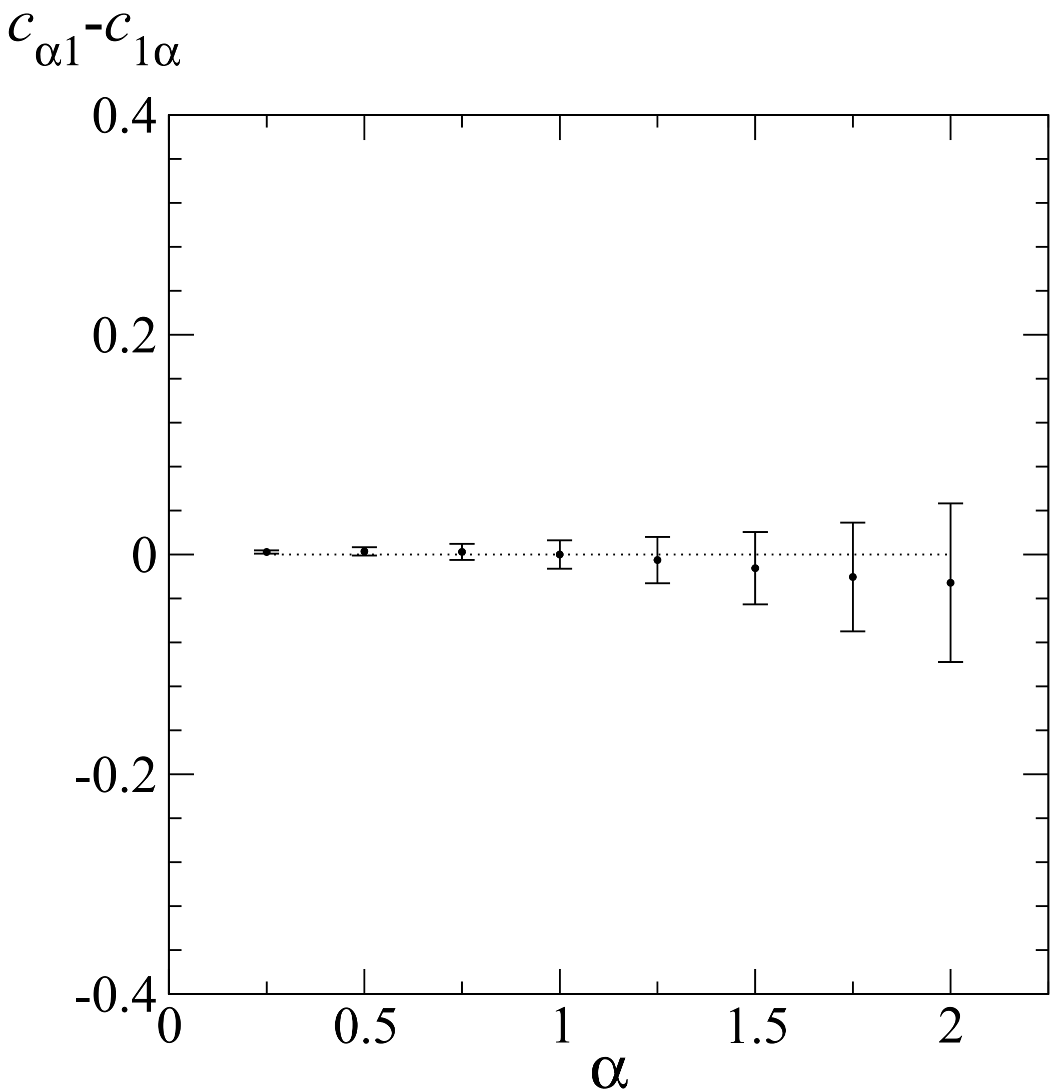
where

$$B_\alpha \equiv \int_{-\infty}^{+\infty} dr \, |r|^\alpha \, \frac{e^{-r^2/2}}{\sqrt{2\pi}}.$$

Symmetry $\alpha \leftrightarrow \beta$



$c_{\alpha\beta}(j)$
α=1
3.5
3
2.5
2
1.5
β=2
1.6
1.5
1.4
1.3
β=1
1.3
1.25
1.2
1.15
1.1
β=0.5
5
10
15
20
j [10 min]

