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Spring School on Superstring Theory and Related Topics

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From Emergent Gauge Theory to Emergent Space

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From Emergent Gauge Theory to Holography

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Background

- Quantum field theory is a universal framework that describes long wavelength fluctuations in quantum many-body systems : low energy effective theory of string theory, condensed matter systems,…
- One goal in CM physics is to have a complete classification of possible phases (IR fixed points) of matter and understand low energy physics using low energy effective field theory

  Ex) Landau Fermi liquid theory for a class of metals
- Data for IR fixed points : contents and quantum numbers of low energy degrees of freedom, symmetry
- IR theory is in principle completely fixed by UV theory (reductionistic), yet it is usually hard to derive IR data from a first principle calculation and they sometimes bear little resemblance to the UV one (emergence)

Background (cont'd)

- Different types of IR fixed point
  - Gapped phases (no IR d.o.f.) : (trivial) insulator, confinement phase in gauge theory
  - Topological phases (sub-extensive IR d.o.f.) : Quantum Hall liquids, Chern-Simons theory
  - Gapless phases (extensive gapless modes) : metal, deconfinement phase in gauge theory, quantum critical points in condensed matter systems
    - Gapless modes can be protected by (emergent) symmetry, or can be achieved by fine tuning (e.g. QCP)

Background (cont'd)

- Different types of gapless phases
  - IR free theory (easiest) : gauge theory in D>4
  - Weakly interacting theory (easy) : 3D O(N) model in the large N limit
  - Strongly coupled in original variables but weakly coupled in dual variables (non-trivial, but easy once duality mapping is found) : strongly coupled 4D gauge theory (electro-magnetic duality)
  - Theory that has no weak coupling description (hardest) : 4D gauge theory at intermediate coupling

Background (cont'd)

- Theories that have weak coupling descriptions in terms of dual variables
  - Original `particles' remain strongly coupled and have short life time, yet they are organized into long-lived (weakly coupled) collective excitations
  - Duality mapping is a change of variable from the original particles to the collective excitations. The non-trivial information on the strong coupling dynamics is encoded in the duality mapping.
  - Dual variable may carry new (sometimes fractional) quantum numbers : fractionalization
  - Dual variable may live in different space : holography
  - Fractionalization and holography are the topics of this lecture

Background (cont'd)

- In fractionalization, the dual description is gauge theory : redundancy in phase (emergent phase)
- In holography, the dual description is gravity : redundancy in space (emergent space)
- Incidentally, gauge theory and gravity are the two fundamental building blocks of our nature
- Fractionalization and holography may shed important insights into gauge theory and gravity themselves
  - Why is nature built on the principle of redundancy?

Plan

I. Fractionalization and emergent gauge theory in strongly correlated systems

II. Holographic description of QFT

   1. Toy model
   2. O(N) model
   3. U(N) gauge theory

I. Fractionalization

This part of lecture is based on a pedagogical model proposed in
[SL and Patrick Lee, PRB (05)]

- Original idea of fractionalization in spin systems was introduced by Fazekas and P.W. Anderson, Phils. Mag. 30, 432 (74).
- Many exactly solvable models : proof of principle [Kitaev; X.-G. Wen; Moessner and Sondhi; Motrunich and Senthil, …]
- String-net (Hamiltonian) picture : M. Levin and X.-G. Wen
- Further reviews and applications to condensed matter systems :
  - P. A. Lee, N. Nagaosa, X.-G. Wen, Rev. Mod. Phys. 78, 17 (2006)
  - S. Sachdev, 0901.4103
  - L. Balents, Nature 464, 199 (2010)

Outline

- Model
- Field theory description of fractionalization
- World line picture of fractionalization
  - Emergence of gauge theory
  - Statistics of fractionalized particles

Model

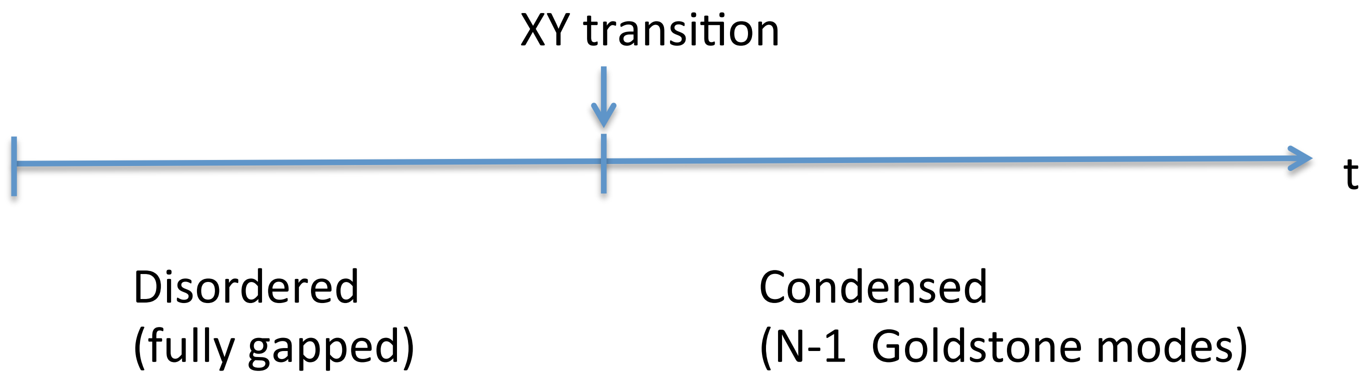
$$S = -t \sum_{<i,j>} \sum_{a,b} \cos\left(\theta_i^{ab} - \theta_j^{ab}\right)$$
$$- K_3 \sum_{i} \sum_{a,b,c} \cos\left(\theta_i^{ab} + \theta_i^{bc} + \theta_i^{ca}\right)$$

- An effective theory for exciton (composite of electron and hole) condensate in multi-band insulator, but let's not worry about where it came from
- i, j : lattice sites in 4D lattice (3 space + discrete time)
- θ_i^{ab} : boson (meson) with flavor a and anti-flavor b
  (a,b=1,2,…,N)
- Constraint : $\theta_i^{ab} + \theta_i^{ba} = 0$ N(N-1)/2 d.o.f.
- $U(1)^{(N-1)}$ global symmetry $\theta_i^{ab} \to \theta_i^{ab} + \phi^a - \phi^b$

Weak coupling phase (K_3<<1)

$$S = -t \sum_{<i,j>} \sum_{a,b} \cos\left(\theta_i^{ab} - \theta_j^{ab}\right)$$
$$- K_3 \sum_{i} \sum_{a,b,c} \cos\left(\theta_i^{ab} + \theta_i^{bc} + \theta_i^{ca}\right)$$

- Weakly coupled N(N-1)/2 bosons
- Gapped phase (t<<1)
- Symmetry broken phase (t>>1) : (N-1) Goldstone modes



Strong coupling limit (K_3>>1)

$$S = -t \sum_{<i,j>} \sum_{a,b} \cos\left(\theta_i^{ab} - \theta_j^{ab}\right)$$

$$\boxed{- K_3 \sum_{i} \sum_{a,b,c} \cos\left(\theta_i^{ab} + \theta_i^{bc} + \theta_i^{ca}\right)}$$

Strong coupling (K3 >> 1) : Dynamical constraints

$$\theta^{ab} \qquad (S^1)^{N(N-1)/2}$$

$$\theta^{ab} + \theta^{bc} + \theta^{ca} = 0 \quad \Downarrow$$

$$\theta^{ab} = \phi^a - \phi^b \qquad \frac{(S^1)^N}{\boxed{S^1}}$$

Parton fields

U(1) redundancy

Effective theory for parton fields

$$S = -t \sum_{<i,j>} \boxed{\left[\sum_a e^{i(\phi_i^a - \phi_j^a)}\right]} \left[\sum_b e^{-i(\phi_i^b - \phi_j^b)}\right] + c.c$$

becomes classical In the large N limit :
introduce collective field

$$e^{tA^*A} = c\int d\eta d\eta^* e^{-t[|\eta|^2 - \eta^* A - \eta A^*]}$$

$$S = t\sum_{a<b}\sum_{<i,j>}\left[|\eta|^2 - \eta e^{-i(\phi_i^b - \phi_j^b)} - \eta^* e^{i(\phi_i^a - \phi_j^a)}\right],$$

$$\eta = |\eta| e^{i a_{ij}}$$: complex auxiliary field

- a_{ij} : gauge field associated with local symmetry $\phi_i^a \to \phi_i^a + \varphi_i$
- Bare gauge coupling is infinite (auxiliary field) $\cancel{+\frac{1}{g^2}F_{\mu\nu}F^{\mu\nu}}$

Fractionalization

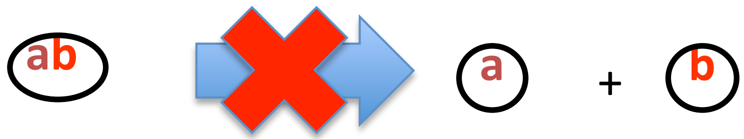
$$S = \int dx \left[|(\partial_\mu - a_\mu)\Phi_a|^2 + V(\Phi_a) + \frac{1}{g^2} F_{\mu\nu} F^{\mu\nu} \right]$$

$$\Phi_a = e^{i\phi^a}$$

- Integrating out high energy modes generates kinetic term for the gauge field
- In the large N limit, renormalized gauge coupling is small : $g^2 \sim 1/N$, and deconfinement phase is stable
- Low energy excitations : fractionalized bosons, gauge boson
- Fractionalized bosons are weakly coupled in the large N limit
- Phase of fractionalized bosons becomes classical in the large N limit (emergence of phase), although not gauge invariant

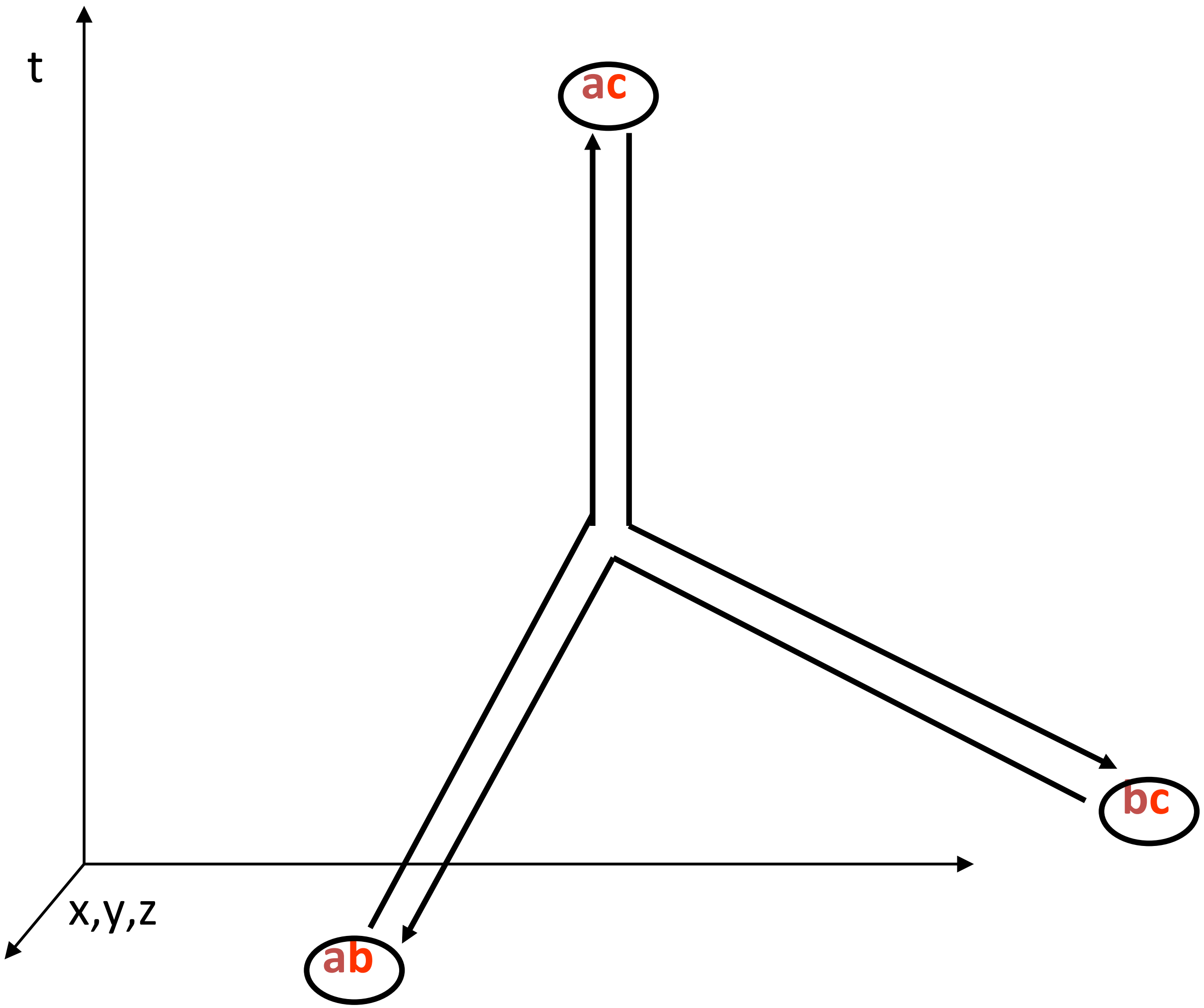
One may ask….

- Bare gauge coupling is infinite and partons are microscopically confined within original `mesons' : one can not physically break one meson into two partons because mesons are fundamental particles (in this model)



- Fractionalization is a low energy phenomenon caused by many-body effects, c.f. composite particle can break into elementary particles at high energy
- How can one have fractionalized particles as low energy excitations ?
- This is a common conceptual issue in all `slave-particle' approaches
- Known exactly solvable models do not address this issue because they have finite gauge coupling at the lattice scale

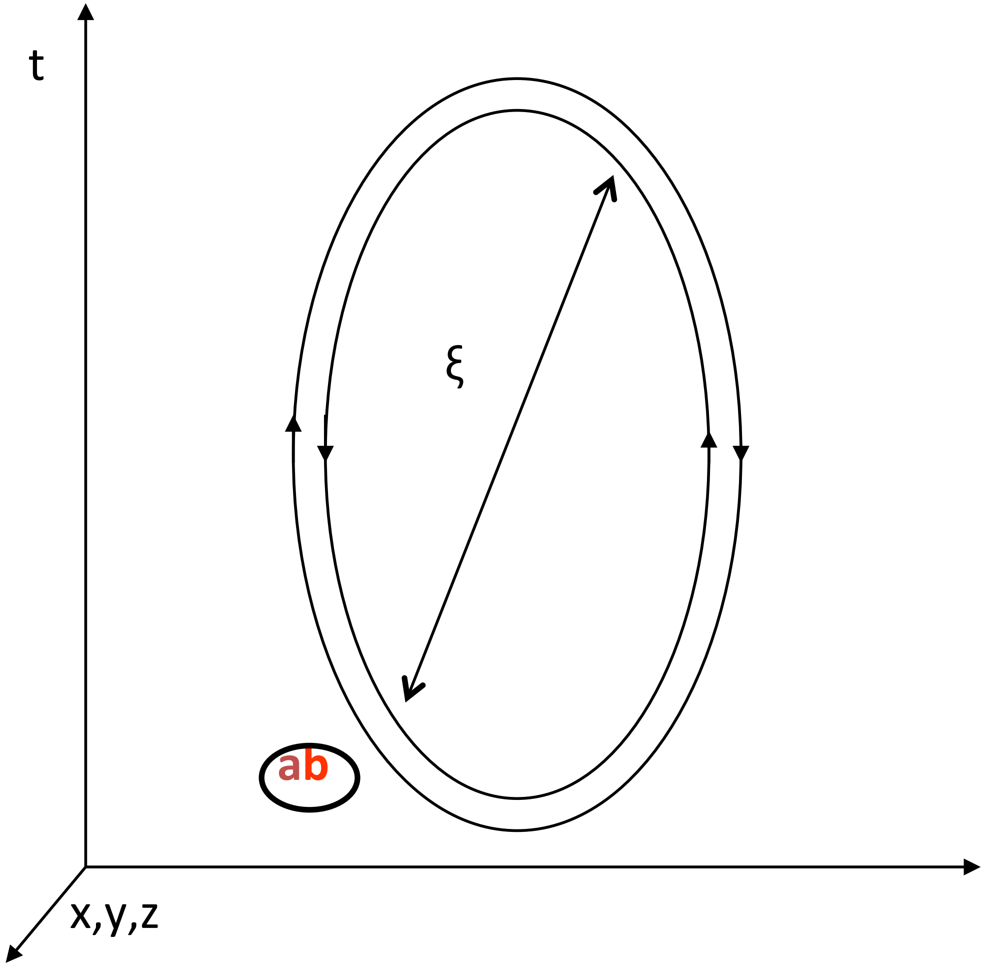
World line action

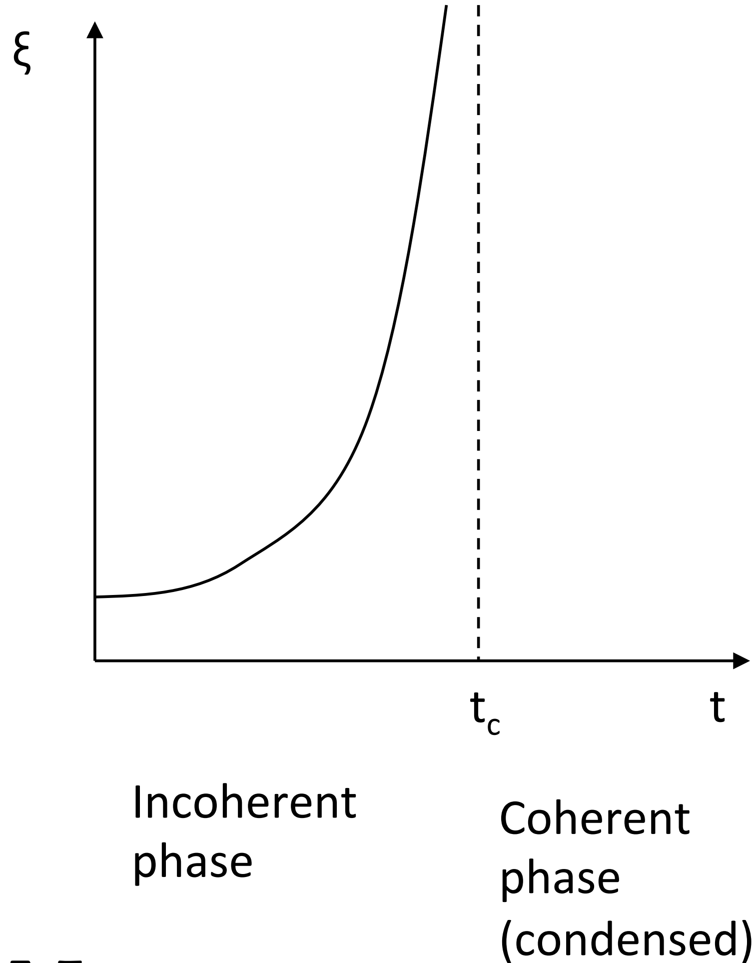


$$S = \frac{1}{t} \sum_i \int ds_i + \frac{1}{K_3} N_v$$

N_v : number of vertices

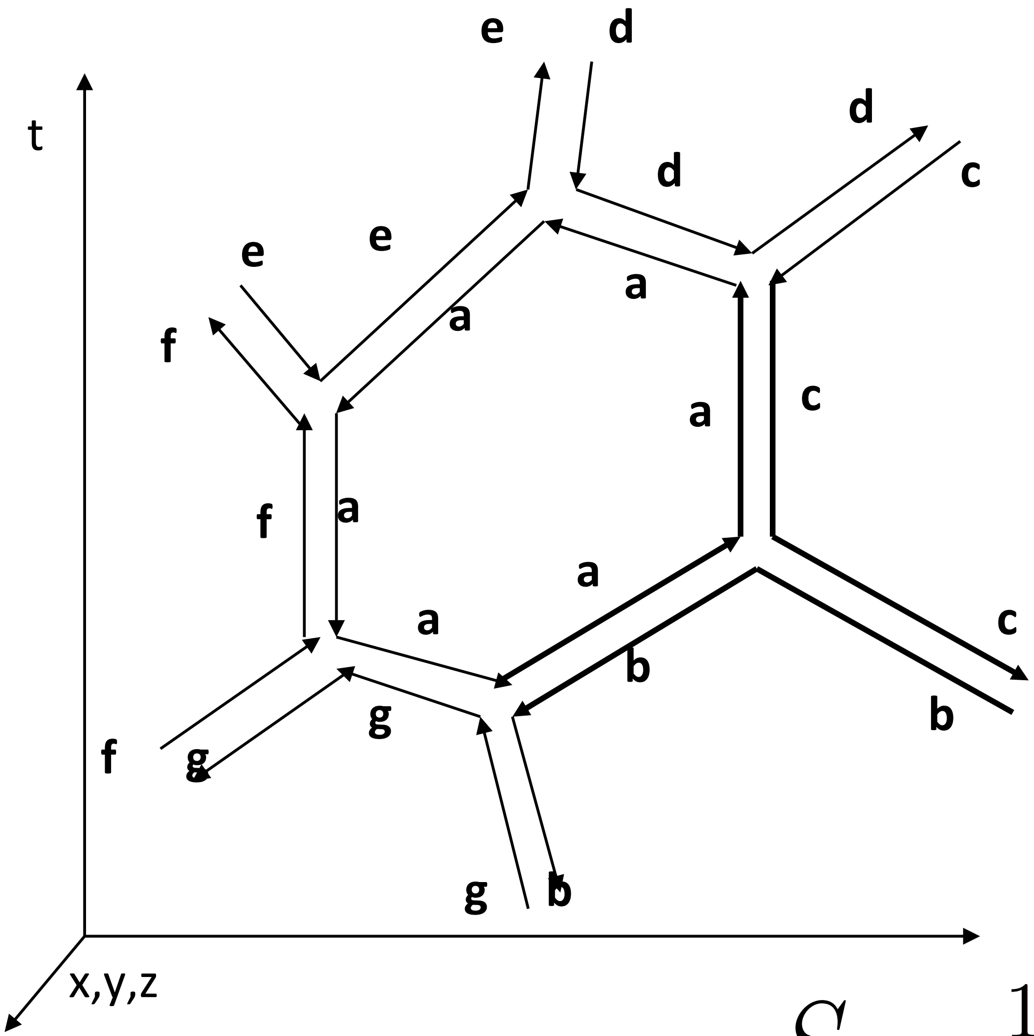
Dynamics (weak coupling : $K_3 << 1$)





