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AdS/CFT and Black Holes - Lecture 3

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A de Sitter Farey Tail

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Overview

The Problem

Quantum cosmology is confusing:

- What are the appropriate observables for eternal inflation?
- What is the meaning and origin of the entropy of a cosmological horizon?
- Is quantum mechanics the right language to describe cosmology?

We were able to answer analogous questions about black hole physics using AdS/CFT.

Let us be bold and apply the same techniques to cosmology.

de Sitter Space

de Sitter space is the maximally symmetric solution of general relativity with a positive cosmological constant.

$$S(g) = \frac{1}{G} \int_M \sqrt{-g} \left(R - \frac{2}{\ell^2} \right)$$

The universe inflates eternally.

The three dimensional theory can be solved exactly, in the sense that the partition function

$$Z = \int \mathcal{D}g \; e^{-S[g]}$$

can be computed exactly.

We will be inspired by AdS/CFT but we will not use it.

The Idea

The saddle point approximation is

$$Z = \int \mathcal{D}g \; e^{-S[g]} = \sum_{g_0} e^{-kS^0 + S^1 + \frac{1}{k}S^2 + \ldots}$$

where $k = \ell/G$ is the coupling. The approximation becomes exact if we can

- Find all classical saddles
- Compute all perturbative corrections around each saddle

We can do it!

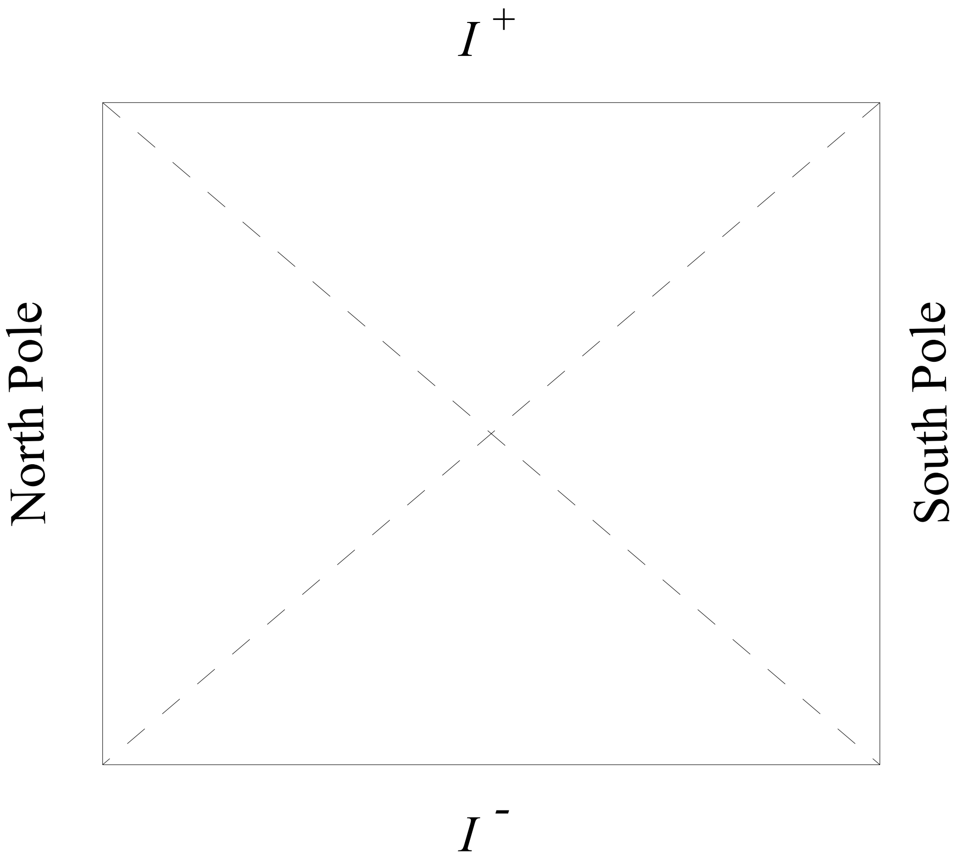
The new classical saddles have a straightforward physical interpretation and lead to quantum gravitational effects for de Sitter observers.