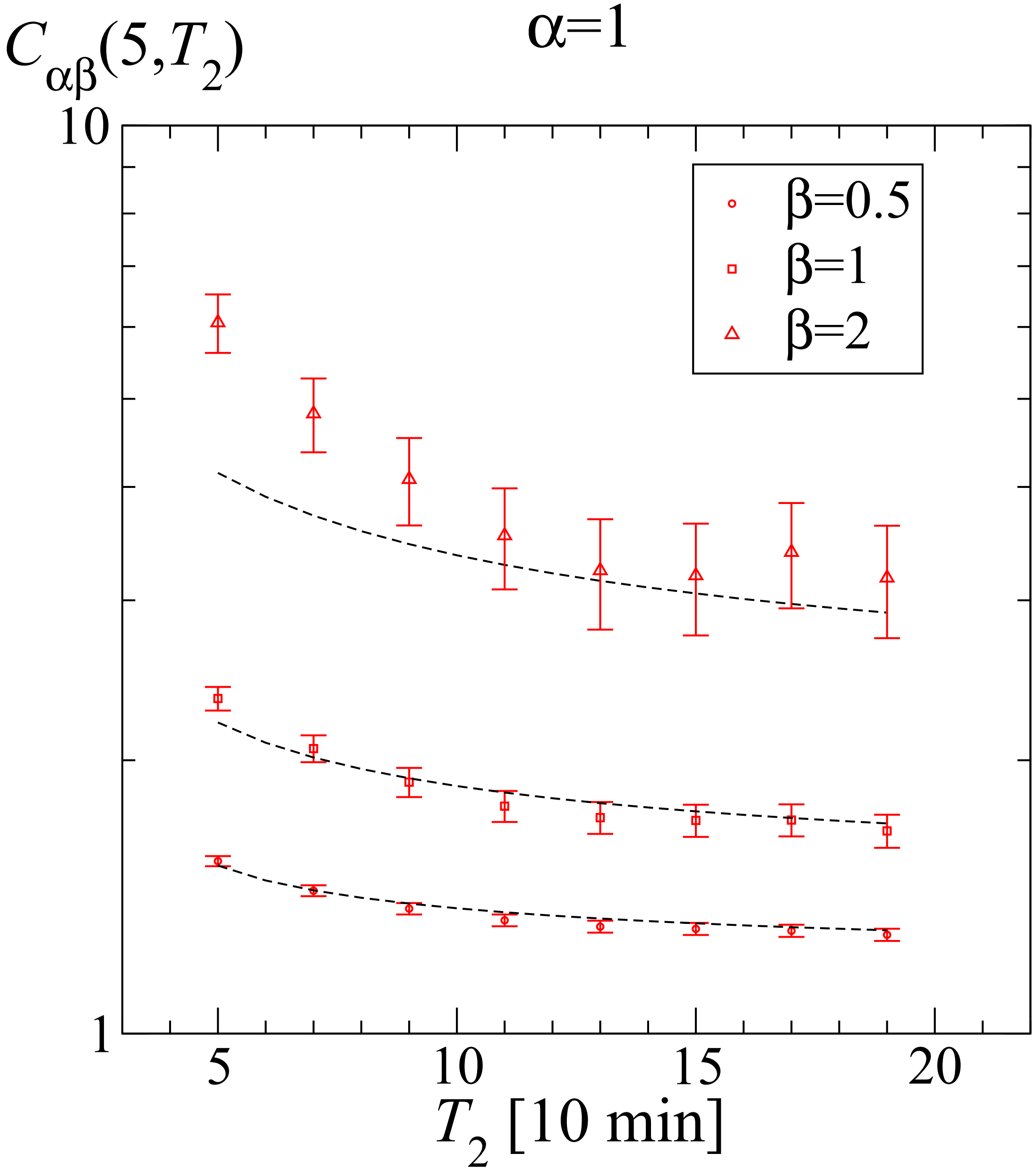


Symmetry $\alpha \leftrightarrow \beta$ verified.

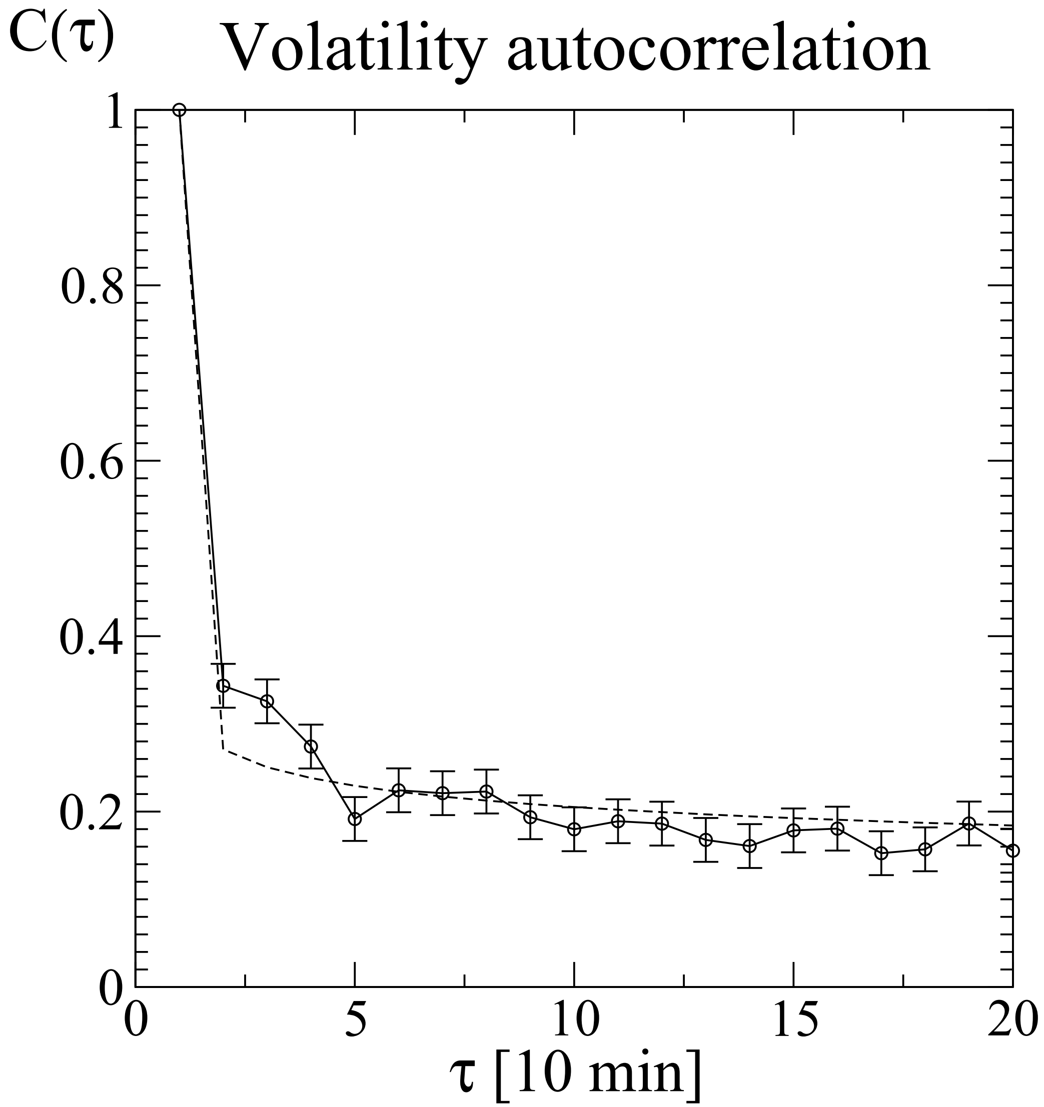
$$C_{\alpha\beta}(T_1, T_2) \equiv \frac{\langle |Y_{T_1}|^{\alpha} \, |Y_{T_2}|^{\beta} \rangle}{\langle |Y_{T_1}|^{\alpha} \rangle \langle |Y_{T_2}|^{\beta} \rangle}$$