$$S = \frac{1}{t} \sum_i \int ds_i + \frac{1}{K_3} N_v$$

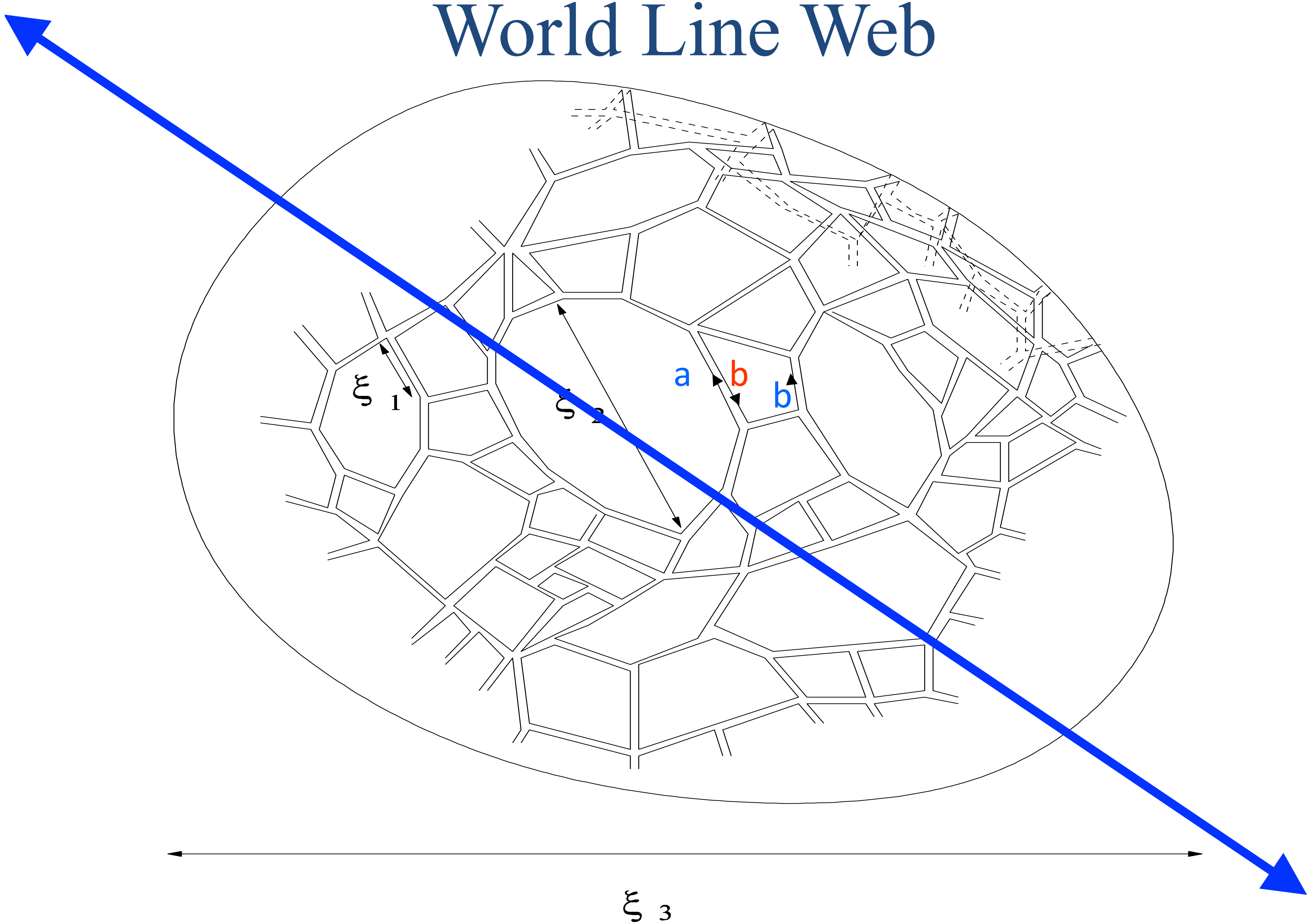
Dynamics (strong coupling : $K_3 >> 1$)



Frequent exchange
of partner

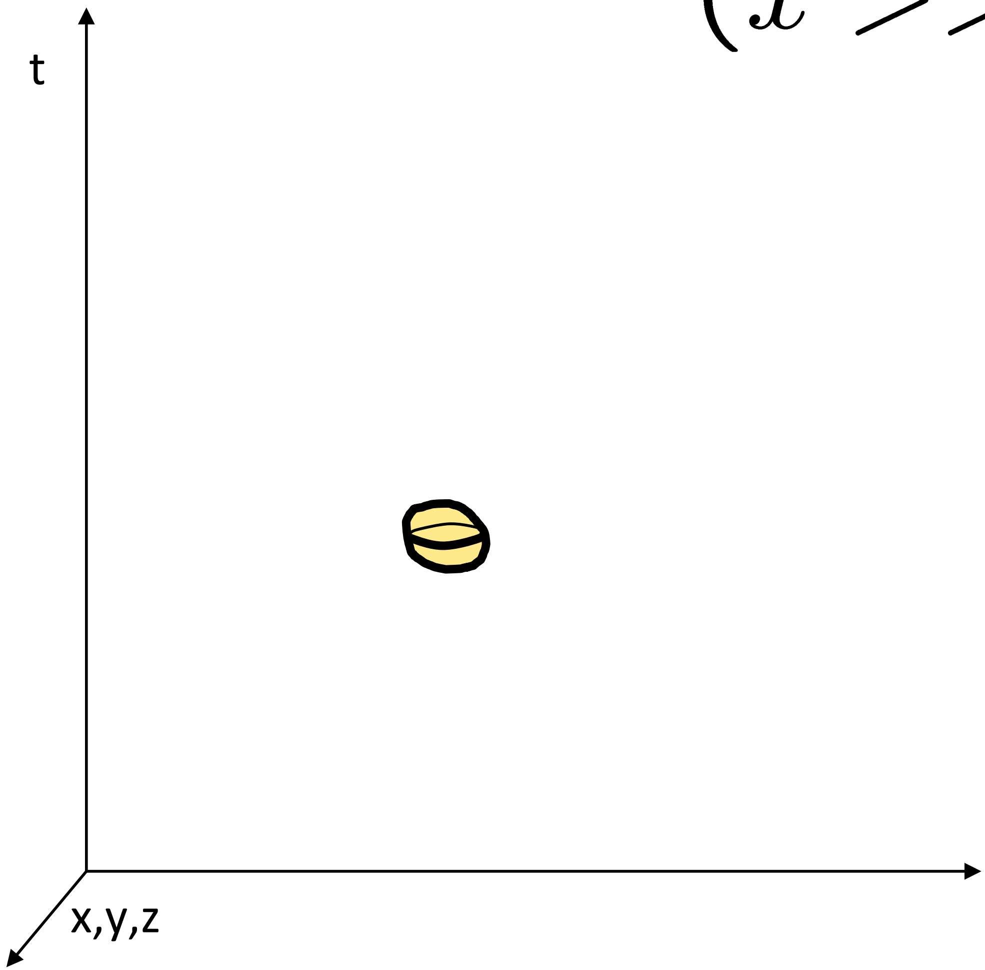
$$S = \frac{1}{t}\sum_i \int ds_i + \frac{1}{K_3} N_v$$

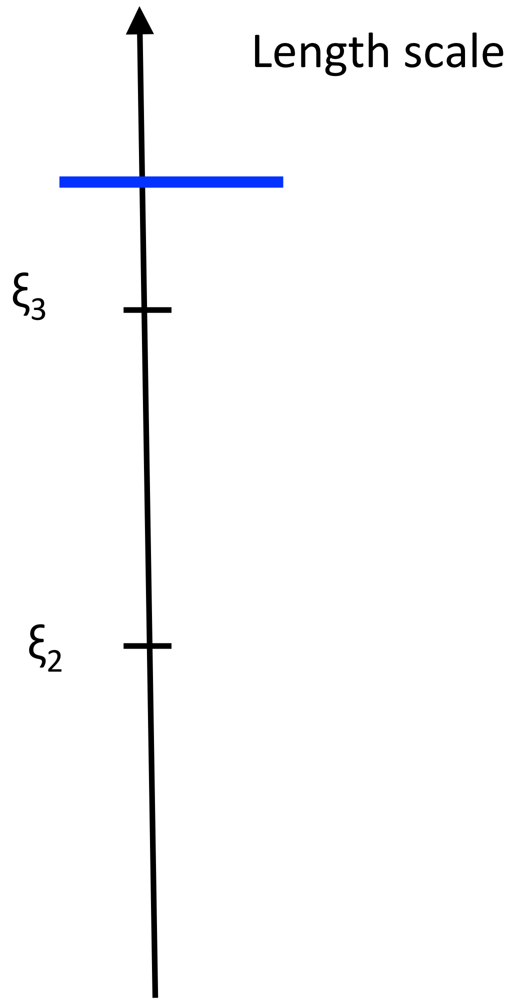
World Line Web



physics @ largest length scale

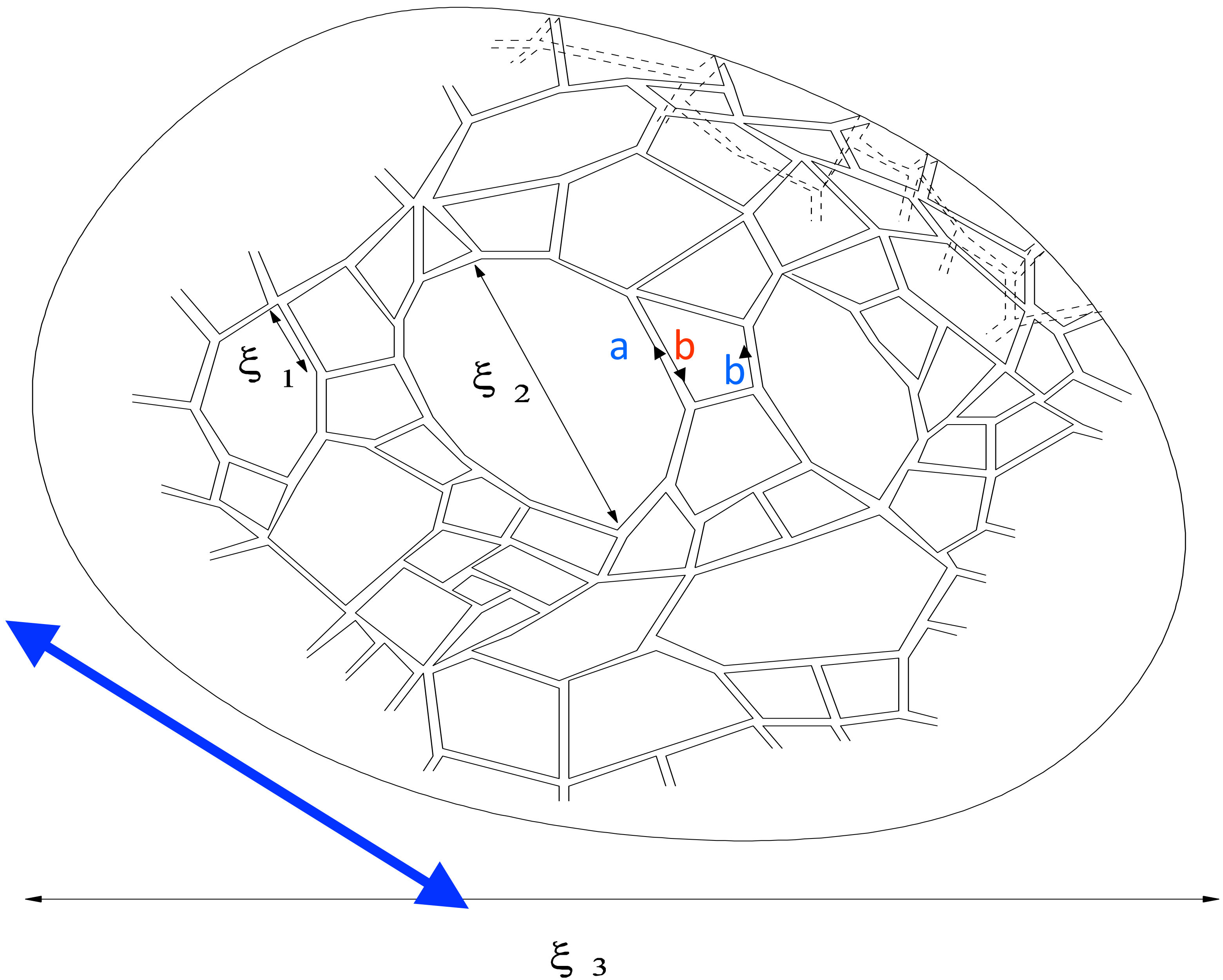
$$(x >> \xi_3)$$





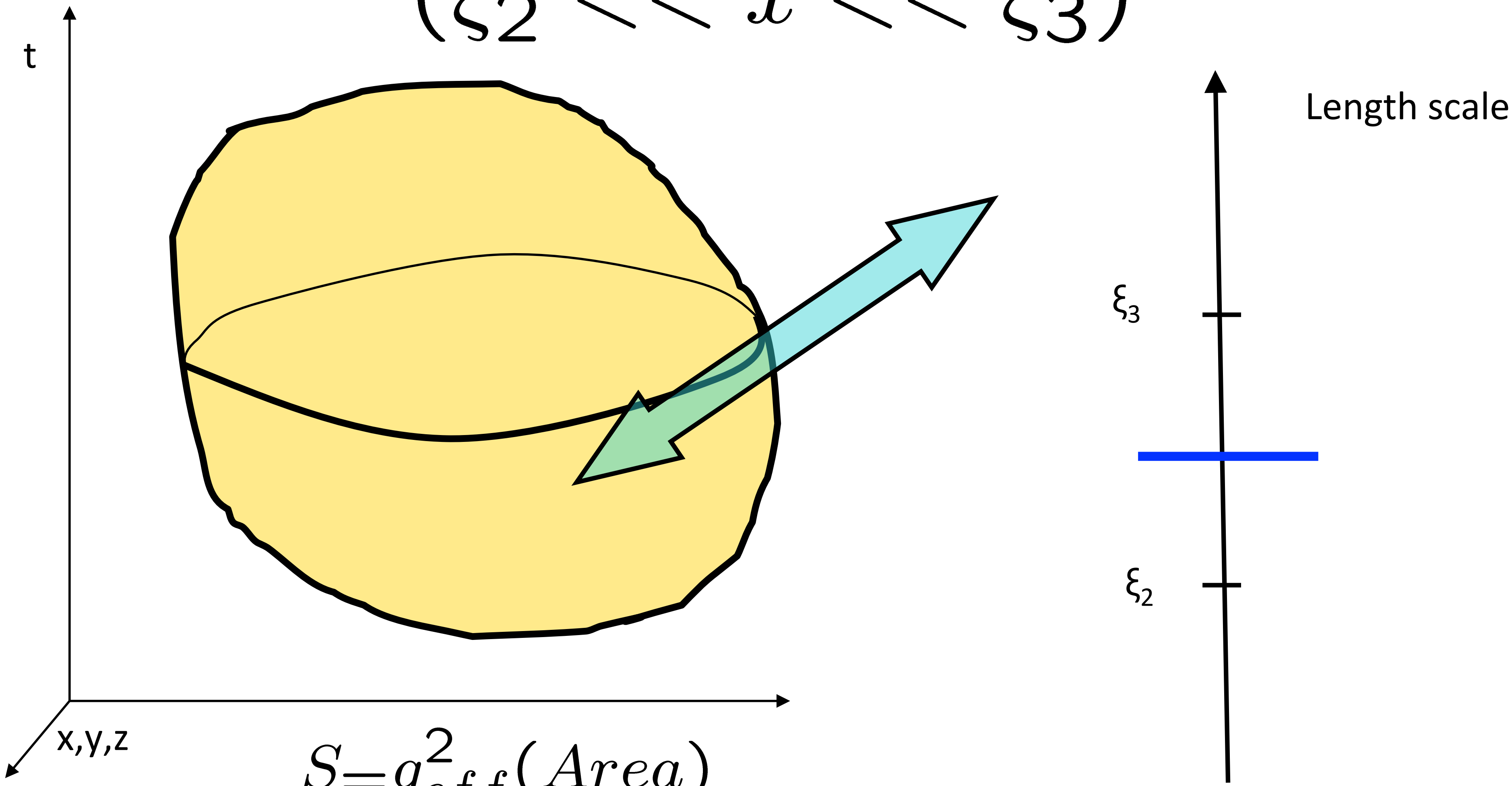
Only small bubbles of world line web exist
Boring vacuum : no charge, no massless particle
(disordered phase)

World Line Web



physics @ intermediate length scale

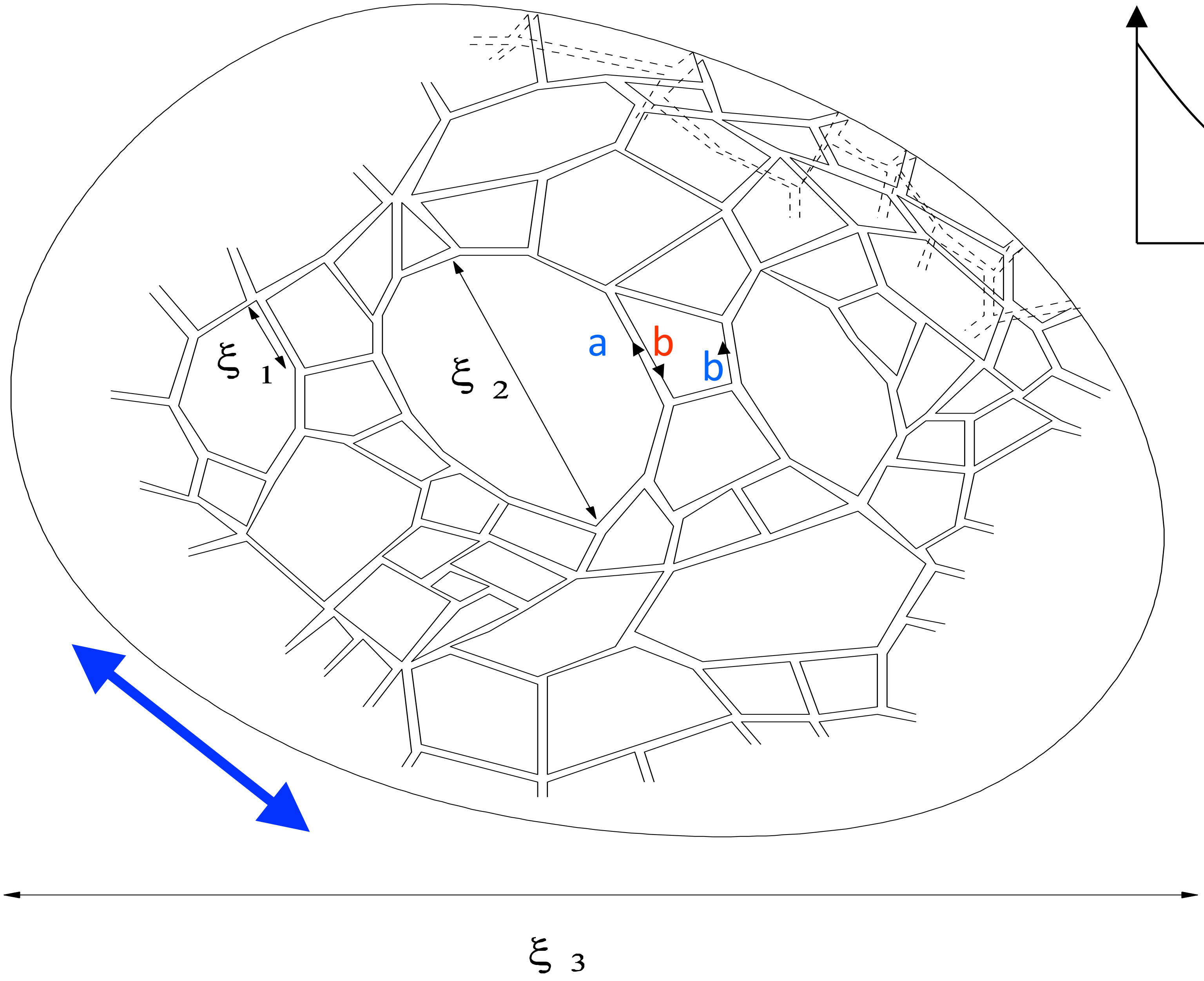
$$(\xi_2 << x << \xi_3)$$

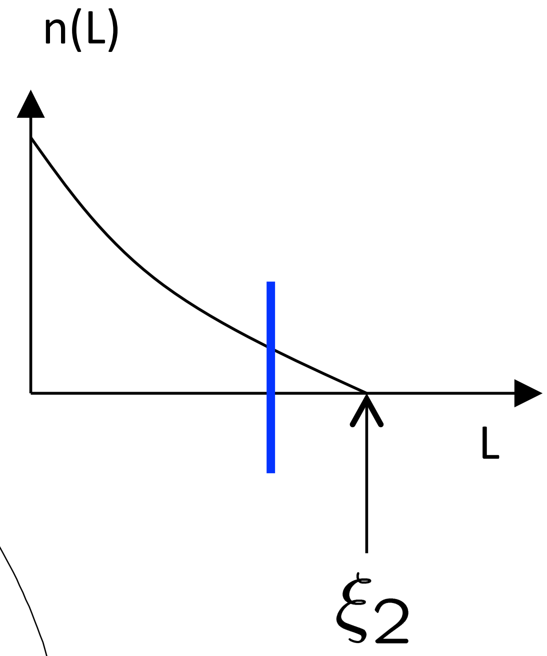


$$S = g_{eff}^2 (Area)$$

- World line web looks like a smooth membrane which fluctuate wildly
- One can understand this membrane as world sheet of electric flux line of U(1) gauge theory