The Plan for Today:

- The de Sitter Farey Tail
- The Partition Function
- Discussion

Causal Structure of de Sitter

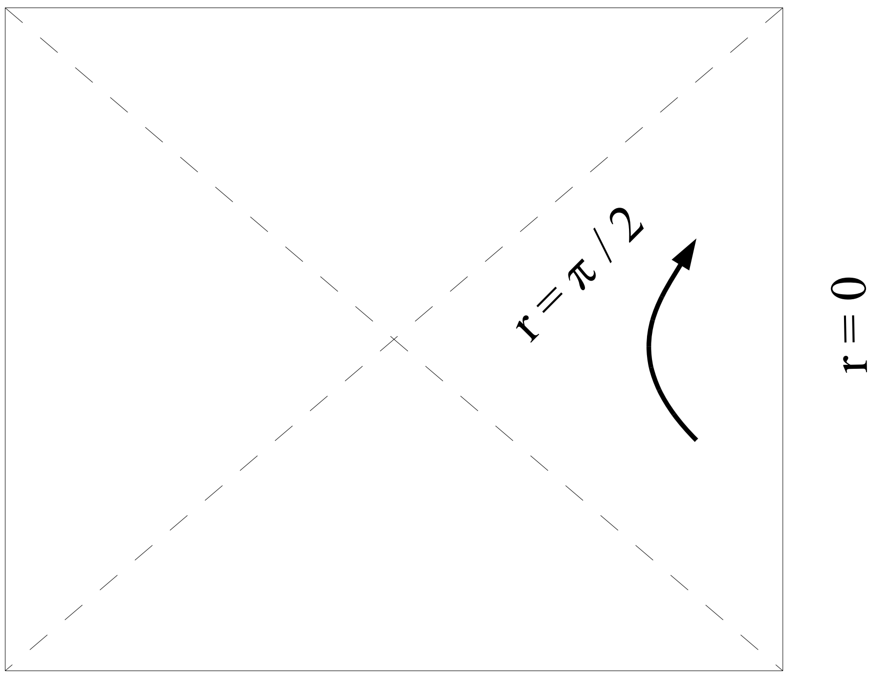
Eternal inflation is the ultimate existential nightmare.



The Causal Patch

A single observer can access only a small portion of the de Sitter geometry. The metric in this patch is

$$ds^2 = dr^2 - \cos^2 r \; dt^2 + \sin^2 r \; d\phi^2$$



Quantum Field Theory in de Sitter

There is no global time-like Killing vector, so no global definition of energy. But there is a notion of energy associated with this observer.

The two killing vectors

- $H = \partial_t$ generates time translations
- $J = \partial_\phi$ generates rotations.

States are labelled by a fixed energy H and angular momentum J.

The operators H and J can be constructed explicitly for free QFT.

The definition of the QFT vacuum state requires some care. It is typically constructed by Wick rotation $t = it_E$.

Euclidean Continuation

The Euclidean metric

$$ds^2 = dr^2 - \cos^2 r \; dt_E^2 + \sin^2 r \; d\phi^2 \ .$$

is smooth only if we identify $t_E \sim t_E + 2\pi n$.

Since $\phi \sim \phi + 2\pi m$ the geometry is compact. It is the sphere S^3.

The identification is generated by the operator

$$\rho = e^{-\beta H}, \qquad \beta = 2\pi$$

so field theory correlators obtained by analytic continuation are in canonical ensemble at finite temperature.

This is the Hawking temperature of de Sitter.

Euclidean Continuation II

But the metric is also smooth if we identify

$$(t_E, \phi) \sim (t_E, \phi) + 2\pi \left(\frac{1}{p} n, m + \frac{q}{p} n \right)$$

for any $(p, q) = 1$.