Our model:

$$C_{\alpha\beta}(T_1, T_2) \propto \frac{1}{T_1^{\alpha D} \, T_2^{\beta D}} \frac{\langle |X_1|^{\alpha+\beta} \rangle}{\langle |X_1|^{\alpha} \rangle \, \langle |X_1|^{\beta} \rangle}$$



$C_{\alpha\beta}(5,T_2)$
α=1
10
1
β=0.5
β=1
β=2
5
10
15
20
T_2 [10 min]


C(τ)
Volatility autocorrelation
1
0.8
0.6
0.4
0.2
0
0
5
10
15
20
τ [10 min]

Self-similar process used to generate single, long time series ($D = 1/2$):

$$p^c(x_t|x_1, x_2, \ldots, x_{t-1}) = \int_0^{+\infty} d\sigma_t \ \rho^c(\sigma_t|x_1, x_2, \ldots, x_{t-1}) \ G_{\sigma_t}(x_t),$$

where

$$\rho^c(\sigma_t|x_1, x_2, \ldots, x_{t-1}) \equiv \frac{\rho(\sigma_t) \ \prod_{j=1}^{t-1} G_{\sigma_t}(x_j)}{\int_0^{+\infty} d\sigma' \ \rho(\sigma') \ \prod_{j=1}^{t-1} G_{\sigma'}(x_j)}$$

Dynamics of stochastic volatility type obtained by extracting

$$(\sigma_1, x_1), \ \ (\sigma_2, x_3), \ \ \ldots, \ \ (\sigma_t, x_t), \ \ \ldots$$

If trajectory in (σ, x)-space ergodic, empirical return PDF extracted from it obeys anomalous scaling with $D = 1/2$ and $g(y) = \int d\sigma \; \rho(\sigma) \; G_\sigma(y)$.

However, trajectory not ergodic due to memory accumulation mechanism:

$$\rho^c(\sigma_t | x_1, x_2, \ldots, x_{t-1}) \underset{t\to\infty}{\longrightarrow} \delta(\sigma - \overline{\sigma})$$

($\overline{\sigma}$ chosen with probability density $\rho(\overline{\sigma})$).

Single trajectory does not account for dependence and anomalous scaling as given by ensemble.

Way out: autoregressive simulation with finite memory range T_0:

$$p^c\left(x_{t+T_0} \middle| x_t, x_{t+1}, \ldots, x_{t+T_0-1}\right) \equiv$$

$$\equiv \frac{p^{(T_0+1)}\left(x_t, x_{t+1}, \ldots, x_{t+T_0}\right)}{\int dx'_{t+T_0}\, p^{(T_0+1)}\left(x_t, x_{t+1}, \ldots, x_{t+T_0-1}, x'_{t+T_0}\right)}$$

For long enough trajectory now ergodicity holds and anomalous scaling is well satisfied up to time T_0.

Volatility autocorrelation still a problem (constant in time up to T_0)

Way out: introduce non-stationarity via $D \neq 1/2$ and "inhomogeneity restarts" at random times $\bar{t}_i$:

$$a_{\bar{t}_i+1} = a_1, \; a_{\bar{t}_i+2} = a_2, \ldots$$

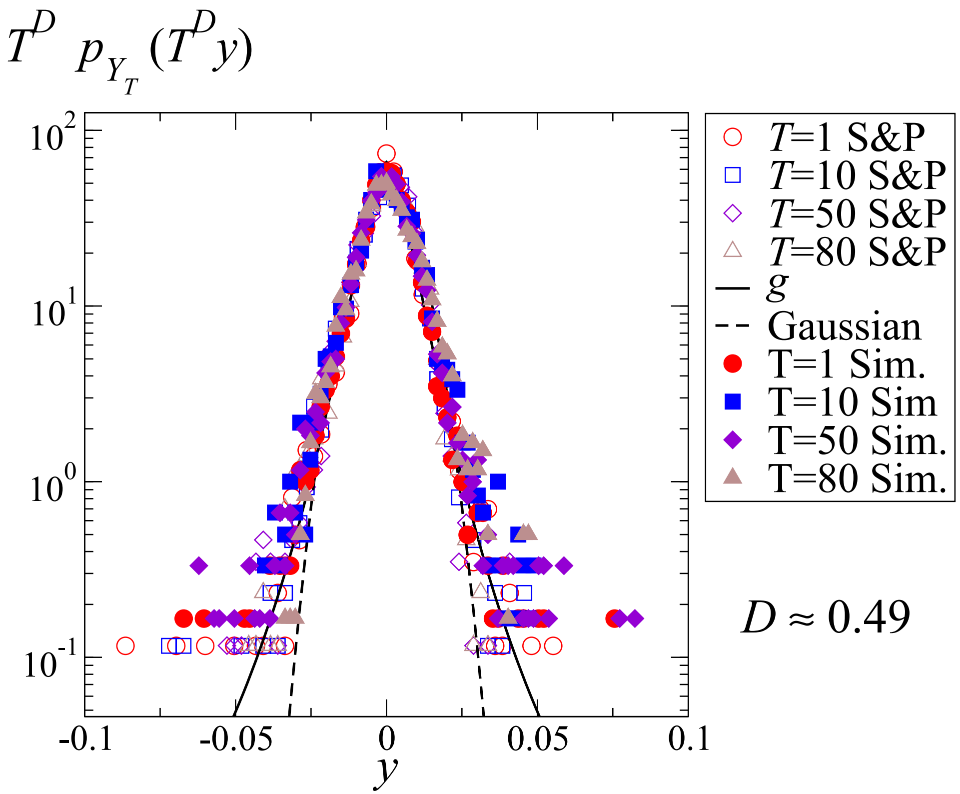
$$\bar{t}_i - \bar{t}_{i+1} \simeq T_1 > T_0$$

Restarts simulate major external perturbations, crashes, ...

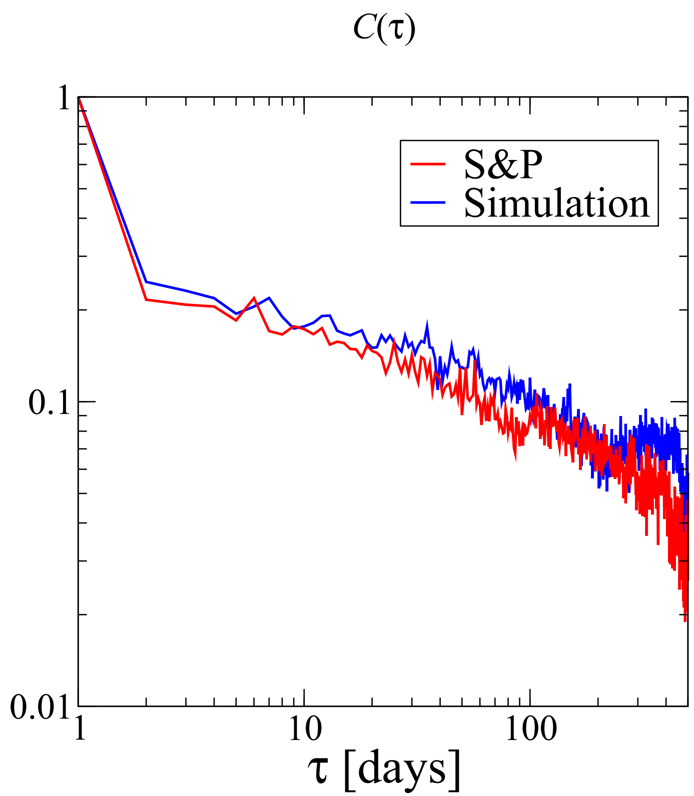
Under appropriate conditions process with restarts stationary

Simulation

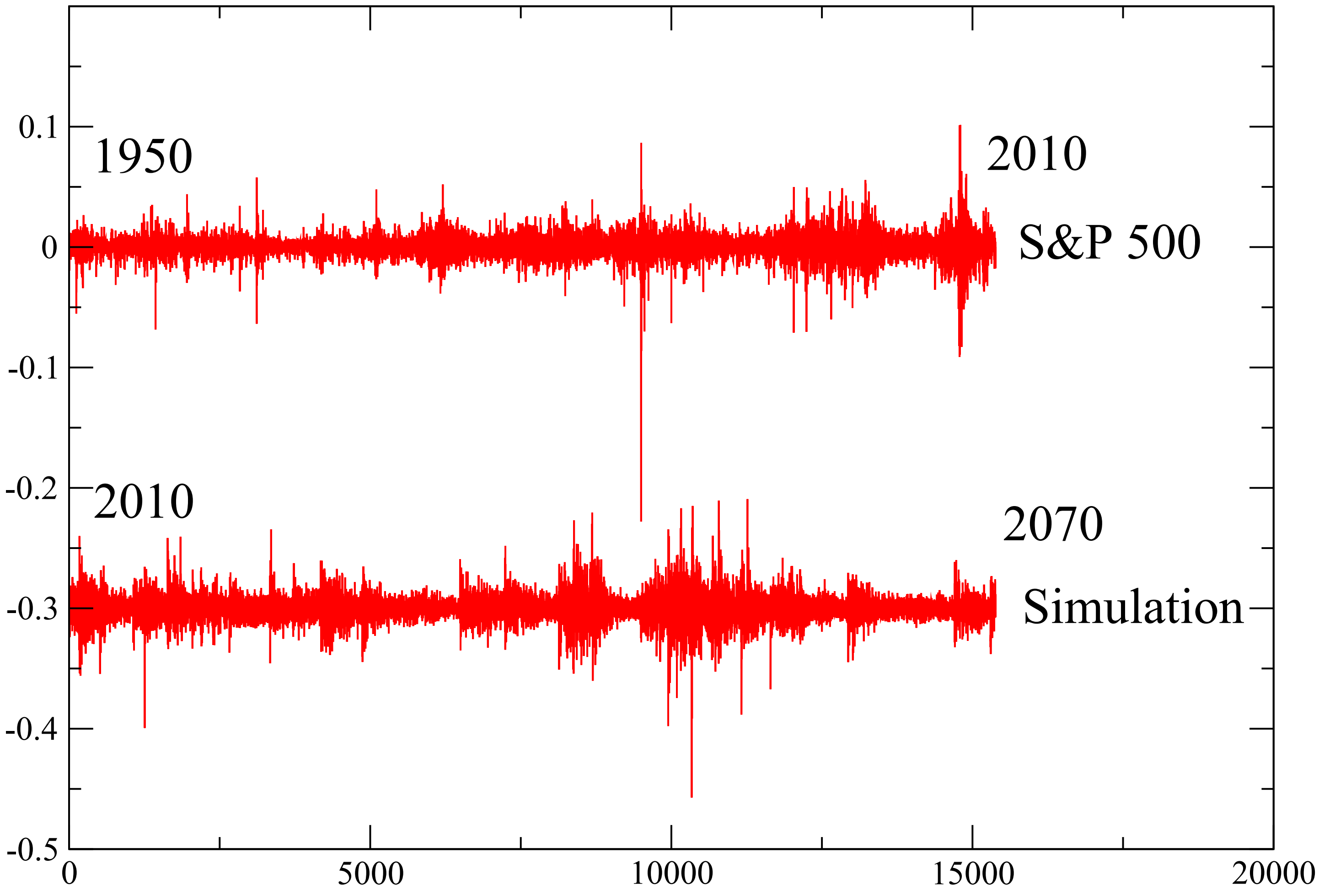
$$D = 0.27, \qquad T_0 = 100 \text{ [days]}, \qquad T_1 = 400 \text{ [days]}$$