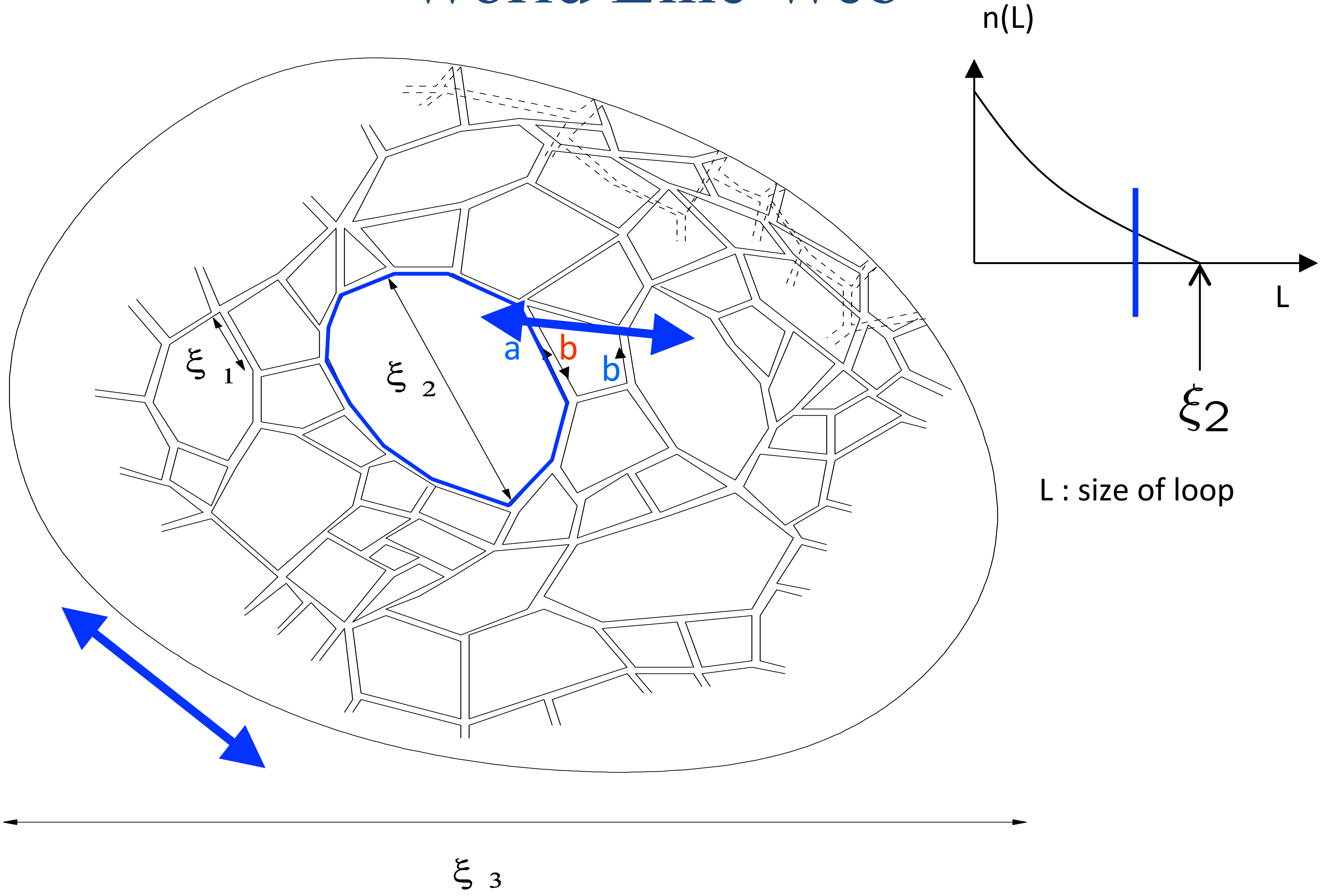
World Line Web





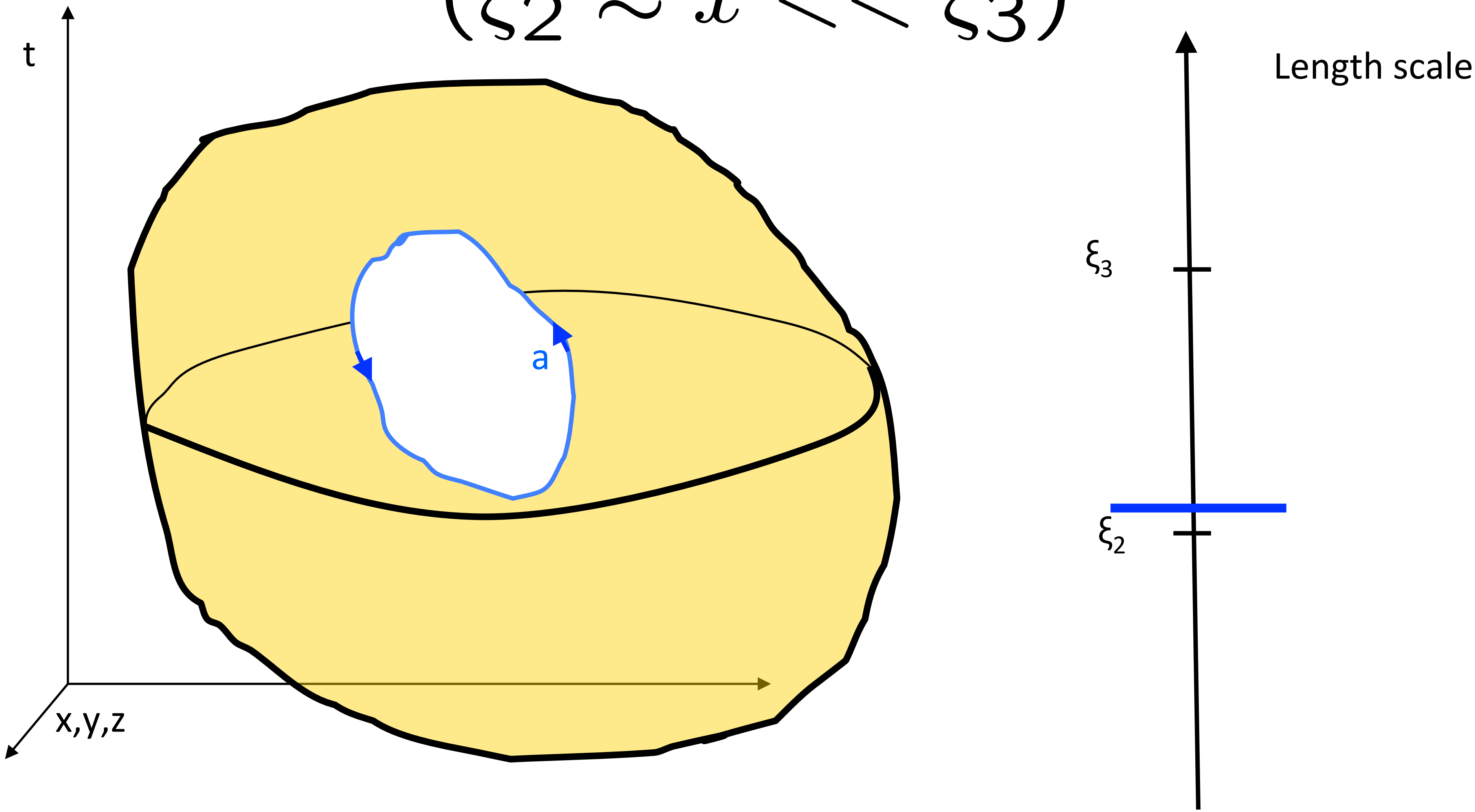
L : size of loop

World Line Web



Physics near loop length scale

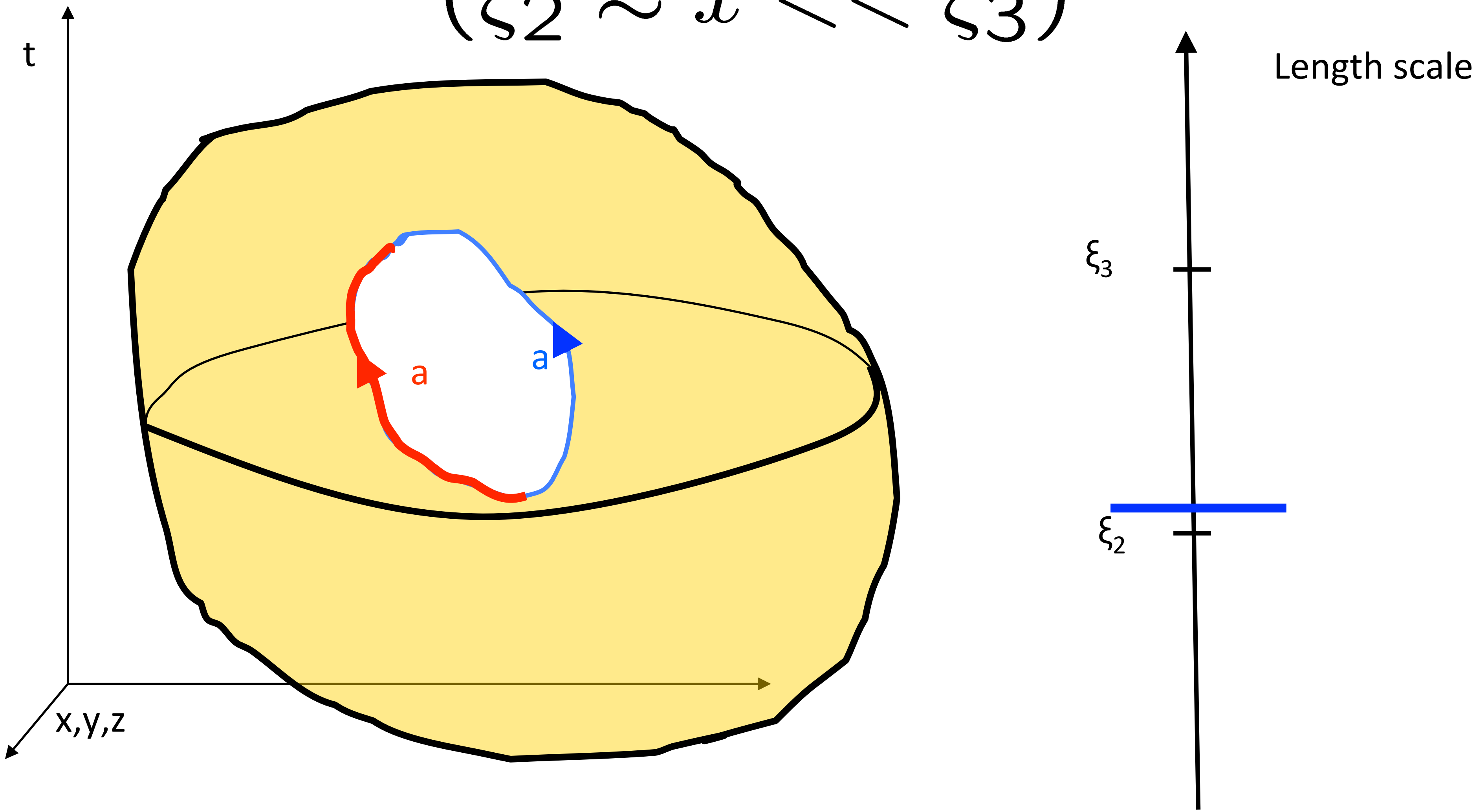
$$(\xi_2 \approx x << \xi_3)$$



Boundary of puncture looks like a world line of particle
World line web looks like world sheet of an electric flux line

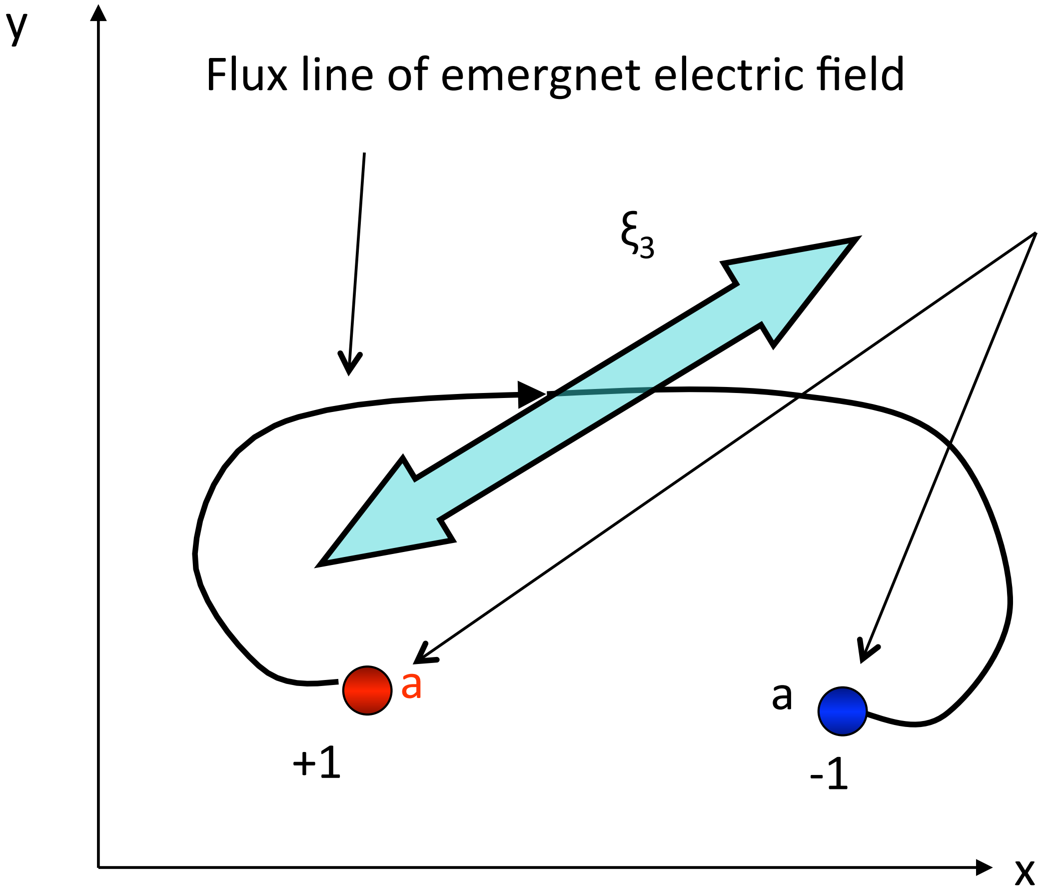
Physics near loop length scale

$$(\xi_2 \approx x << \xi_3)$$



Propagation in negative time (energy) :
anti-particle in positive time (energy)

Emergence of new charge & new electric field



Profile of emergent particle :

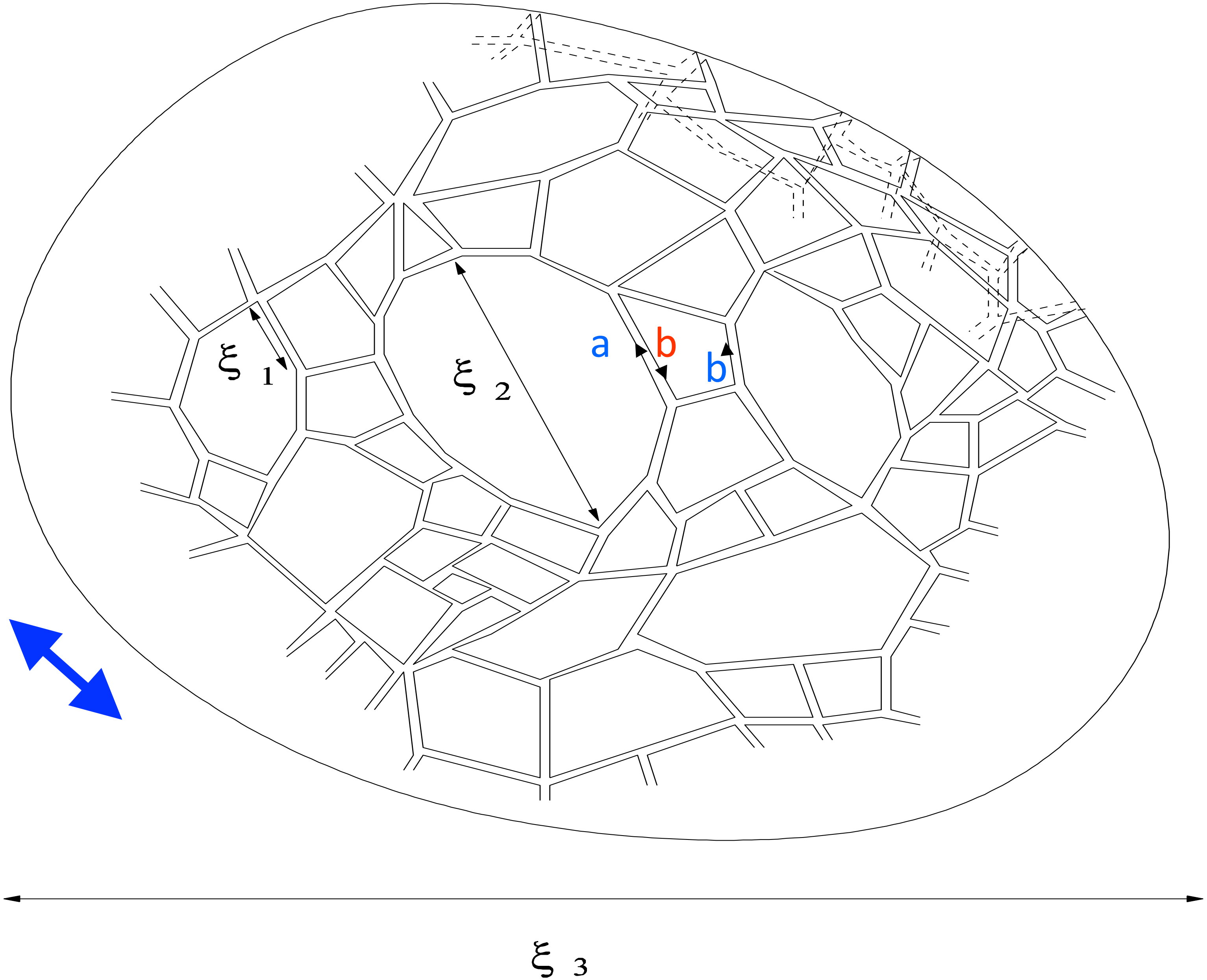
Boson : no (-1) phase for loop
Emergent `electric' charge : ±1
Flavor number : a or a
half of exciton flavor

Fractionalization!