This is the lens space $L(p, q) = S^3/\mathbb{Z}_p$.

The identification is generated by the operator

$$\rho = e^{-\beta H + \theta J}, \qquad \beta = \frac{2\pi}{p}, \quad \theta = 2\pi i \frac{q}{p}$$

so field theory correlators are in grand canonical ensemble at finite temperature and chemical potential.

Hartle-Hawking State

At the level of QFT in a fixed background these geometries are all equally good "Euclidean continuations of de Sitter."

Which one should we use? All of them!

The Hartle-Hawking state

$$\psi(h) \sim \int_{g|_{\partial M = h}} \mathcal{D}g \; e^{-S}$$

includes contributions from all geometries.

In the semi-classical $G \to 0$ limit the sphere dominates and the state is approximately thermal.

The lens spaces lead to (exponentially subleading) quantum gravity deviations from the standard Bunch-Davies vacuum.

Partition Function

Partition Function

Let's compute the norm of the Hartle-Hawking state. This is the partition function

$$Z = \int \mathcal{D}g \; e^{-S[g]} = \sum_{g_0} e^{-kS^0 + S^1 + \frac{1}{k}S^2 + \ldots}$$

The classical saddles are lens spaces $L(p,q) = S^3/\mathbb{Z}_p$. The classical action is

$$kS^0 = -\frac{2\pi k}{p} \; ,$$

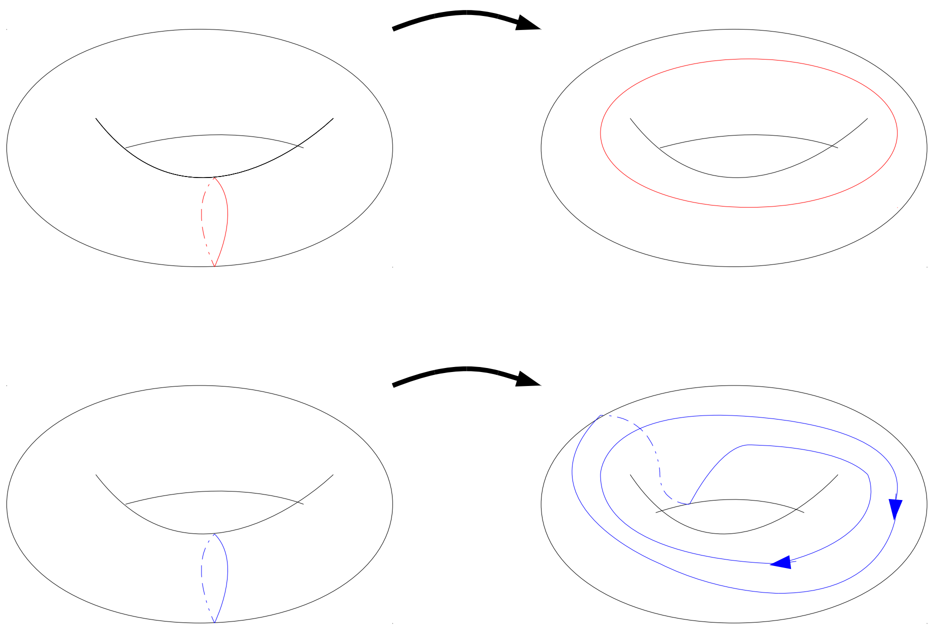
So the tree level sum diverges

$$Z \sim \sum_{p,q} e^{2\pi k/p} \to \infty$$

Can it be regulated? We need to include quantum effects!

A de Sitter Farey Tail

This is a sum over the modular group $SL(2, \mathbb{Z})$, just like in AdS/CFT. It is a sum over ways of “filling in” the Euclidean horizon.



Perturbative Corrections

Even though 3D gravity has no local degrees of freedom there are still "global" degrees of freedom associated with the topology of space-time. These give quantum contributions to the effective action.