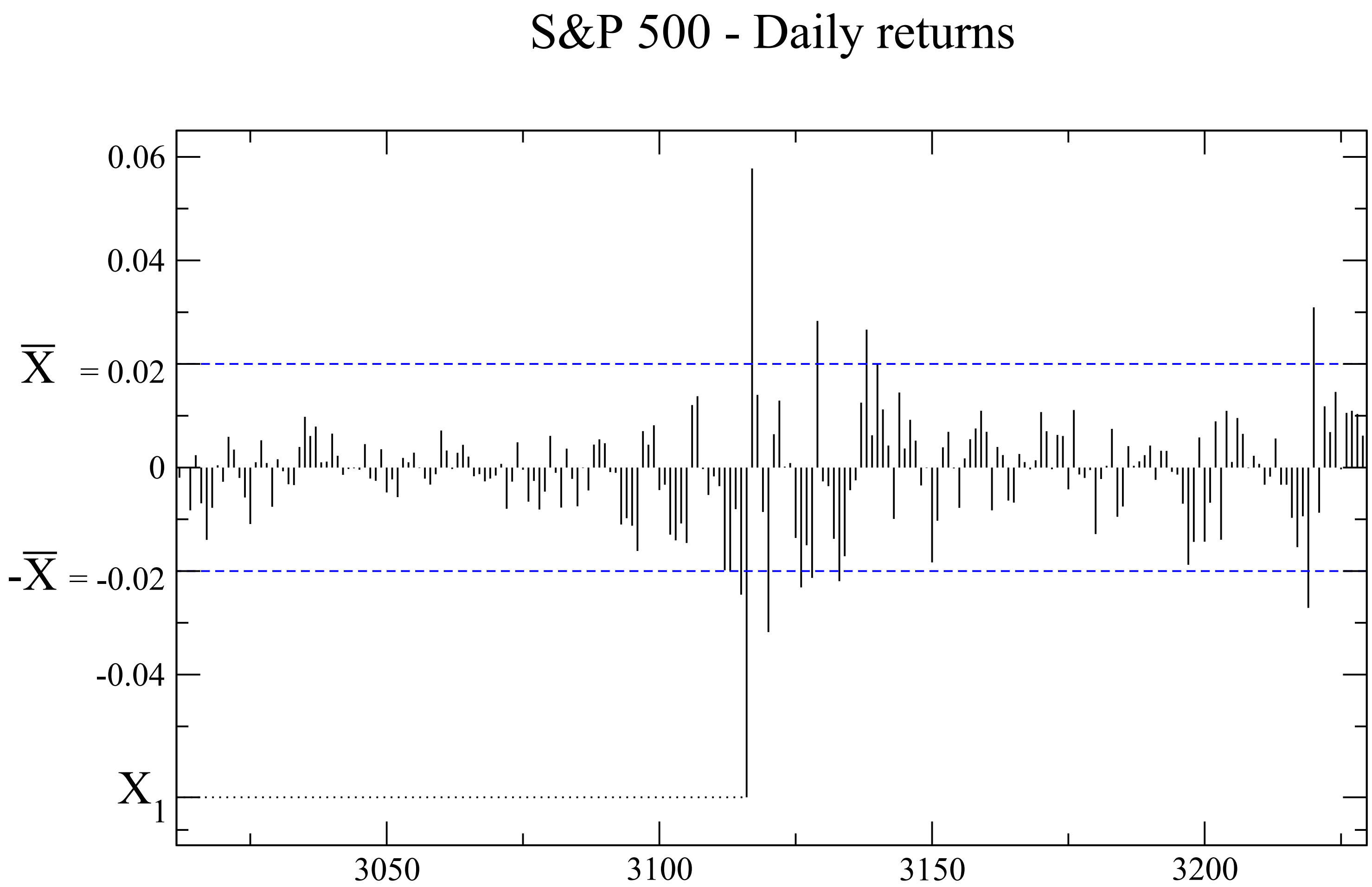
$D \approx 0.49$



S&P 500 - daily returns



Non-stationarity and after-crashes



X_1: Shock
$\overline{X}$: Threshold for aftershocks

Aftershocks obey Omori Law:

$$N(T) = K \left[(T+\tau)^{(1-p)} - \tau^{(1-p)} \right] / (1-p)$$

where

- $N(T)$: # of aftershocks up to time T after a main shock (crash)