$$V(r) \sim 1/r$$

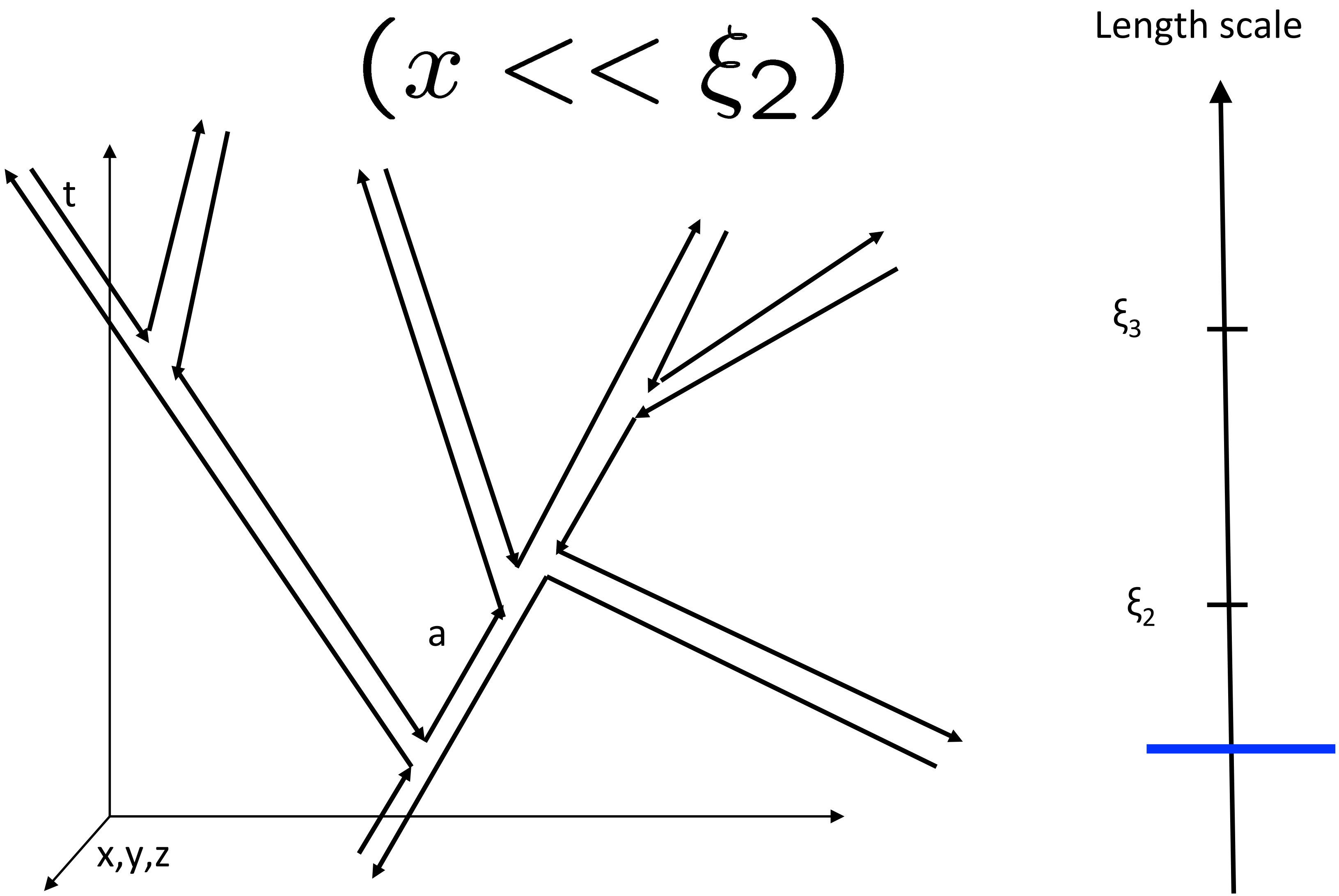
deconfinement

World Line Web



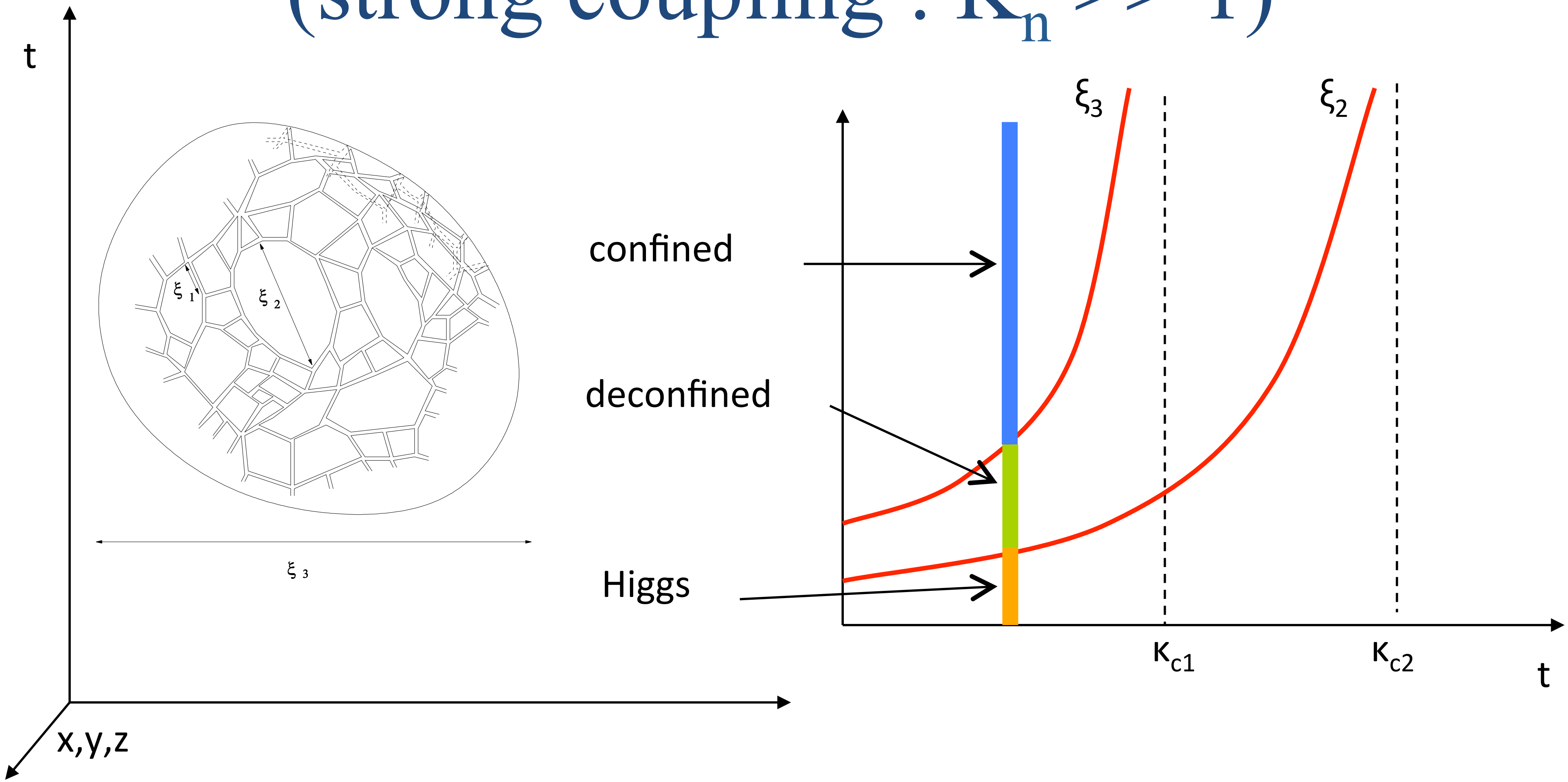
physics @ smallest length scale

$$(x << \xi_2)$$



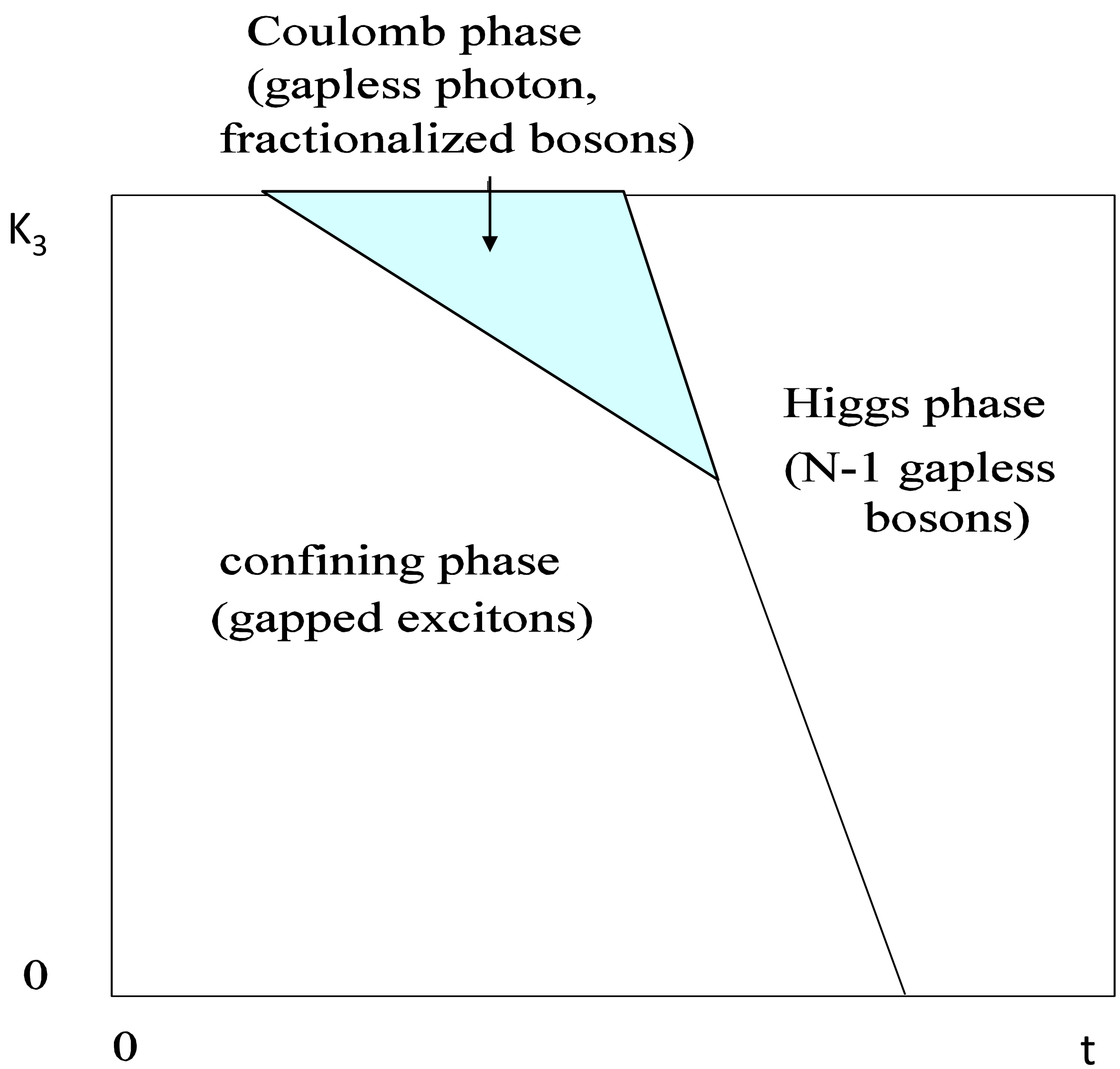
- Loops of single line looks very large : fractionalized particle condense (Higgs phase)
- One fractionalized mode is eaten up by gauge field
- (N-1) gapless modes : they are nothing but the Goldstone modes in the condensed phase

Dynamics of off-diagonal exciton (strong coupling : $K_n >> 1$)



In the large N limit, one can make the tension of the world sheet much smaller than the mass of the fractionalized particles

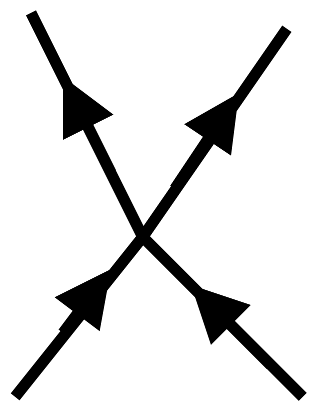
Phase diagram I



Fractionalized fermion

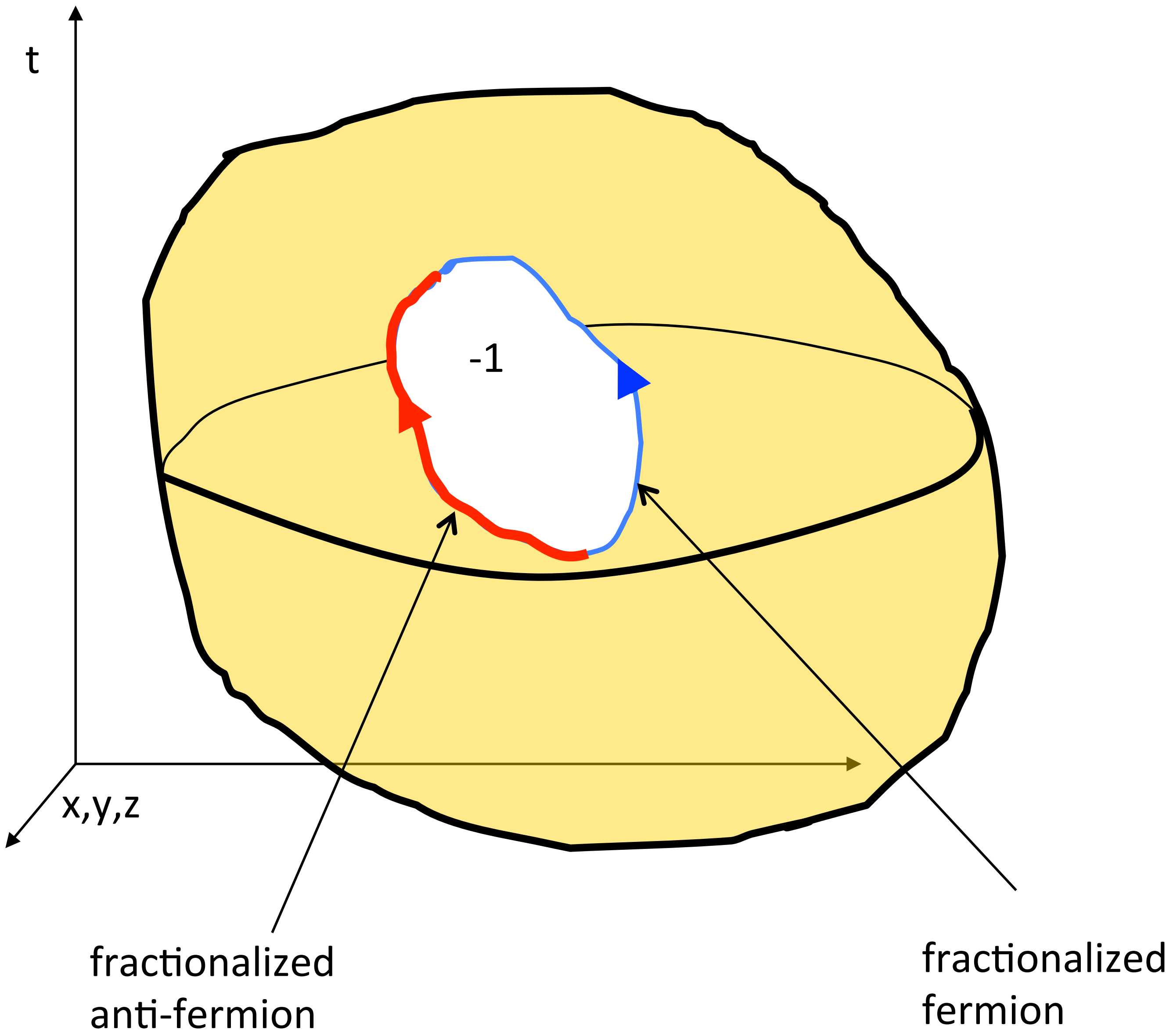
$$S = -t \sum_{<i,j>} \sum_{a,b} \cos\left(\theta_i^{ab} - \theta_j^{ab}\right)$$
$$- K_4 \sum_{i} \sum_{a,b,c,d} \cos\left(\theta_i^{ab} + \theta_i^{bc} + \theta_i^{cd} + \theta_i^{cd}\right)$$

[SL, Patrick Lee(05)]

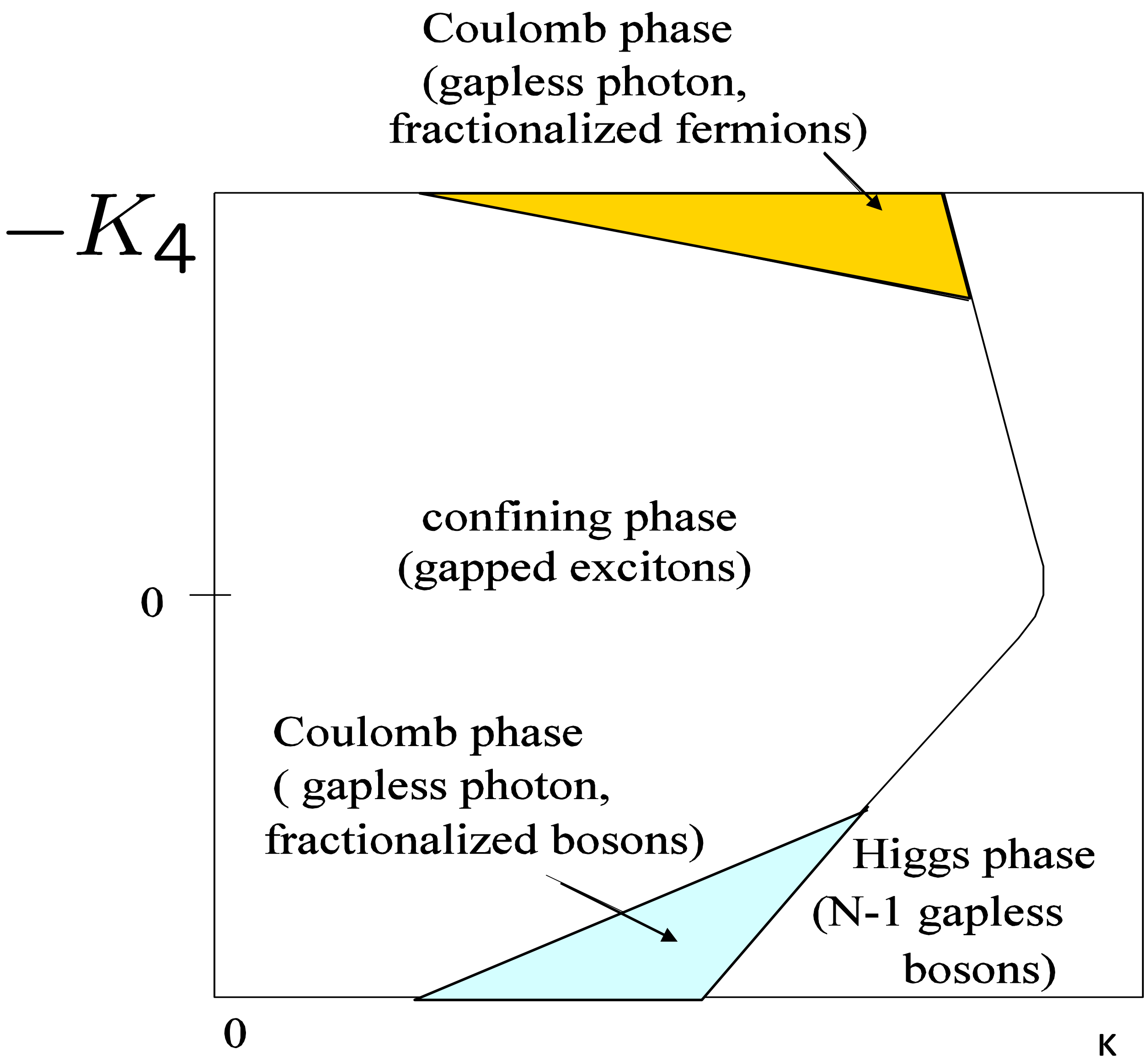


$K_4 < 0$: each vertex contributes (-1)

Fermion loop



Phase diagram II



Summary

- Fractionalized particles and emergent gauge boson can arise as low energy excitations
- In large N limit, fractionalized particles are weakly coupled
- Statistics (spin) is dynamically determined

II. Holographic Description of QFT; O(N) vector model and Large N gauge theory

Ref : 0912.5223, 1011.1474

AdS/CFT correspondence

[Maldacena]

- N=4 supersymmetric SU(N) gauge theory in 3+1D is dual to the type IIB string theory in $AdS_5 x S^5$
- Strongly coupled QFT is mapped into a classical (weakly coupled) theory in the large N limit
- Dual theory becomes local in the strong coupling limit
- Dual theory lives in one-higher (extended) dimensional space
- Additional dimension is an energy scale
- An example of emergent space

Dictionary in AdS/CFT

[Gubser, Klebanov, Polyakov; Witten]

Low energy

(D+1)-dim anti-de Sitter space

$$\hat{O}(x) \;\rightarrow\; \phi_O(x,z)$$

$$\Delta_O \;\rightarrow\; m_O$$

z

High energy

x

D –dim flat space

RG and AdS/CFT

Saddle point equation of motions + regularity condition in the bulk corresponds to beta function(al)s of boundary field theory in the large N limit [Verlinde]

Low energy

EOM → RG flow

$$J(x,z) \;=\; J(x,0)$$

z

$$S_0 + \int dx\, J(x) O(x)$$

High energy

x

D –dim flat space

Holographic description of QFT

$$
\begin{aligned}
Z[J(x)] &= \int D\phi(x) e^{-S_{field\ \ theory}[\phi]} \\
&= \int D\ \ \text{``}J(x,z)\text{''}\, e^{-S'[J(x,z)]}\Big|_{J(x,0)=J(x)}
\end{aligned}
$$

D-dimension

(D+1)-dimension

- Is there a general prescription to derive holographic dual for general QFT ?
- When do the dual theories become classical and local?