For example, the one loop determinants of the ghost and linearized metric fluctuations

$$e^{S^1} = \frac{\det \Delta_{ghost}}{\det \Delta_{graviton}}$$

do not quite cancel.

To compute these perturbative corrections we can either

- Compute an infinite number of all-loop Feynman diagrams
- Cheat

Let's cheat!

Chern-Simons Theory

Classically, 3D gravity is a Chern-Simons theory. The frame fields and spin connection

$$e^a = e^a{}_\mu dx^\mu \qquad \omega^a = \omega_{\mu bc}\epsilon^{abc} dx^\mu$$

are combined into two connections

$$A^a_\pm = \omega^a \pm e^a$$

The equations of motion

$$de^a - \epsilon^a{}_{bc}\omega^b \wedge e^c = 0 \qquad d\omega^a - \epsilon^a{}_{bc}\left(\omega^b \wedge \omega^c + e^b \wedge e^c\right) = 0$$

imply that $A_\pm$ are flat $SU(2)$ connections

$$F = dA + A \wedge A = 0, \qquad A = A^a T^a$$

These are the equations of motion of Chern-Simons theory.

Chern-Simons Theory II

General relativity is an $SU(2) \times SU(2)$ Chern-Simons theory with imaginary levels $k_\pm = \pm ik$

$$S_{GR} = kI_{cs}[A_+] - kI_{cs}[A_-]$$

where I_{cs} is the Chern-Simons invariant

$$I_{cs}[A] = \mathrm{Tr} \int \left(A \wedge dA + \frac{2}{3} A \wedge A \wedge A \right)$$

In the Minkowski and AdS cases this formulation is subtle since both the gauge group and space-time are not compact.

But we are now studying CS theory with a compact gauge group on a compact manifold.

The CS partition function can be computed exactly using TQFT methods.

Chern-Simons Theory $\neq$ Gravity

The two theories are not equivalent at the quantum level.

The path integral of CS is a sum over all flat connections on a manifold of fixed topology. The path integral of gravity is a sum over topologies, with a specific flat connection (a metric) for each topology.

But they are equivalent at all orders in perturbation theory around a given classical saddle.

To compute the gravity perturbative corrections we must isolate the contribution to the CS path integral which comes from the flat connection of gravity.

We can check that

- Tree level: Einstein-Hilbert action $\leftrightarrow$ CS invariant
- One-loop level: GR determinant $\leftrightarrow$ CS determinant (hard!)

The Answer

We now compute the complete path integral, including all perturbative (loop) and non-perturbative (instanton) corrections from to lens spaces.

The answer is divergent due to the sum over geometries with small volume.

$$Z = 24\zeta(1) + \dots$$

The Hartle-Hawking state of Einstein gravity is non-normalizable.

This divergence cannot be regulated using standard field theory techniques.

A similar divergence appeared the AdS/CFT sum over geometries, but it could be regulated provided the bulk theory satisfied certain consistency conditions.

Discussion

Interpretation

What does it all mean?

Perhaps:

- de Sitter gravity does not exist.
- de Sitter gravity exists but we have not done the path integral correctly.
- de Sitter gravity exists only if we include more interesting degrees of freedom.
- de Sitter gravity exists but we are computing the wrong thing.

Speculations

We do not know the correct observables for eternal inflation.

- Perhaps the correct observables are those whose expectation values are finite. No observer could ever measure the norm of the Hartle-Hawking state.

We do not understand de Sitter entropy.

- Perhaps the sum over lens spaces is responsible for the modular invariance of dS/CFT. This explains the derivation of dS entropy via a Cardy formula.

We can not explain the small value of the cosmological constant.

- Perhaps the proper sum over geometries will only converge when the coupling constant obeys certain (number theoretic) conditions. This would lead to a quantization of the cosmological constant and might explain its small value.