[Lillo and Mantegna (2003)]

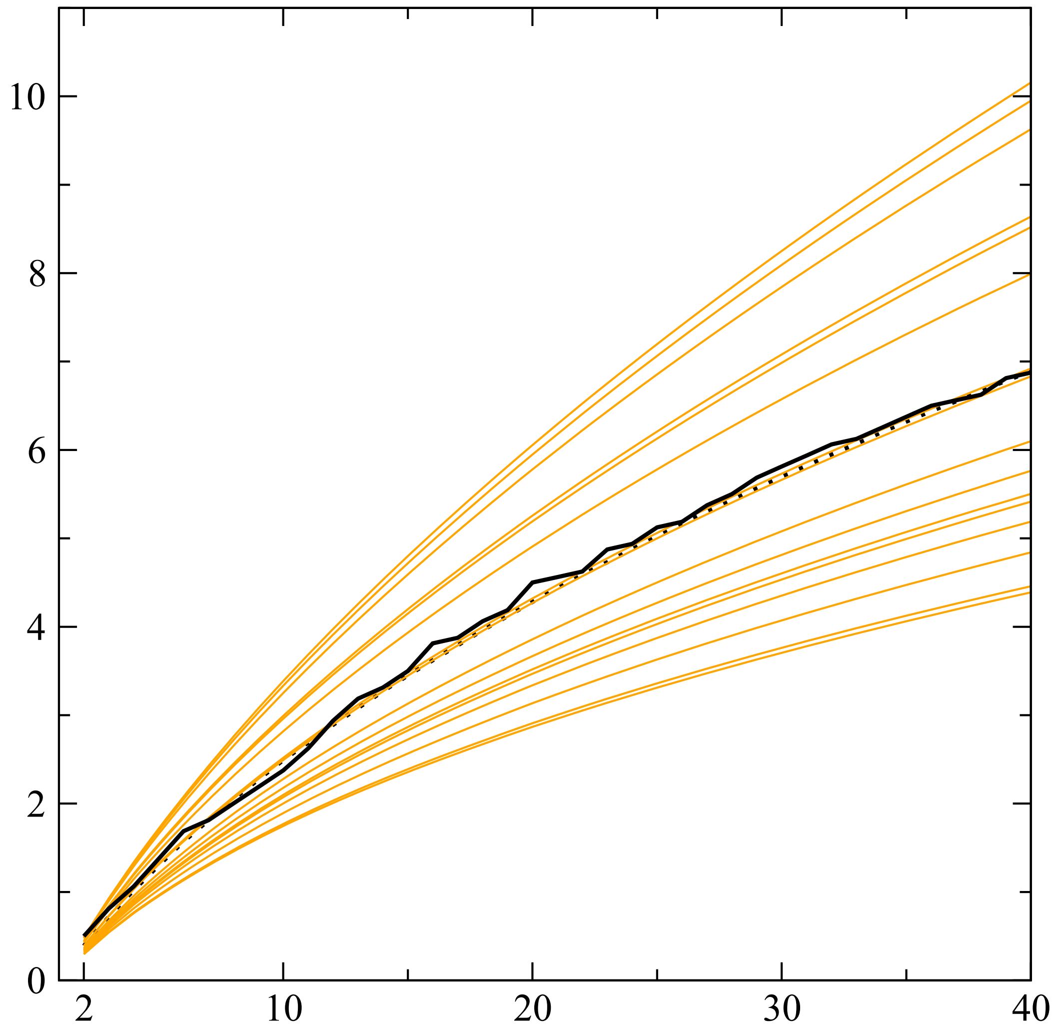
Our model:
Assume that shock x_1 corresponds to “restart”($a_1 = 1$)

$$P(|x_t| > \overline{x}|x_1) = \frac{\int d\sigma\ \rho(\sigma) \left[1 - \mathrm{erf}_\sigma \left(\frac{\overline{x}}{a_t}\right)\right]\ G_\sigma(x_1)}{g(x_1)}$$