[Das, Jevicki; Gopakumar; Heemskerk, Penedones, Polchinski; Faulkner, Liu, Rangamani; Douglas, Mazzucato, Razamat]

Outline

- Holographic description of QFT
  1. Toy model
  2. O(N) vector model
  3. U(N) gauge theory
- Summary

1. Toy model

Toy model

Zero dimensional field theory :

$$Z[\mathcal{J}] = \int d\Phi \; e^{-S[\Phi]}$$

$$S_M[\Phi] = M^2 \Phi^2$$

$$S_{\mathcal{J}}[\Phi] = \sum_{n=1}^{\infty} \mathcal{J}_n \Phi^n$$

S_M : bare action with mass M

$S_{\mathcal{J}}$: deformation with source $\mathcal{J}$

Step 1 : introduce an auxiliary field

$$Z[\mathcal{J}] \;=\; \mu \int d\Phi d\tilde{\Phi}\; e^{-(S_M + S_{\mathcal{J}} + \boxed{\mu^2\tilde{\Phi}^2})},$$

`low energy' field

$$\begin{aligned} \Phi &= \phi + \tilde{\phi}, \\ \tilde{\Phi} &= A\phi + B\tilde{\phi} \end{aligned}$$

`high energy' field

[Polchinski; Polonyi]

$$\begin{aligned} Z[\mathcal{J}] &= \int d\phi d\tilde{\phi}\; e^{-(S_{\bar{\mathcal{J}}}[\phi+\tilde{\phi}] + M'^2\phi^2 + m'^2\tilde{\phi}^2)}, \\ M'^2 &= M^2 e^{2\alpha dz} \\ m'^2 &= \frac{M^2}{2\alpha dz}. \end{aligned}$$

infinitesimally small parameter

positive constant

(rate of coarse graining)

Step 2 : rescale the fields

$$\phi \to e^{-\alpha dz}\phi, \qquad \tilde{\phi} \to e^{-\alpha dz}\tilde{\phi}$$

$$\begin{aligned} Z[J] &= \int d\phi d\tilde{\phi}\ e^{-(S_j[\phi+\tilde{\phi}]+M^2\phi^2+m^2\tilde{\phi}^2)}, \\ & S_j[\phi+\tilde{\phi}] = \sum_{n=1}^{\infty} j_n(\phi+\tilde{\phi})^n, \\ & j_n = \mathcal{J}_n e^{-n\alpha dz}, \quad m = m' e^{-\alpha dz}. \end{aligned}$$

Step 3 : expand the action in power of low energy field

$$
\begin{aligned}
S_j[\phi+\tilde{\phi}] \;=\; & S_j[\tilde{\phi}] \\
& +(j_1+2j_2\tilde{\phi}+3j_3\tilde{\phi}^2+4j_4\tilde{\phi}^3)\phi \\
& +(j_2+3j_3\tilde{\phi}+6j_4\tilde{\phi}^2)\phi^2 \\
& +(j_3+4j_4\tilde{\phi})\phi^3 \\
& +j_4\phi^4.
\end{aligned}
$$

- Usually, one integrates out high energy modes to obtain an effective action for the low energy mode with renormalized couplings
- Instead, we interpret high energy field as fluctuating sources for the low energy field

Step 4 : decouple low energy field from high energy field

$$
\begin{aligned}
S_j' \;=\; & S_j[\tilde{\phi}] \\
& + iP_1 J_1 - iP_1(j_1 + 2j_2\tilde{\phi} + 3j_3\tilde{\phi}^2 + 4j_4\tilde{\phi}^3) + J_1\phi \\
& + iP_2 J_2 - iP_2(j_2 + 3j_3\tilde{\phi} + 6j_4\tilde{\phi}^2) + J_2\phi^2 \\
& + iP_3 J_3 - iP_3(j_3 + 4j_4\tilde{\phi}) + J_3\phi^3 \\
& + iP_4 J_4 - iP_4 j_4 + J_4\phi^4 .
\end{aligned}
$$

- J : fluctuating sources for low energy fields
- P : physical fluctuation of operator

$$P_n \sim i\phi^n$$

- J, P : auxiliary fields (have no dynamics)

Step 5 : integrate out high energy mode

$$Z[\mathcal{J}] \;=\; \int d\phi \Pi_{n=1}^{4}(dJ_n dP_n)\; e^{-(S_J[\phi]+M^2\phi^2+S^{(1)}[J,P])},$$

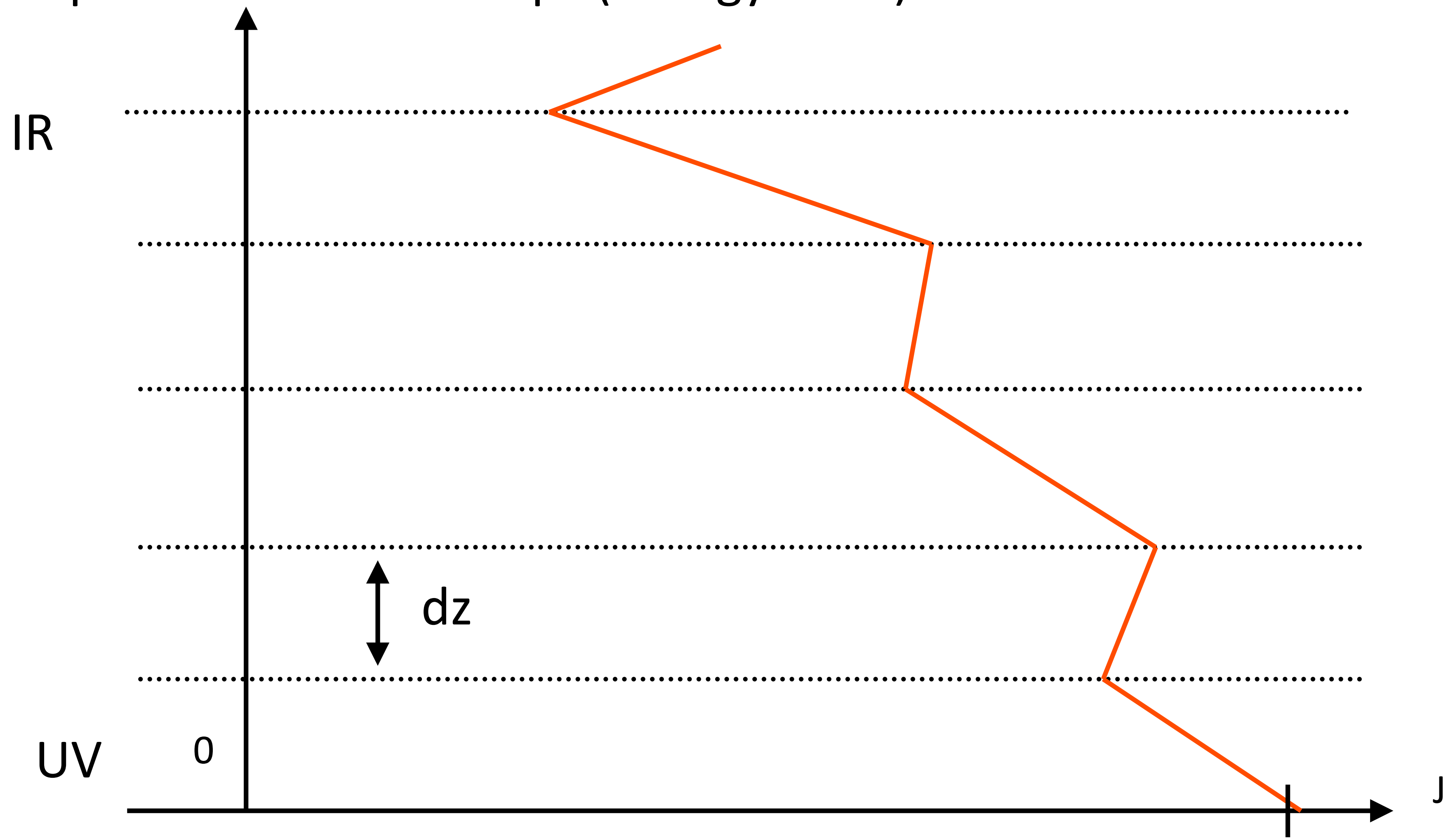
$$\tilde{\mathcal{J}}_n = \frac{\mathcal{J}_n + J_n}{2}$$

$$S^{(1)}[J,P] \;=\; \sum_{n=1}^{4} i(J_n - \mathcal{J}_n + n\alpha dz \mathcal{J}_n)P_n$$
$$+\frac{\alpha dz}{2M^2}(i\tilde{\mathcal{J}}_1 + 2P_1\tilde{\mathcal{J}}_2 + 3P_2\tilde{\mathcal{J}}_3 + 4P_3\tilde{\mathcal{J}}_4)^2$$

- This can be done exactly to the order of dz, using the large mass of the high energy mode
- J, P acquire dynamics
- The action for the low energy field comes back to its original form with fluctuating sources

Step 6 : repeatedly apply the steps 1-5 to the low energy field

z : parameterize RG steps (energy scale)



Boundary value : fixed by UV coupling

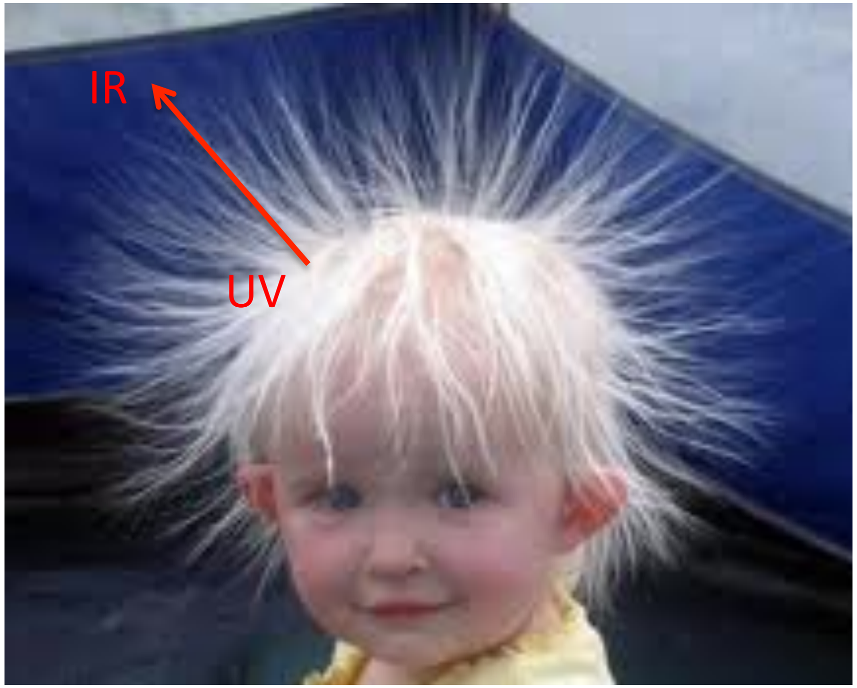
1-dimensional theory

$$
\begin{aligned}
Z[J] &= \int DJDPe^{-S[J,P]}, \\
S[J,P] &= \int_0^\infty dz[i(\partial_z J_n + n\alpha J_n)P_n \\
&\quad + \frac{\alpha}{2M^2}(iJ_1 + 2P_1J_2 + 3P_2J_3 + 4P_3J_4)^2
\end{aligned}
$$

- 1D Path integral of source fields J and conjugate fields P on [0,∞)

Boundary conditions

- Two boundary conditions needed for J_n, P_n
- UV boundary condition : $J_n(0) = \mathcal{J}_n$
- IR boundary condition is imposed dynamically



Physical meaning of α

$$\begin{aligned} L &= i(\partial_z J_n + n\alpha J_n)P_n \\ &\quad + \frac{\alpha}{2M^2}(iJ_1 + 2P_1J_2 + 3P_2J_3 + 4P_3J_4)^2 \end{aligned}$$

$$L = i\dot{J}_n P_n + \boxed{\alpha} H$$

- α : rate at which high energy modes are eliminated
  (\`speed' of RG flow)
- Can be made to be z-dependent α(z)
- Interpret z as time : α(z) becomes the lapse function $\sqrt{g_{zz}}$
- α(z) is NOT integrated : H=0 NOT imposed
- $l = \int_0^\infty \alpha(z)dz$ total proper \`time' is fixed to be infinite
- The theory can be viewed as 1D gravitational theory with fixed size along the z-direction

1D gravitation theory

$$S[J,P] = \int_0^\infty dz[i(\partial_z J_n + n\alpha J_n)P_n + \frac{\alpha}{2M^2}(iJ_1 + 2P_1 J_2 + 3P_2 J_3 + 4P_3 J_4)^2$$

- The theory is local in the `bulk', yet quantum fluctuations are still important in the bulk (no large N)
- Only one set of independent mode because the original theory has only one propagating mode
- Freedom to choose independent fields (say, J_3, P_3)

2. O(N) vector model

O(N) vector model in D dimensions

$$Z[\mathcal{J}] \;=\; \int D\Phi_a \; e^{-(S_M[\Phi]+S_{\mathcal{J}}[\Phi])}$$

$$S_M[\Phi] \;=\; \int d\mathbf{x}d\mathbf{y}\; \Phi_a(\mathbf{x})G_M^{-1}(\mathbf{x}-\mathbf{y})\Phi_a(\mathbf{y}),$$

$$S_{\mathcal{J}}[\Phi] \;=\; \int d\mathbf{x}\,\Big[\mathcal{J}_a\Phi_a + \mathcal{J}_{ab}\Phi_a\Phi_b + \mathcal{J}_{abc}\Phi_a\Phi_b\Phi_c + \mathcal{J}_{abcd}\Phi_a\Phi_b\Phi_c\Phi_d$$
$$+\mathcal{J}_{ab}^{ij}\partial_i\Phi_a\partial_j\Phi_b + \mathcal{J}_{abc}^{ij}\Phi_a\partial_i\Phi_b\partial_j\Phi_c + \mathcal{J}_{abcd}^{ij}\Phi_a\Phi_b\partial_i\Phi_c\partial_j\Phi_d\Big].$$

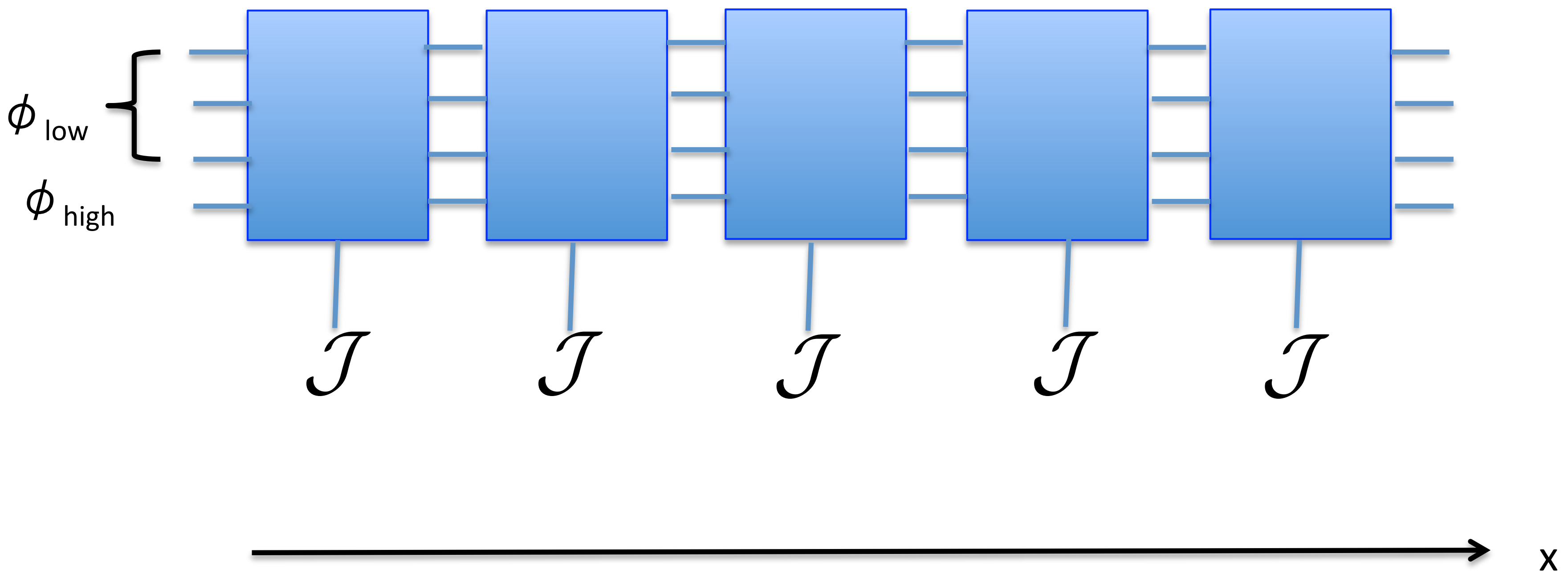
S_M : quadratic action with UV cut-off M

$S_{\mathcal{J}}$: deformation with sources

$\mathcal{J}_{ab..}$: flavor dependent sources (fully symmetric)

Construction of holographic theory

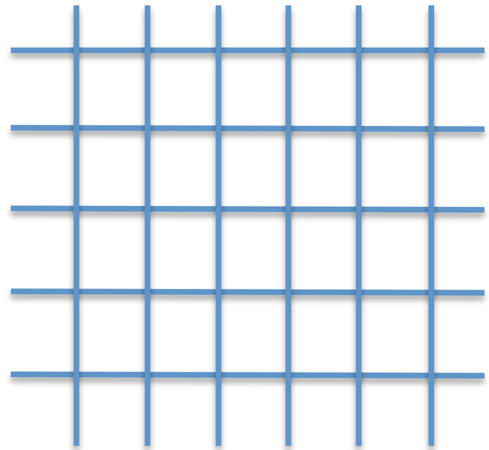
Partition function can be viewed as contractions of an D-dimensional array of matrices which depend on external sources

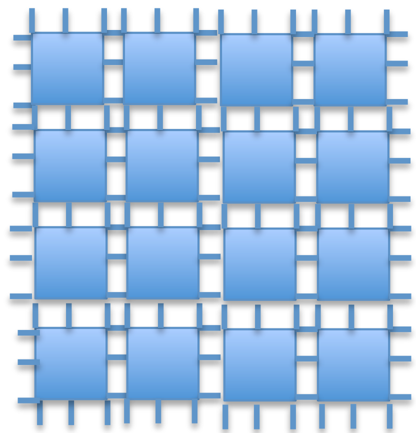


Construction of holographic theory

$$Z[\mathcal{J}] = \int D\phi(x)\ e^{-S[\phi] - \int \mathcal{J}(x)\phi(x)}$$

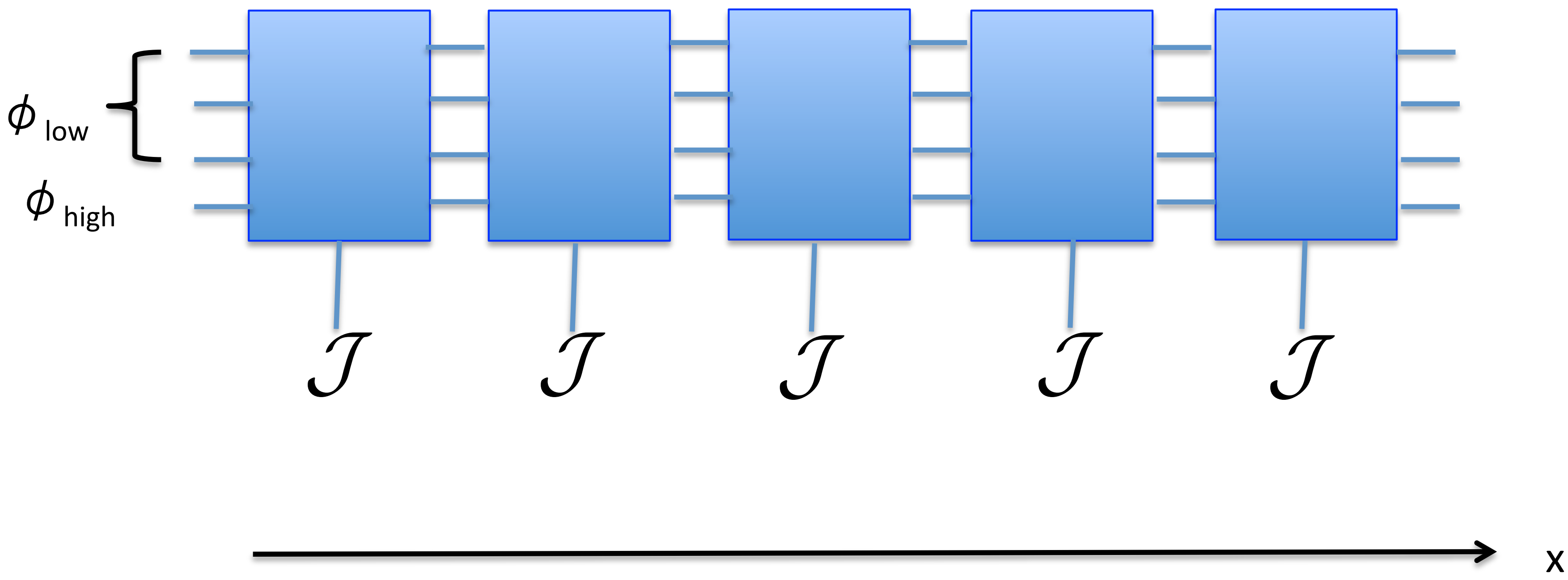
- Divide space into small boxes
- Integrate out fields interior of the boxes
- Write down the partition function as functional integrals of fields on the boundaries of the boxes
- This can be viewed as contractions of an D-dimensional array of matrices





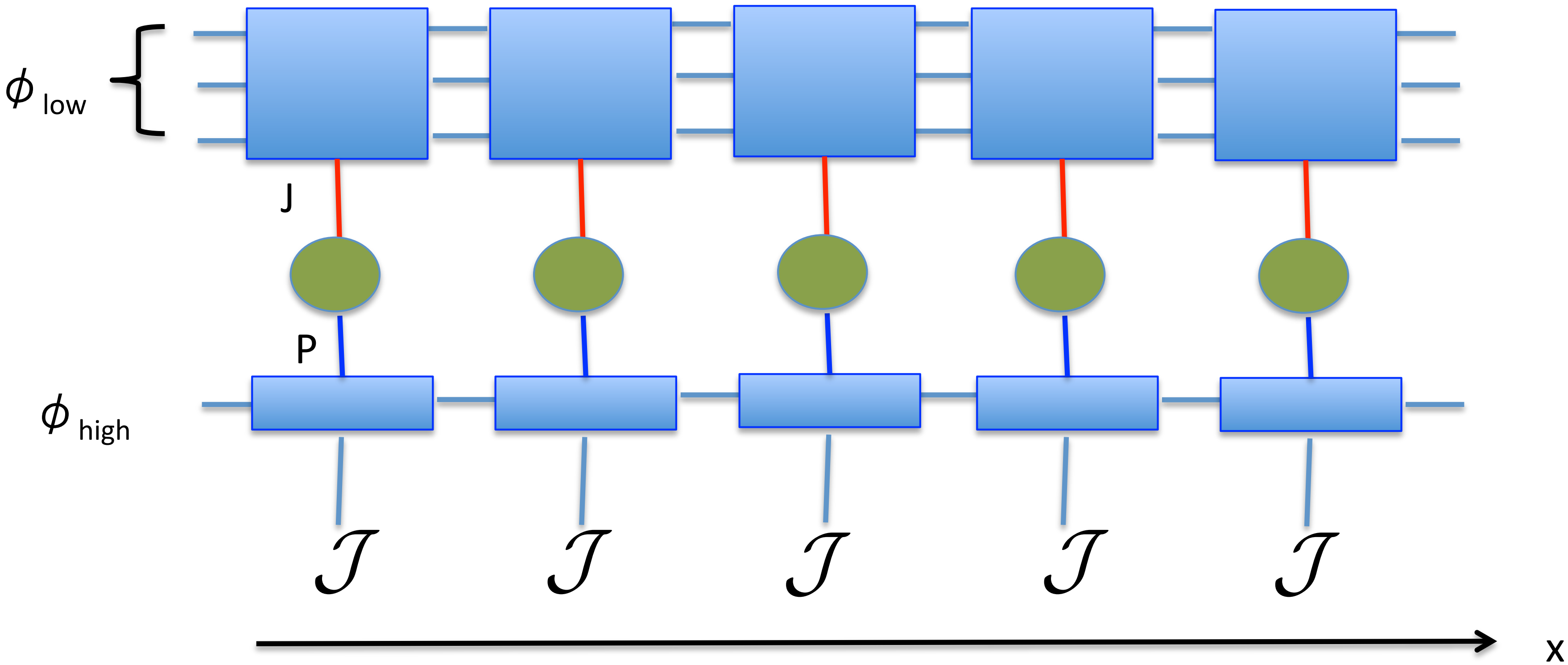
Construction of holographic theory

Partition function can be viewed as contractions of an D-dimensional array of matrices which depend on external sources



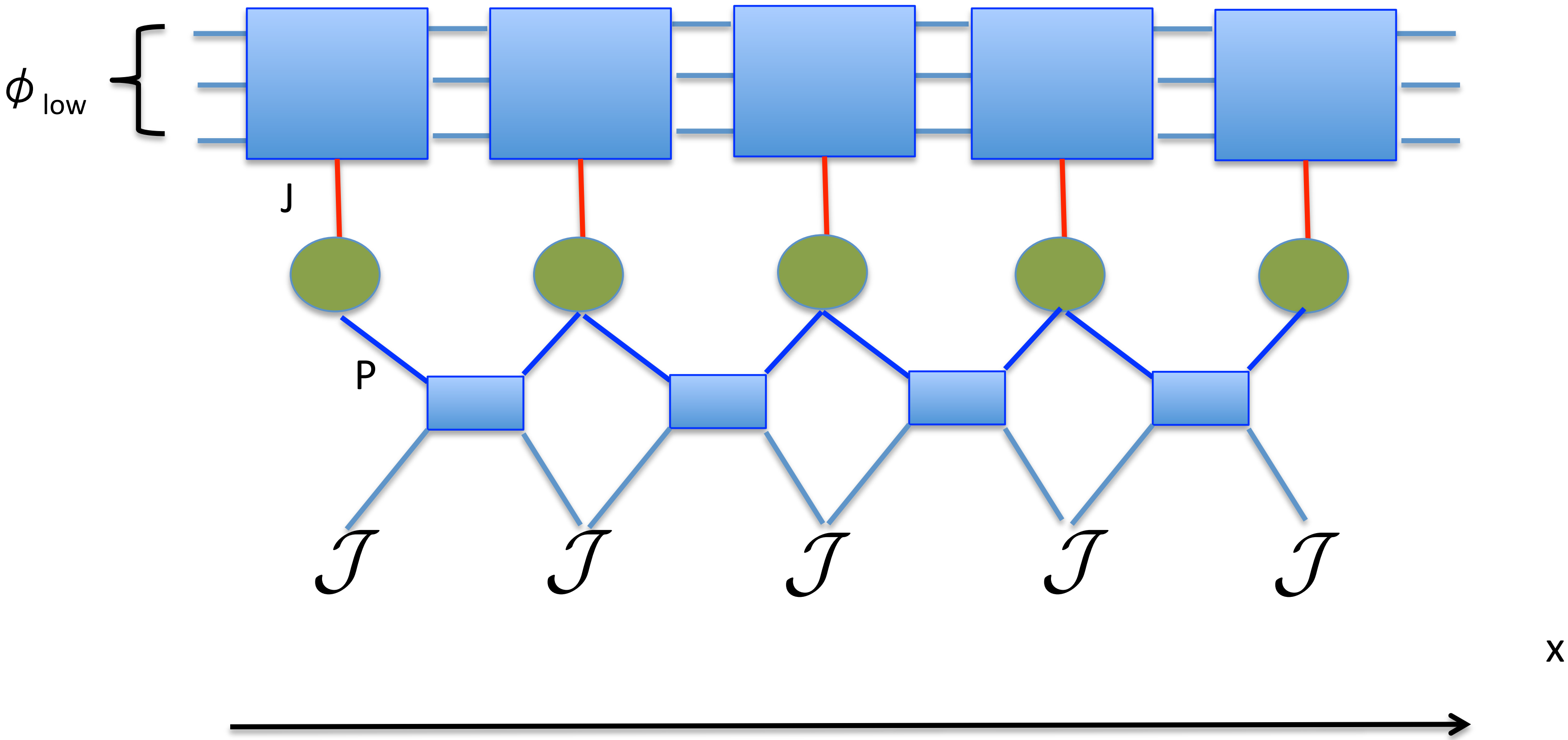
Construction of holographic theory

- High energy fields can be viewed as fluctuating sources for the low energy fields



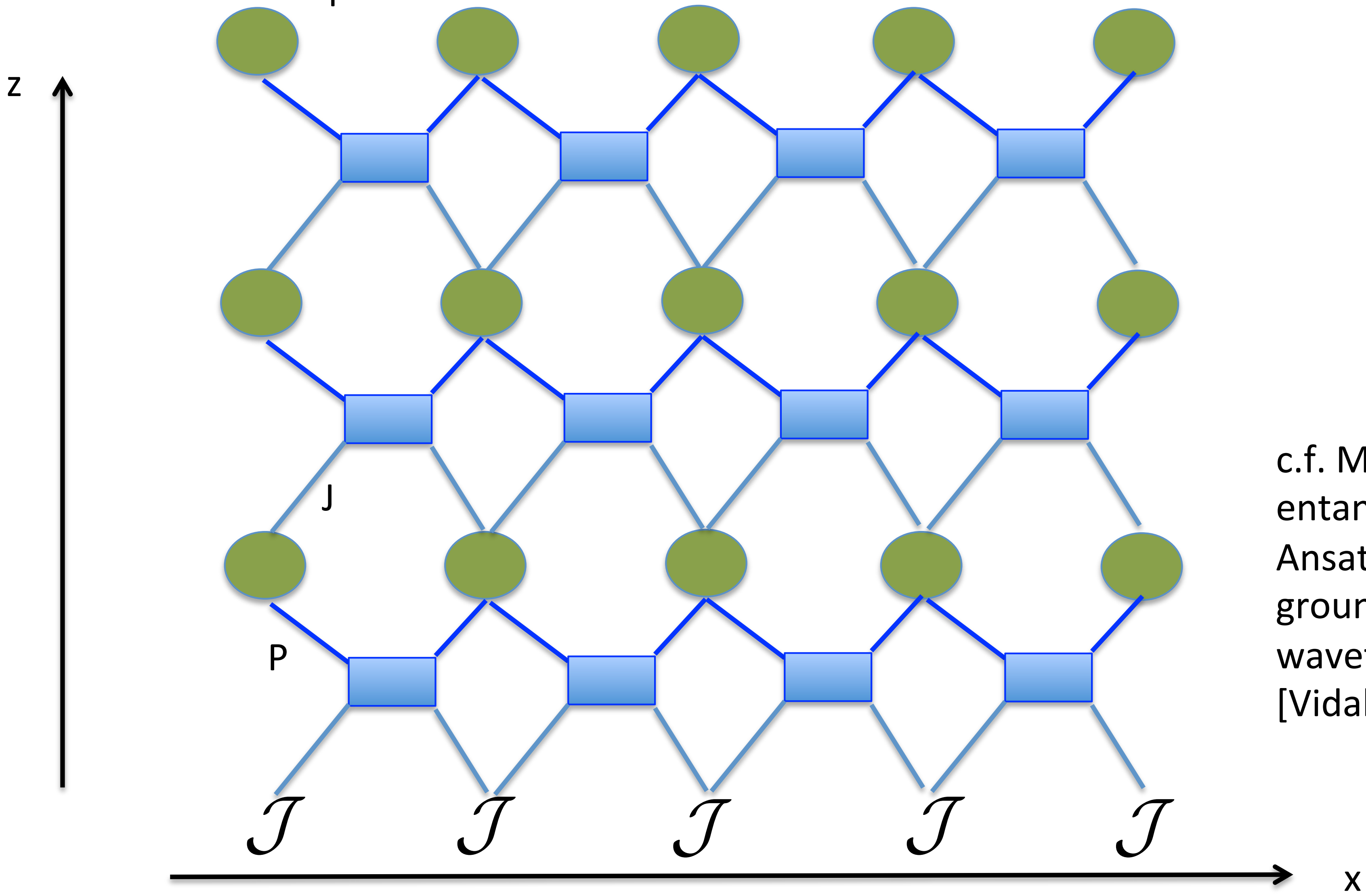
Construction of holographic theory

Integrating out high energy modes generate dynamical action for J, P



Construction of holographic theory

Repetition of these step leads to contractions of (D+1)-dimensional array of matrices for the partition function



c.f. Multi-scale entanglement Ansatz for ground state wavefunction [Vidal]

Dual theory to the O(N) vector model

$$Z[J] \quad = \quad \int DJDP\ e^{-S[J,P]},$$

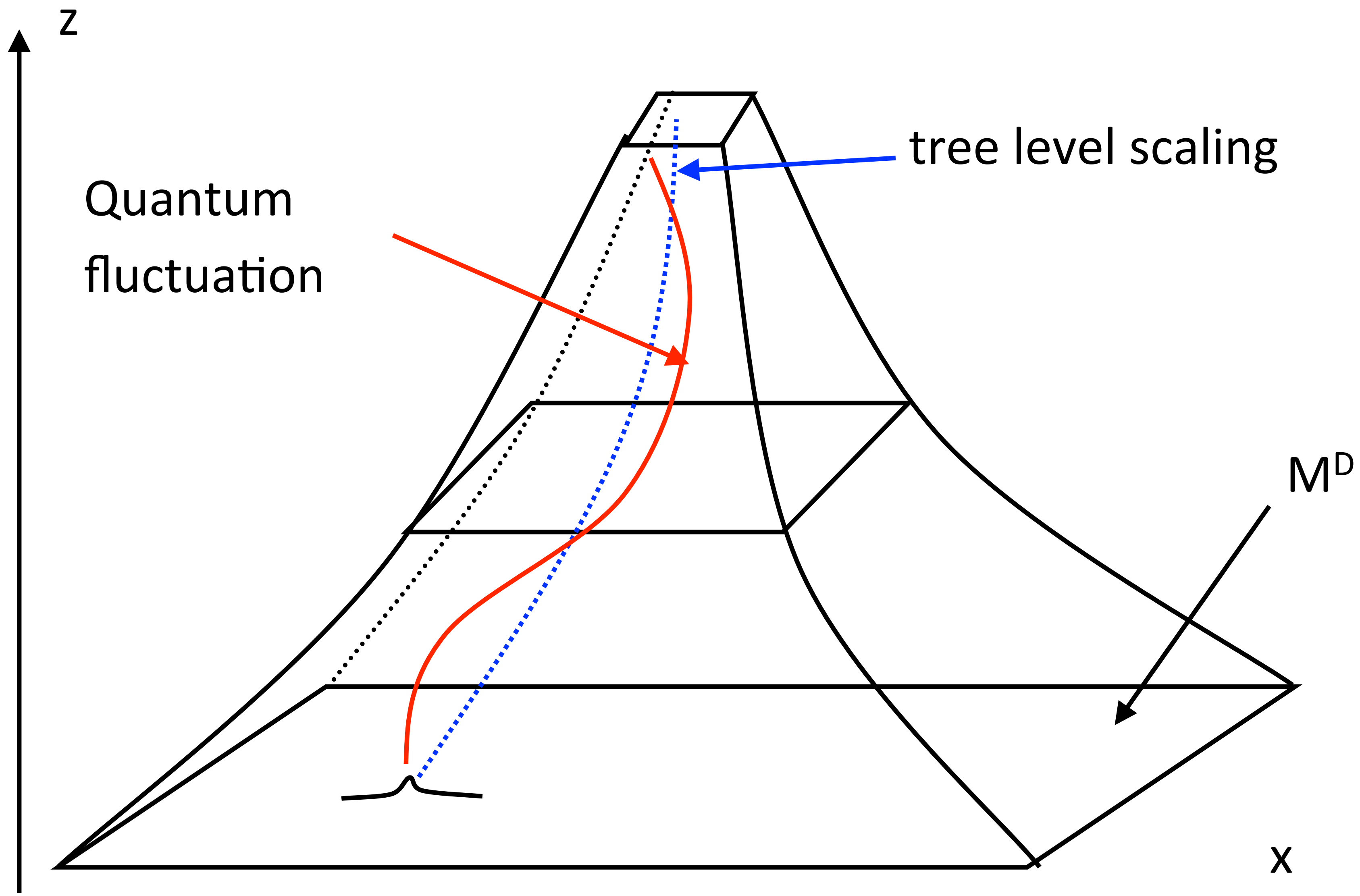
$$\begin{aligned}
S[J,P] \quad = \int dxdz \quad & \{iP_a(\partial J_a - \frac{2+D}{2}\alpha J_a) + iP_{ab}(\partial J_{ab} - 2\alpha J_{ab}) + iP_{ab,ij}(\partial J^{ij}_{ab}) \\
& + iP_{abc}(\partial J_{abc} - \frac{6-D}{2}\alpha J_{abc}) + iP_{abc,ij}(\partial J^{ij}_{abc} - \frac{2-d}{2}\alpha J^{ij}_{abc}) \\
& + iP_{abcd}(\partial J_{abcd} - (4-D)\alpha J_{abcd}) + iP_{abcd,ij}(\partial J^{ij}_{abcd} - (2-d)\alpha J^{ij}_{abcd}) \\
+\frac{1}{4}\int dxdydz \quad & \{\alpha M s_a(x)\partial_M G_M(x-y)s_a(y).\},
\end{aligned}$$

$$\begin{aligned}
s_a \quad = \quad & [iJ_a + 2P_bJ_{ab} - 2\partial_j(J^{ij}_{ab}\partial_i P_b) + 3P_{bc}J_{abc} - \partial_j(J^{ij}_{abc}\partial_i P_{bc}) \\
& + P_{bc,ij}J^{ij}_{abc} + 4P_{bcd}J_{abcd} - \frac{2}{3}\partial_j(J^{ij}_{abcd}\partial_i P_{bcd}) + 2J^{ij}_{abcd}P_{bcd,ij}]
\end{aligned}$$

$$\partial = \partial_z - \alpha x_\mu \frac{\partial}{\partial x_\mu}$$

- (D+1)- dimensional local quantum theory
- Path integral of J and P on M^D *[0,∞)
- Boundary value of J fixed by the bare couplings

(D+1)-dimensional Bulk Space



Large N limit

$$\begin{aligned}
J_{ab}(z) &= J_2(z)\delta_{ab} + \frac{1}{N}\bar{J}_{ab}(z), \\
J_{abc}(z) &= \frac{1}{N}\left[J_{3a}(z)\delta_{bc} + J_{3b}(z)\delta_{ac} + J_{3c}(z)\delta_{ab}\right] + \frac{1}{N^2}\bar{J}_{abc}(z), \\
J_{abcd}(z) &= \frac{J_4(z)}{N}(\delta_{ab}\delta_{cd} + \delta_{ac}\delta_{bd} + \delta_{ad}\delta_{bc}), \\
P_{ab}(z) &= P_2(z)\delta_{ab} + \bar{P}_{ab}(z), \\
P_{abc}(z) &= P_{3a}(z)\delta_{bc} + P_{3b}(z)\delta_{ac} + P_{3c}(z)\delta_{ab} + \bar{P}_{abc}(z), \\
P_{abcd}(z) &= P_4(z)(\delta_{ab}\delta_{cd} + \delta_{ac}\delta_{bd} + \delta_{ad}\delta_{bc}), \\
J^{ij}_{ab}(z) &= J^{ij}_2(z)\delta_{ab} + \frac{1}{N}\bar{J}^{ij}_{ab}(z), \\
J^{ij}_{abc}(z) &= \frac{1}{N}\left[J^{ij}_{3a}(z)\delta_{bc} + J^{ij}_{3b}(z)\delta_{ac} + J^{ij}_{3c}(z)\delta_{ab}\right] + \frac{1}{N^2}\bar{J}^{ij}_{abc}(z), \\
J^{ij}_{abcd}(z) &= \frac{J^{ij}_4(z)}{N}(\delta_{ab}\delta_{cd} + \delta_{ac}\delta_{bd} + \delta_{ad}\delta_{bc}), \\
P_{ab,ij}(z) &= P_{2,ij}(z)\delta_{ab} + \bar{P}_{ab,ij}(z), \\
P_{abc,ij}(z) &= P_{3a,ij}(z)\delta_{bc} + P_{3b,ij}(z)\delta_{ac} + P_{3c,ij}(z)\delta_{ab} + \bar{P}_{abc,ij}(z), \\
P_{abcd,ij}(z) &= P_{4,ij}(z)(\delta_{ab}\delta_{cd} + \delta_{ac}\delta_{bd} + \delta_{ad}\delta_{bc})
\end{aligned}$$

We consider the limit where N goes to infinity with fixed J_a, J_2, $\bar{J}_{ab}$, J_{3a}, $\bar{J}_{abc}$, J_4, J^{ij}_2, $\bar{J}^{ij}_{ab}$, J^{ij}_{3a}, $\bar{J}^{ij}_{abc}$, J_4, J^{ij}_4.

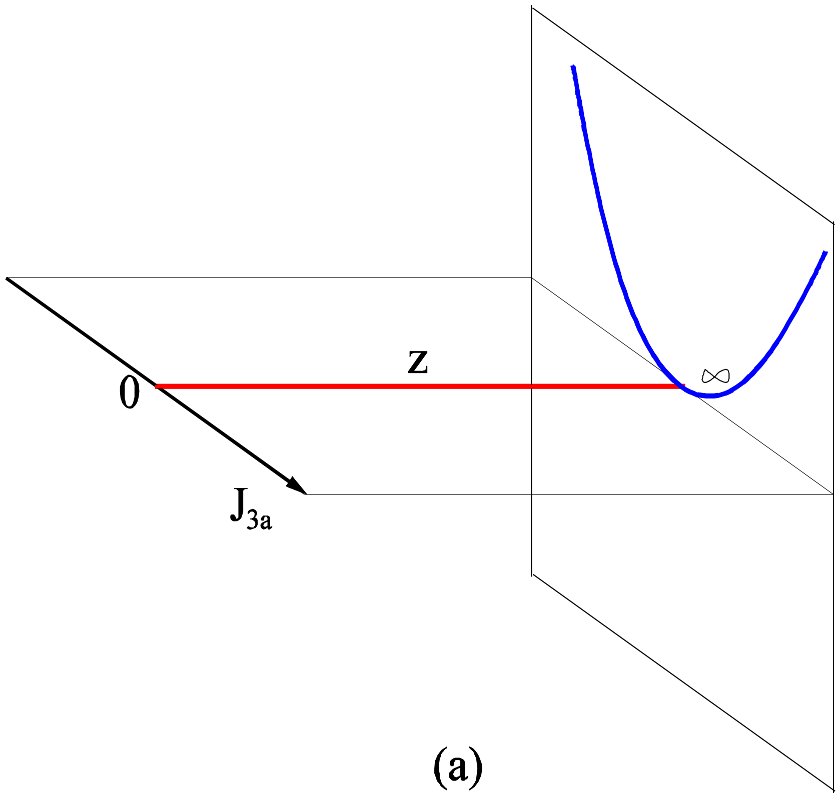
Large N limit (cont'd)

- Action is proportional to N
- Singlet fields become classical in the large N limit
- Quantum fluctuations of non-singlet fields are still important
- Holographic dual is not very useful : one still needs to functional integration
- Integrating out non-singlet fields in the bulk introduce non-local terms for singlet fields : reduces to higher spin gauge theory [Vasiliev; Klebanov-Polyakov] (?)