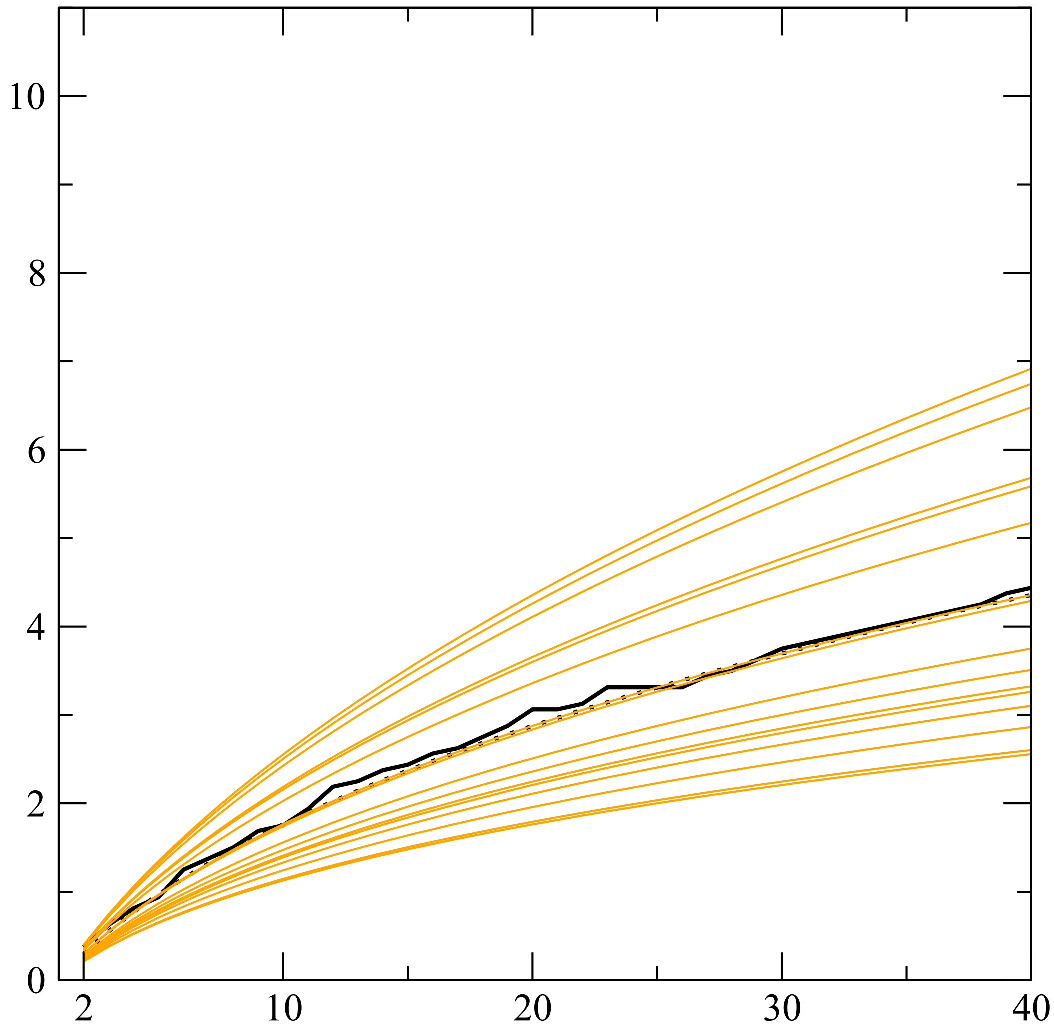
Putting together M different shocks to enrich statistics

$$N(T) = \frac{1}{M} \sum_{i=1}^{M} \sum_{t=2}^{T} P(|x_t| > \overline{x}|x_{1,i})$$

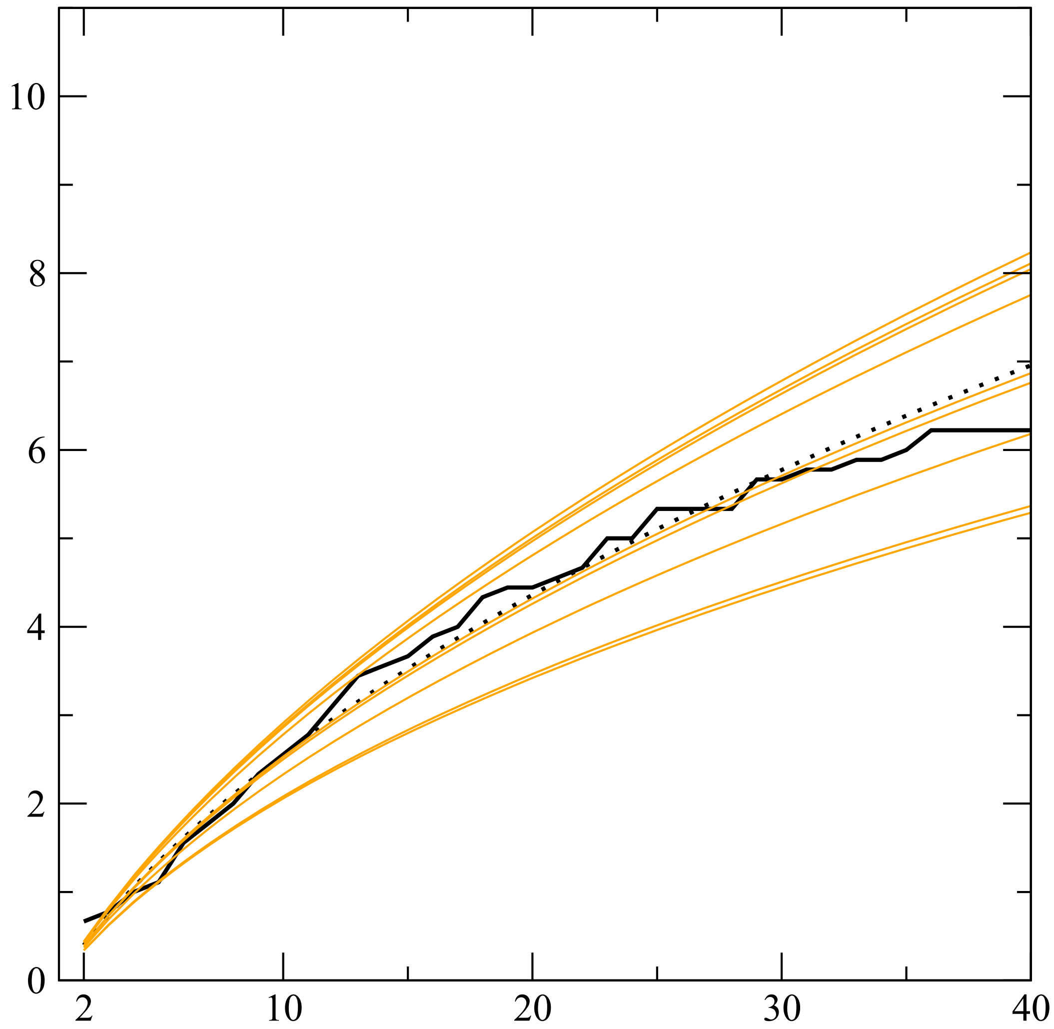
$X_1 > 0.044$ (16 events) $\overline{X} = 0.018$ $D = 0.27$