Spontaneous symmetry breaking

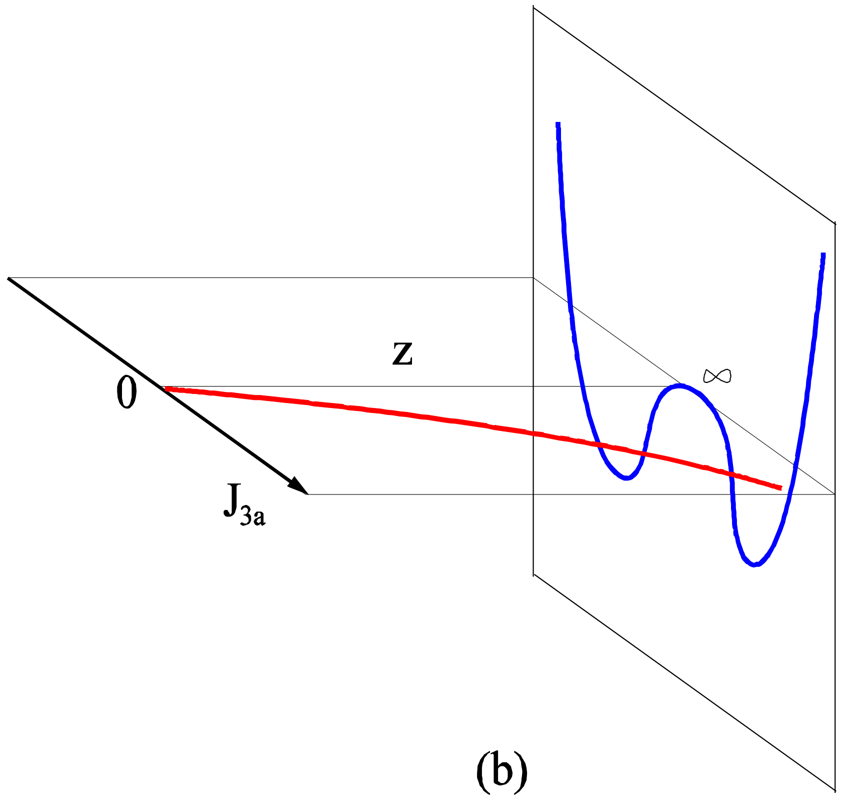
- Non-singlet conjugate field spontaneously develop expectation value at IR

$P_{3a} \sim <\phi^2\phi_a> = 0$



(a)

$P_{3a} \sim <\phi^2\phi_a> \neq 0$



(b)

Critical exponent

Turn on x-dependent sources at UV boundary :

$$\mathcal{J}_2(\mathbf{x}) = \mathcal{J}_2^c + \mathcal{J}_2^{'}(\mathbf{x})$$

Integrate out all bulk fields (both singlets and non-singlets) consistent with UV boundary condition.

EOM :

$$\partial_z \left(e^{-2\alpha z} G^{'-1}(pe^{\alpha z}) \partial_z f_a(\mathbf{p}, z) \right) = 0.$$

$$J_{3a}(\mathbf{x}, z) = \int d\mathbf{p} \quad f_a(\mathbf{p}, z) e^{ipxe^{\alpha z} + (6-D)\alpha z/2}.$$

Critical exponent (cont'd)

Solution consistent with UV B.C. : f_a (p,z=0) = 0 :

$$f_a(\mathbf{p}, z) \;=\; y_a(\mathbf{p}) \left[1 - \frac{K\left(pe^{\alpha z}/M\right)}{K\left(p/M\right)}\right]$$

One arbitrary parameter should be fixed by additional boundary condition.

IR boundary condition is imposed dynamically by the boundary action (which is φ^4-theory itself) :

$$\begin{aligned} S \;=\; & \frac{1}{16\mathcal{J}_4^2}\int d\mathbf{p} \;\; p^2 K^{-1}(p/M)|y_a(\mathbf{p})|^2 \\ & + \frac{1}{16\mathcal{J}_4^2}\int d\mathbf{p}_1 d\mathbf{p}_2 \;\; \left[\mathcal{J}_2^c \delta(\mathbf{p}_1+\mathbf{p}_2) + \mathcal{J}_2^{'}(-\mathbf{p}_1-\mathbf{p}_2)\right] y_a(\mathbf{p}_1) y_a(\mathbf{p}_2) \\ & + \frac{3}{256 N \mathcal{J}_4^3}\int d\mathbf{p}_1 d\mathbf{p}_2 d\mathbf{p}_3 d\mathbf{p}_4 \;\; \delta(\mathbf{p}_1+\mathbf{p}_2+\mathbf{p}_3+\mathbf{p}_4) y_a(\mathbf{p}_1) y_a(\mathbf{p}_2) y_a(\mathbf{p}_3) y_a(\mathbf{p}_4). \end{aligned}$$

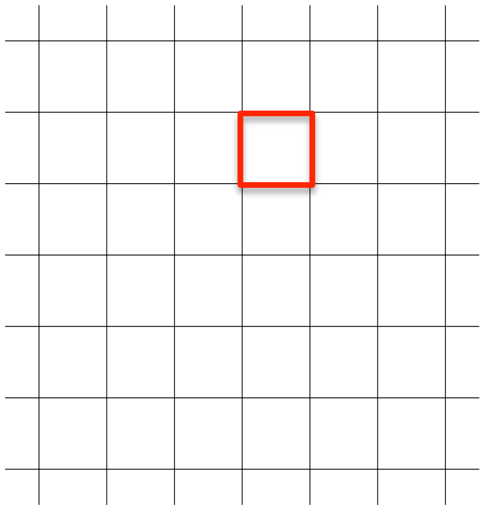
$[\phi^2]$=2 (D=3) consistent with field theory prediction

3. U(N) gauge theory

U(N) lattice gauge theory

$$S[U, \mathcal{J}] = -N\mathcal{J}\sum_{\mathrm{C}} W_{\mathrm{C}}$$

$$\mathcal{J} = \frac{1}{Ng^2}$$: Inverse of ‘t Hooft couplings

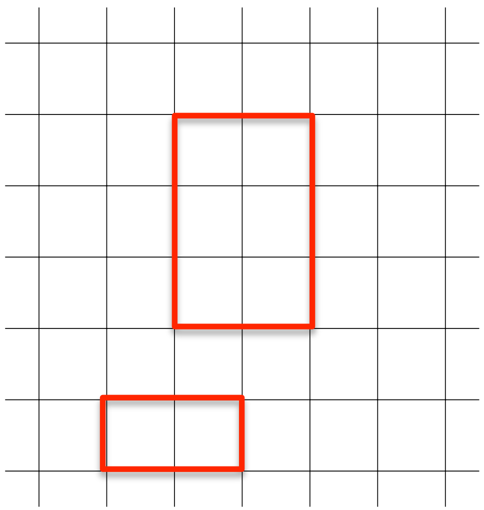


D-dimensional Euclidean hypercubic lattice

$$W_C = \prod_{<i,j>\in C} U_{ij}$$: Wilson loop

U(N) lattice gauge theory

$$S[U, \mathcal{J}] = -N \sum_{C} \mathcal{J}_C W_C$$



$\mathcal{J}_C$: loop-dependent inverse 'tHooft coupling

U(N) lattice gauge theory

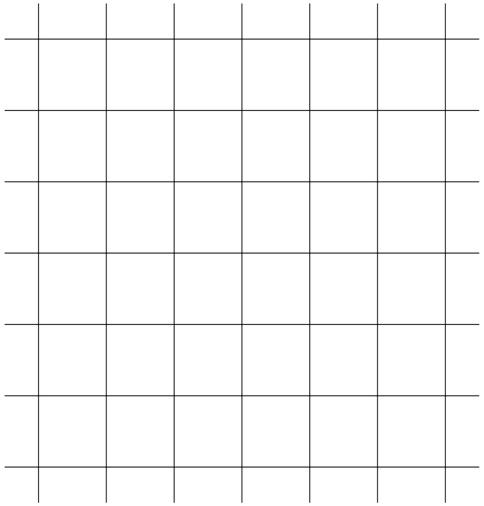
$$S[U;\mathcal{J}] = -\sum_{n=1}^{\infty} \sum_{\{C_1,..,C_n\}} N^{2-n} \mathcal{J}_{\{C_1,..,C_n\}} \prod_{i=1}^{n} W_{C_i}\,.$$

$\mathcal{J}_C$: Sources for single-trace operators

$\mathcal{J}_{C_1,C_2}$: Sources for double-trace operators

$\vdots$

$\mathcal{J}_{\{C_1,..,C_n\}}$: Sources for general multi-trace operators

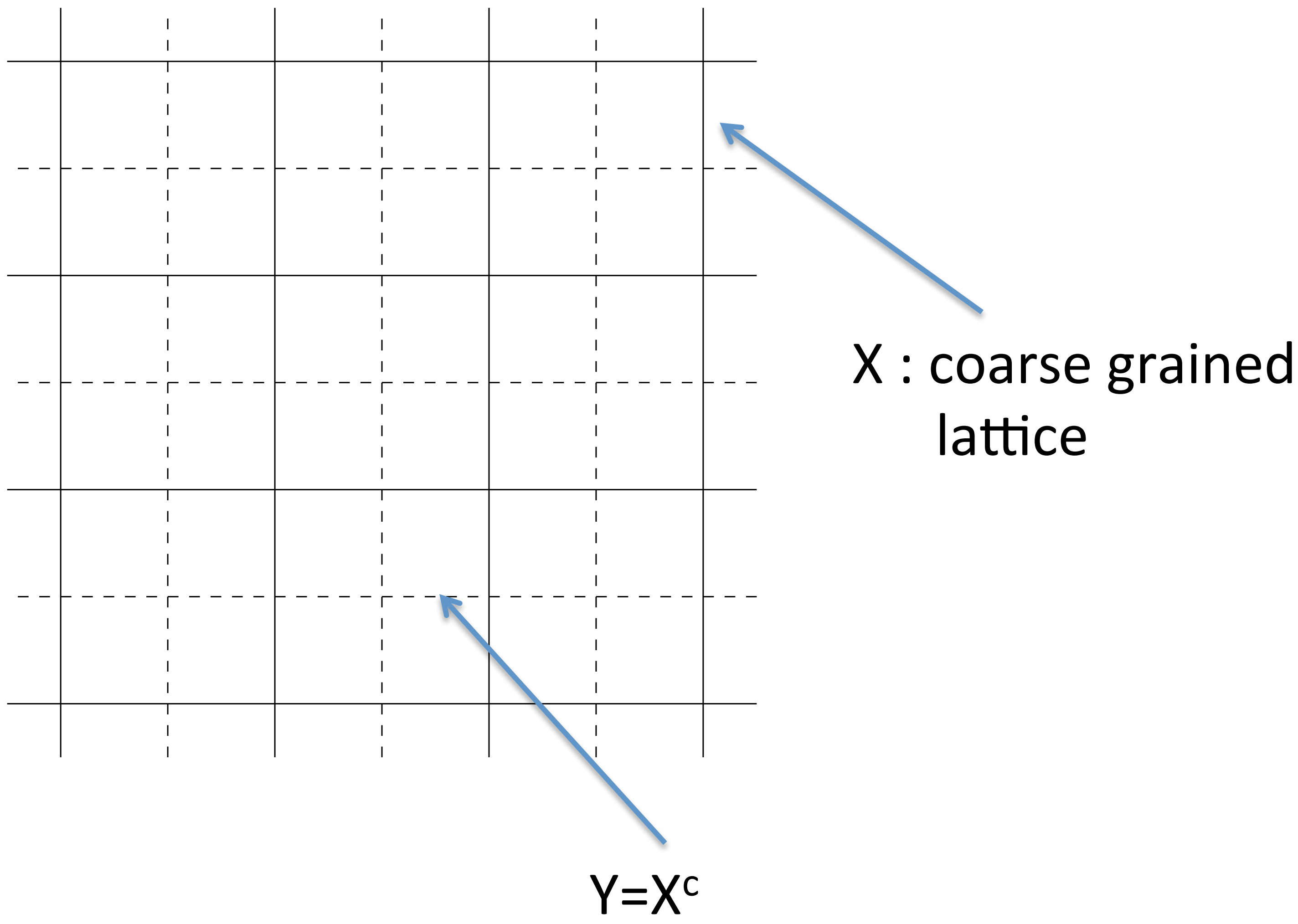


U(N) lattice gauge theory

$$S[U;\mathcal{J}] = -\sum_{n=1}^{\infty} \sum_{\{C_1,..,C_n\}} N^{2-n} \mathcal{J}_{\{C_1,..,C_n\}} \prod_{i=1}^{n} W_{C_i}.$$

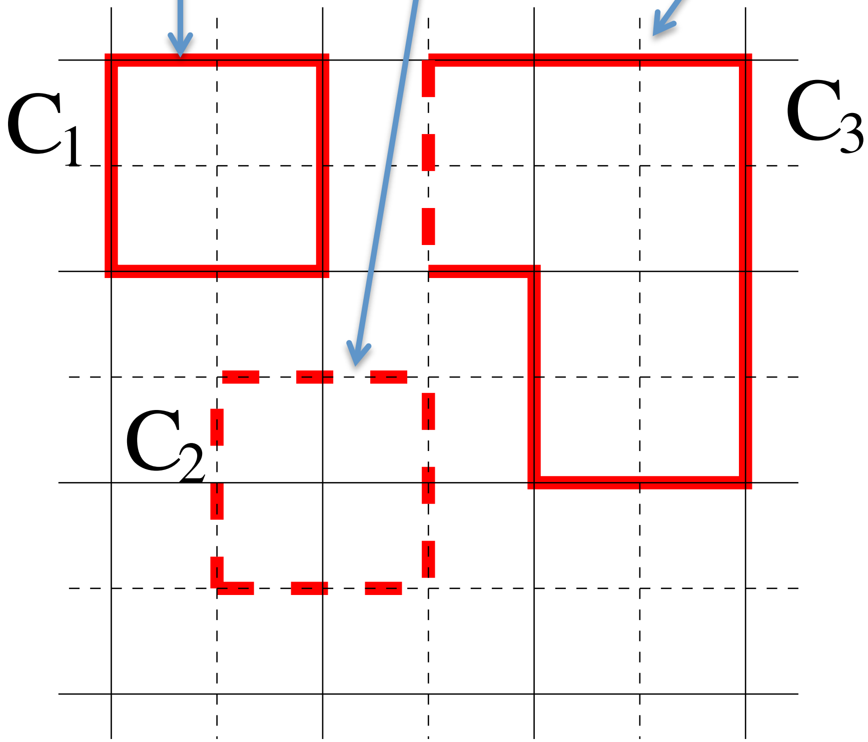
- Large N limit with fixed $\mathcal{J}$: S ~ N^2
- It is expected that there is a classical variable (master field) underlying the theory [Witten, Coleman]
- Classical variable is NOT gluon field when J is small [‘t Hooft]
- It turns out that the classical degrees of freedom are closed loops (strings) living in one higher dimesion

Divide lattice into two sub-lattices



Wilson loop operators

$$S[U,\tilde{U};\mathcal{J}] = S_X[U] + S_Y[\tilde{U}] + S_{XY}[U,\tilde{U}],$$



Gauge invariant coupling

$$Z[\mathcal{J}] = \int dU e^{-S_X[U]} \left\langle e^{-S_{XY}[U,\tilde{U}]} \right\rangle_Y$$

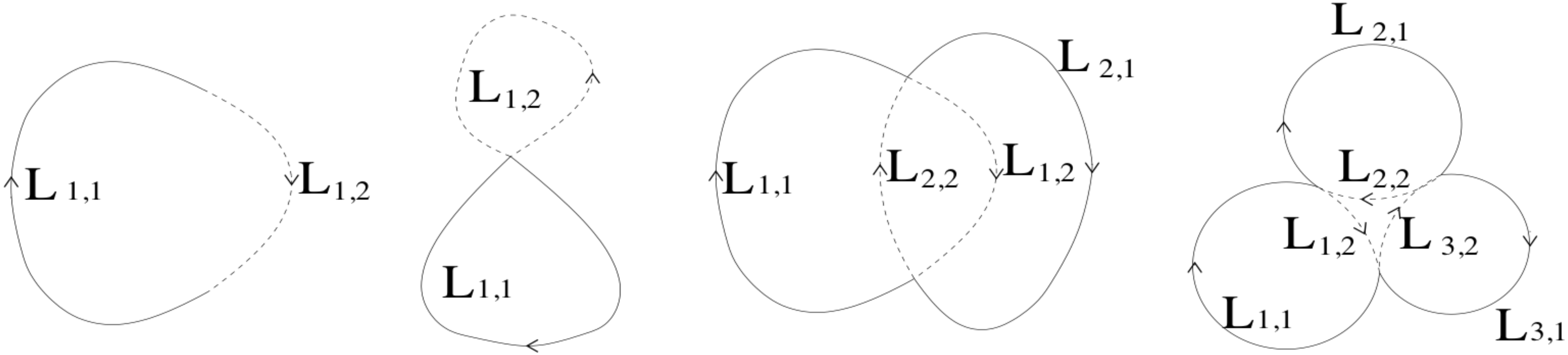
$$\langle O \rangle_Y = \int d\tilde{U} O e^{-S_Y[\tilde{U}]}$$

$$= \int dU e^{-S_X[U]} \left[1 - < S_{XY} >_Y + \frac{1}{2} < S_{XY}^2 >_Y + \ldots \right]$$

- S_X and S_Y are gauge invariant in each sub-lattice
- In $< \ >_{Y,}$ only those Wilson loops that have no open ends in Y survive due to the gauge invariance in sub-lattice Y

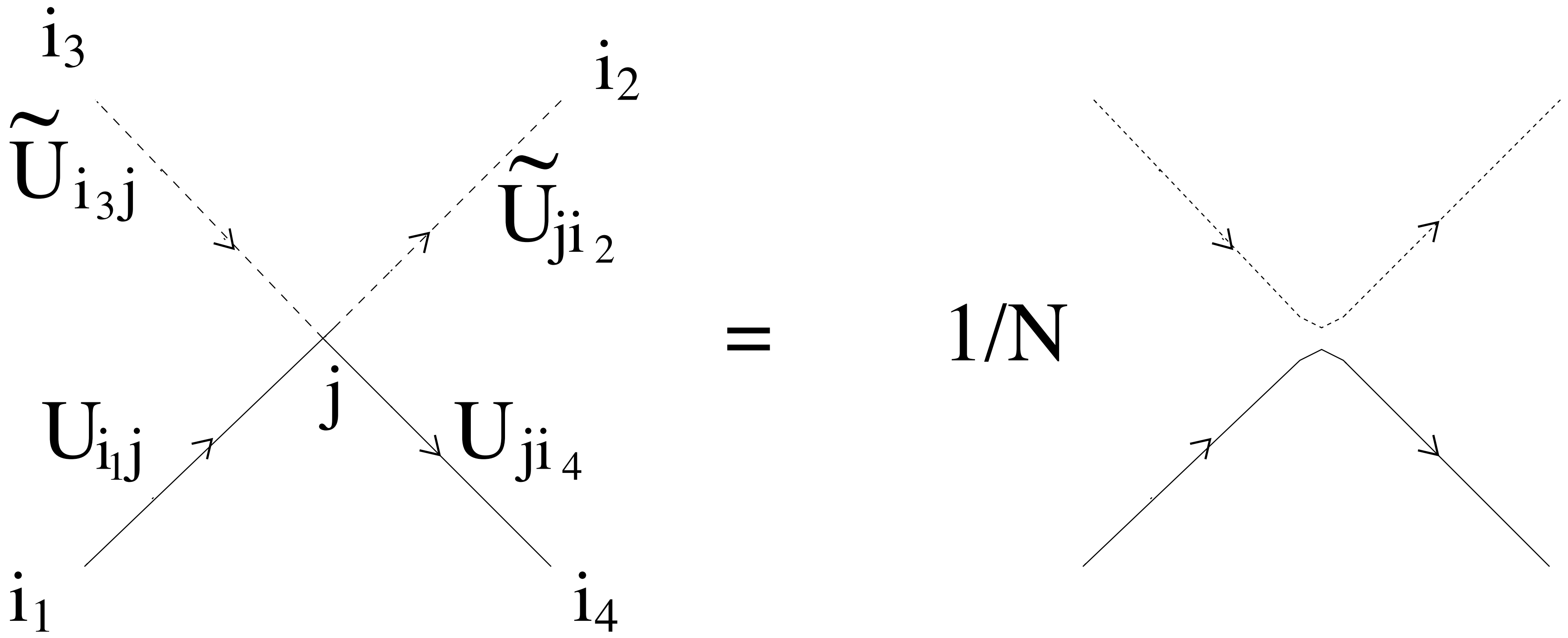
Gauge invariant coupling

$$Z = \int dU e^{-S_X[U]} \left[1 - < S_{XY} >_Y + \frac{1}{2} < S_{XY}^2 >_Y + \ldots \right]$$



(a) (b) (c) (d)