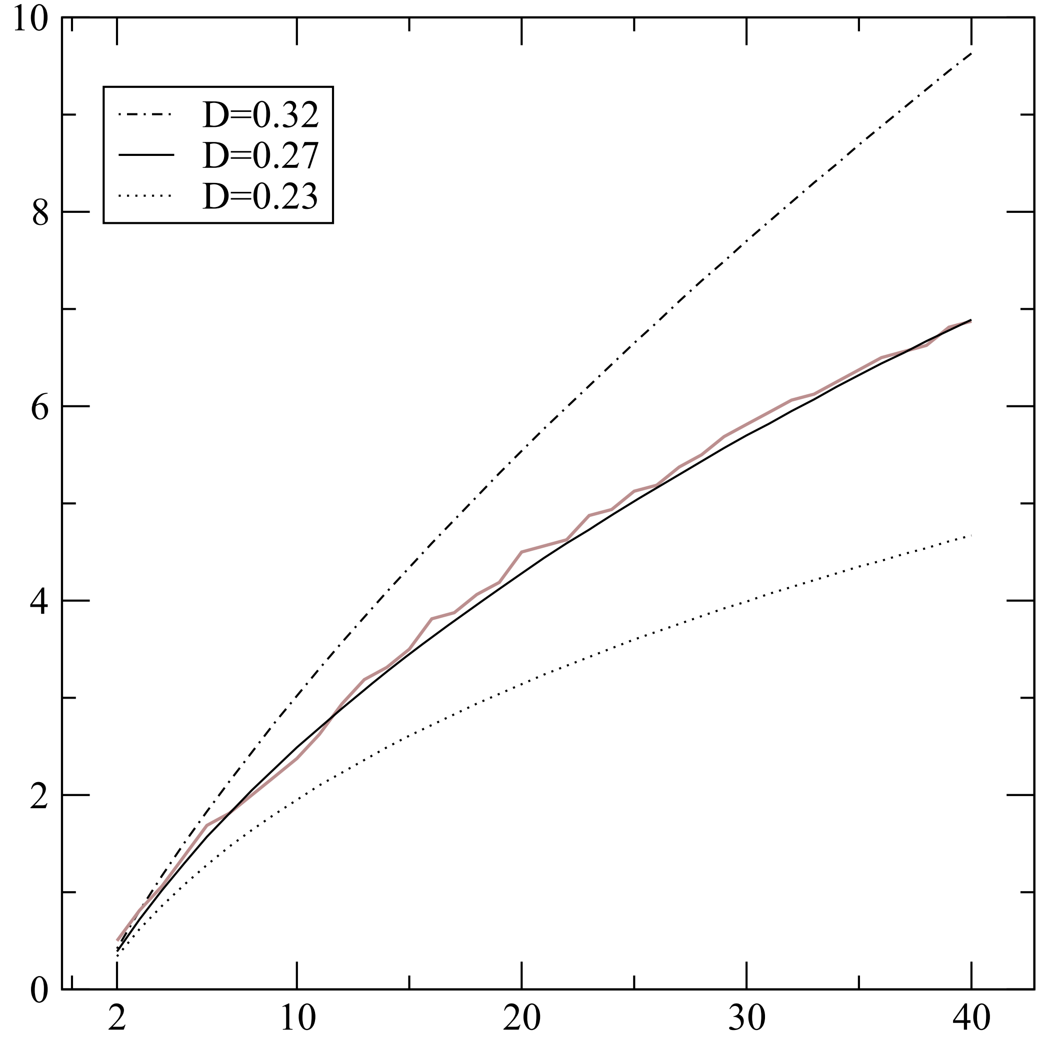
$X_1 > 0.044$ (16 events) $\overline{X} = 0.022$ D=0.27



$X_1 > 0.054$ (9 events) $\overline{X} = 0.020$ D=0.27



$X_1 > 0.044$ (16 events) $\overline{X} = 0.018$



Our curves depend on D very sensibly [$\rho(\sigma)$ already fixed by scaling of return PDF].

Remarkable consistency with $D \simeq 0.27$ as determined by autocorrelation decay.

Our curves very well reproduced by (3-parameter) Omori law fits ($p \simeq 0.8$).

Conclusions

- Correlated stability allows definition of anomalously self-similar, time-inhomogeneous, non-Markovian processes.
- Processes successful in describing high-frequency dynamics of exchange rates, indexes and single stocks as recorded by ensembles of histories.
- Generation of long time series consistent with most stylized facts possible on the basis of autoregressive, stochastic volatility scheme.