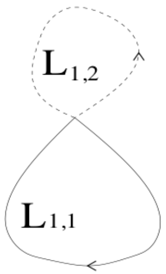
Operation X



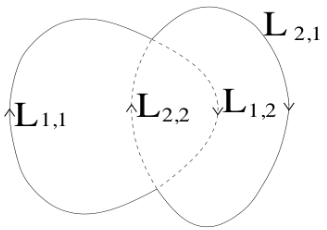
This is true only inside $< \; >_Y$, not as an operator identity

Gauge invariant coupling (cont'd)

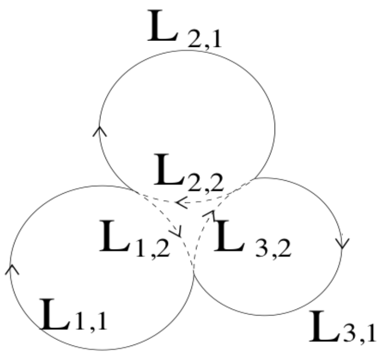
- Non-vanishing contributions in $<e^{-Sxy}>_Y$ can be written as products of Wilson loops in X and Y



$$< \frac{1}{N} W_{L_{1,1}} \tilde{W}_{L_{1,2}} >_y$$



$$< \frac{1}{N^2} W_{L_{1,1}+L_{2,1}} \tilde{W}_{L_{1,2}+L_{2,2}} >_y$$



$$< \frac{1}{N^3} W_{L_{1,1}+L_{2,1}+L_{3,1}} \tilde{W}_{L_{1,2}+L_{3,2}+L_{2,2}} >_y$$

Gauge invariant coupling (cont'd)

- Exponentiate the polynomial of Wilson loops

$$Z[\mathcal{J}] = \int_X dU \int_Y d\tilde{U} e^{-S_Y[\tilde{U}] + \sum_{n=1}^{\infty} \sum_{\{C_1,..,C_n\} \in X} N^{2-n} (\mathcal{J}_{\{C_1,..,C_n\}} + f_{\{C_1,..,C_n\}}[\tilde{W}]) \prod_{i=1}^{n} W_{C_i}}$$

$$f_{\{C_1,..,C_n\}}[\tilde{W}] = \sum_{m=1}^{\infty} N^{-m} \sum_{\{\tilde{C}_1,..,\tilde{C}_m\} \in Y} h_{\{C_1,..,C_n\};\{\tilde{C}_1,..,\tilde{C}_m\}}[\mathcal{J}] \prod_{k=1}^{m} W_{\tilde{C}_k}$$

- The action is polynomial of Wilson loops in X and Y
- This theory can be viewed as a gauge theory on X with dynamical sources : Wilson loops in Y play the role of fluctuating sources for Wilson loops in X
- In the large N limit, Wilson loops are classical : we can introduce collective fields for the dynamical sources

Fields of loops

$$Z[\mathcal{J}] = \int dU\, d\tilde{U}\, dJ\, dP e^{-\left(S_Y[\tilde{U}] + S''[J,P,W,\tilde{W}]\right)}$$

$$S''[J,P,W,\tilde{W}] = iN^2 \sum_{n=1}^{\infty} \sum_{\{C_1,..,C_n\}\in X} P_{\{C_1,..,C_n\}} \left(J_{\{C_1,..,C_n\}} - \mathcal{J}_{\{C_1,..,C_n\}} - f_{\{C_1,..,C_n\}}[\tilde{W}]\right)$$

$$-\sum_{n=1}^{\infty} \sum_{\{C_1,..,C_n\}\in X} N^{2-n} J_{\{C_1,..,C_n\}} \prod_{i=1}^{n} W_{C_i}$$

- J_C : dynamical sources for Wilson loops in X
- P_C : dynamical vev of Wilson loops :

$$i < P_{C_1,\ldots,C_n} >= \frac{1}{N^n} \left\langle \prod_i W_{C_i} \right\rangle$$

Fields of loops

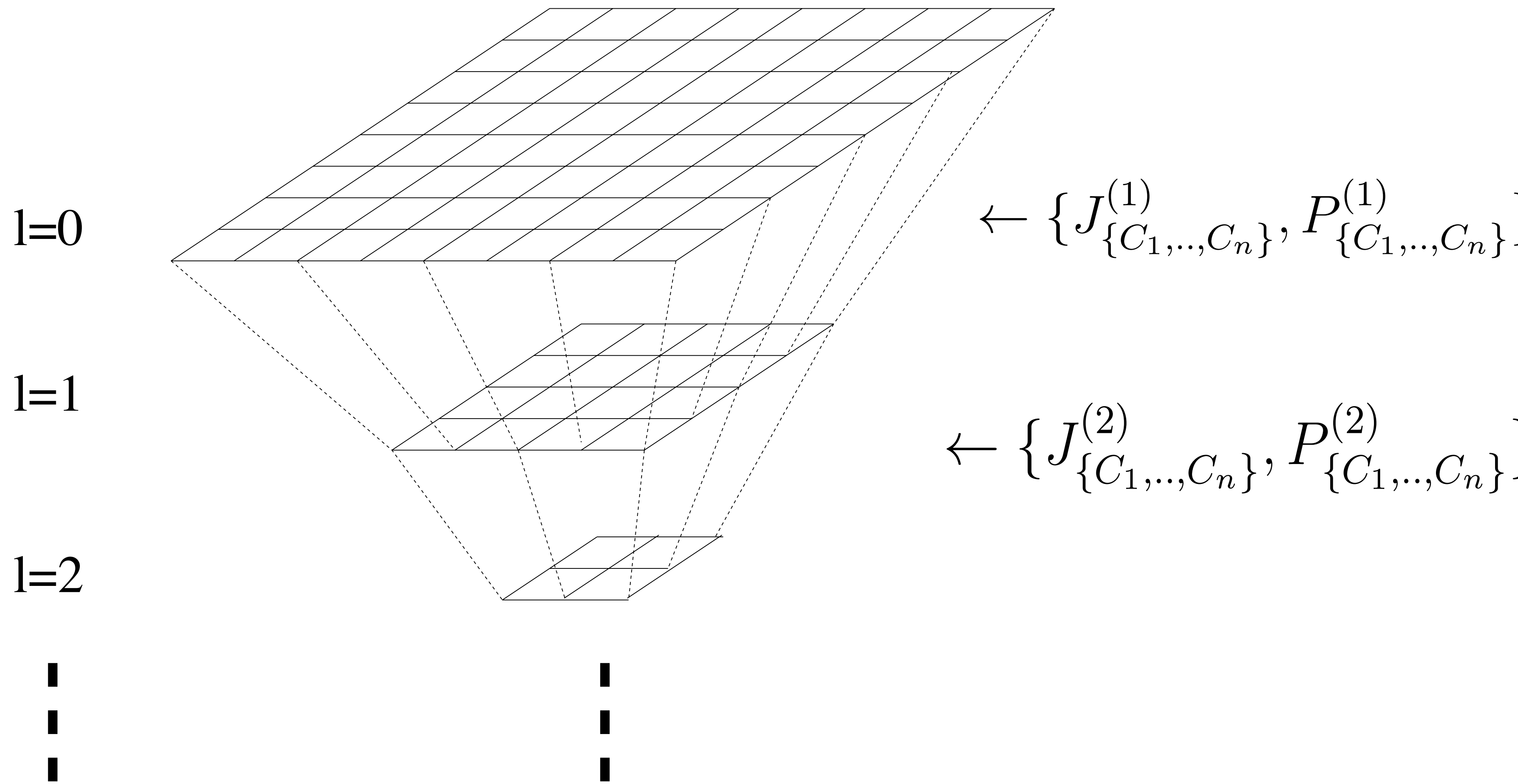
$$Z[\mathcal{J}] = \int dU dJ dP e^{-S}$$

$$S = iN^2 \sum_{n=1}^{\infty} \sum_{\{C_1,...,C_n\} \in X} P_{\{C_1,..,C_n\}}(J_{\{C_1,..,C_n\}} - \mathcal{J}_{\{C_1,..,C_n\}})$$

$$- \sum_{n=1}^{\infty} \sum_{\{C_1,...,C_n\} \in X} N^{2-n} J_{\{C_1,..,C_n\}} \prod_i W_{C_i} + N^2 \mathcal{H}[\mathcal{J}, P]$$

- J_C, P_C acquires dynamics once gauge field in Y is integrated out

(D+1)-dim Lattice



Lattice Loop Field Theory

$$Z[\mathcal{J}] = \int \prod_{l=1}^{\infty} \left[dJ^{(l)} dP^{(l)} \right] e^{-S_{LLFT}}$$

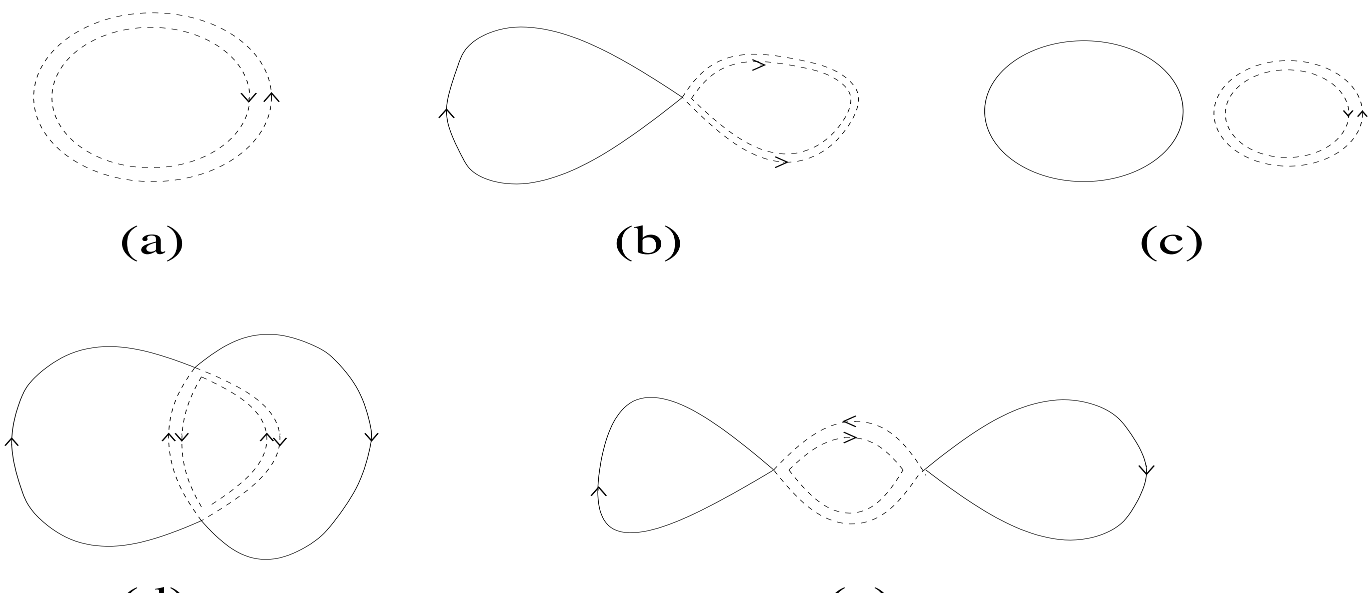
$$S_{LLFT} = N^2 \sum_{l=0}^{\infty} \left[i \sum_{n=1}^{\infty} \sum_{\{C_1,..,C_n\}} P^{(l+1)}_{\{C_1,..,C_n\}} \left(J^{(l+1)}_{\{C_1,..,C_n\}} - J^{(l)}_{\{T[C_1],..,T[C_n]\}} \right) + \mathcal{H}[J^{(l)}, P^{(l+1)}] \right]$$

- (D+1)-dimensional field theory of closed loops
- Classical in the large N limit with fixed 't Hooft coupling
- Bare coupling fixes UV boundary condition

$$J^{(0)}_{\{C_1,..,C_n\}} = \mathcal{J}_{\{C_1,..,C_n\}}$$

Hamiltonian of loop fields

- It is convenient to interpret l as discrete time
- J_C, P_C : annihilation/ creation operator of loops
- H describes various loop propagation/interactions
- H can be perturbatively computed in J (large ‘t Hooft coupling expansion)



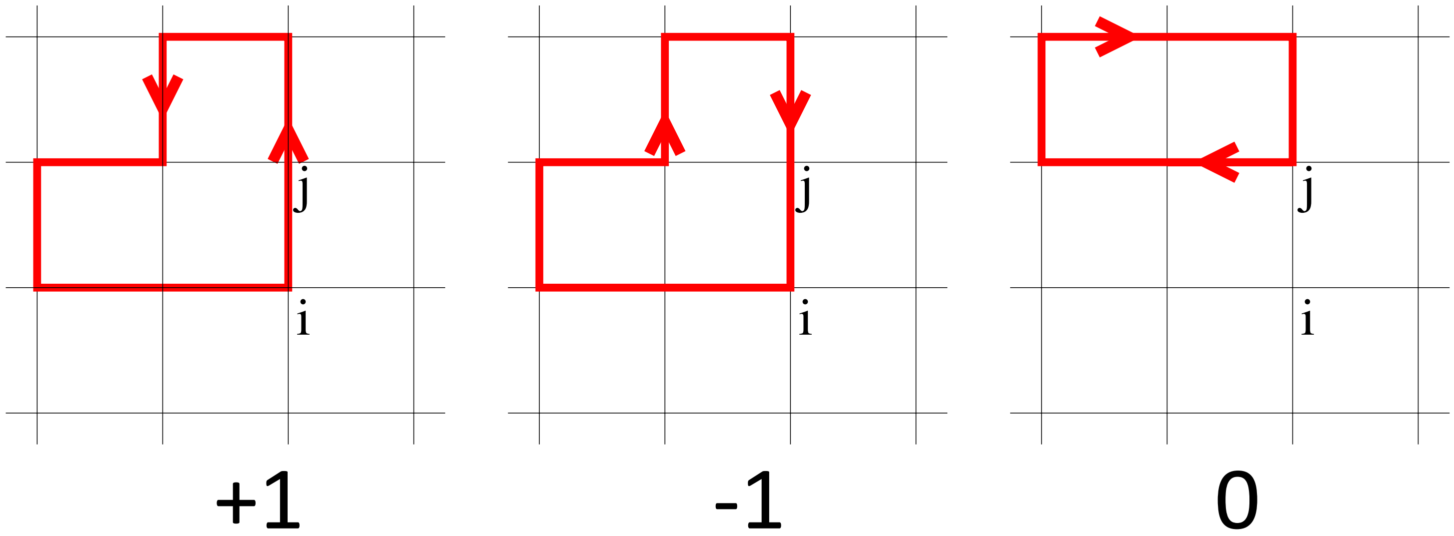
(a) (b) (c) (d) (e)

Hamiltonian

$$\begin{aligned}\mathcal{H}[J,P] = -\Big\{ & J_{\tilde{C}} + iP_{C_1}\left(J_{L_1}\delta_{L_1,T[C_1]+\tilde{C}} + J_{\{T[C_1],\tilde{C}\}} + \frac{1}{2}J_{L_1}J_{L_2}\delta_{L_{1,1}+L_{2,1},T[C_1]}\delta_{L_{1,2}+L_{2,2},\tilde{C}}\right) \\ & + iP_{\{C_1,C_2\}}\Big(J_{L_1}\delta_{L_{1,1},T[C_1]}\delta_{L_{1,2}+L_{1,4},\tilde{C}}\delta_{L_{1,3},T[C_2]} + J_{\{L_1,L_2\}}\delta_{L_{1,1},\tilde{C}}\delta_{L_{1,2},T[C_1]}\delta_{L_2,T[C_2]} \\ & + J_{\{T[C_1],T[C_2],\tilde{C}\}}\Big) + \ldots\Big\} \times \\ \Big\{ & J_{\bar{\tilde{C}}} + iP_{C_3}\left(J_{L_3}\delta_{L_3,T[C_3]+\bar{\tilde{C}}} + J_{\{T[C_3],\bar{\tilde{C}}\}} + \frac{1}{2}J_{L_3}J_{L_4}\delta_{L_{3,1}+L_{4,1},T[C_3]}\delta_{L_{3,2}+L_{4,2},\bar{\tilde{C}}}\right) \\ & + iP_{\{C_3,C_4\}}\Big(J_{L_3}\delta_{L_{3,1},T[C_3]}\delta_{L_{3,2}+L_{3,4},\bar{\tilde{C}}}\delta_{L_{3,3},T[C_4]} + J_{\{L_3,L_4\}}\delta_{L_{3,1},\bar{\tilde{C}}}\delta_{L_{3,2},T[C_3]}\delta_{L_4,T[C_4]} \\ & + J_{\{T[C_3],T[C_4],\bar{\tilde{C}}\}}\Big) + \ldots\Big\} + \ldots, \end{aligned} \tag{34}$$

Symmetry of H

- Infinitely many conserved charges, Q^{ij}



- Broken down to D global charges by UV boundary condition $J^{(0)}_{\{C_1,..,C_n\}} = \mathcal{J}_{\{C_1,..,C_n\}}$

$$Q_\mu = \sum_i Q^{ii+\mu}$$

Saddle point EOM

$$J^{(l+1)}_{\{C_1,..,C_n\}} - J^{(l)}_{\{T[C_1],..,T[C_n]\}} = -\frac{\partial \mathcal{H}[J^{(l)}, P^{(l+1)}]}{\partial \mathcal{P}^{(l+1)}_{\{C_1,..,C_n\}}},$$

$$\mathcal{P}^{(l+1)}_{\{T^{-1}[C_1],..,T^{-1}[C_n]\}} - \mathcal{P}^{(l)}_{\{C_1,..,C_n\}} = \frac{\partial \mathcal{H}[J^{(l)}, P^{(l+1)}]}{\partial J^{(l)}_{\{C_1,..,C_n\}}}.$$

- Saddle point solution becomes exact in the large N limit
- UV boundary condition : $J^{(0)}_{\{C_1,..,C_n\}} = \mathcal{J}_{\{C_1,..,C_n\}}$
- The second boundary condition is imposed dynamically
- Relation to D-dim loop equation [Makeenko, Migdal] ?

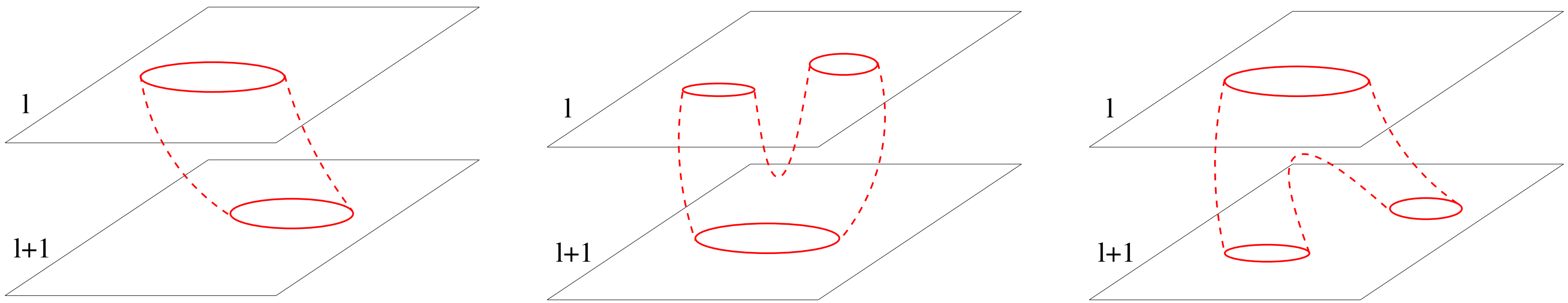
Propagating loop fields

- Fluctuations above vacuum

$$j_{\{C_1,..,C_n\}} = J_{\{C_1,..,C_n\}} - < J_{\{C_1,..,C_n\}} >$$

$$p_{\{C_1,..,C_n\}} = P_{\{C_1,..,C_n\}} - < P_{\{C_1,..,C_n\}} >$$

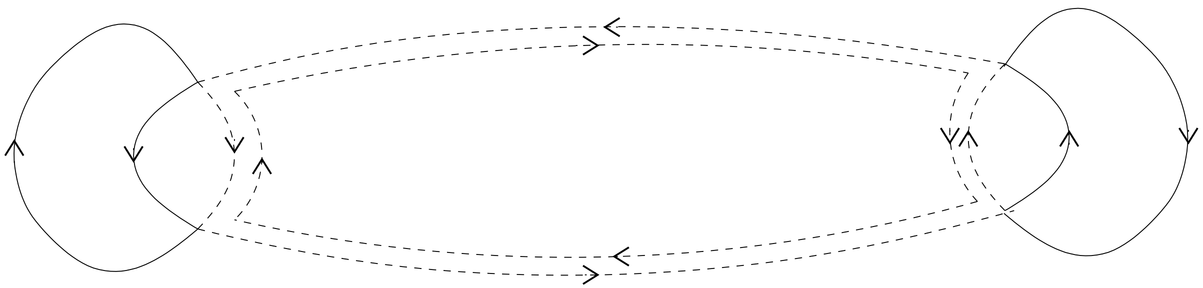
- propagating fields describes small fluctuations of loop fields



(a) (b) (c)

Locality

- Due to the presence of large loops, LLFT is generically non-local



- Large loops become exponentially suppressed in the large 't Hooft coupling limit : a sense of locality emerges

Summary

- General D-dimensional QFT can be formally mapped into (D+1)-dimensional quantum theory for fluctuating source fields and their conjugates
- The holographic dual for O(N) model in general includes non-singlet fields in the bulk, but remains quantum even in the large N limit
- D-dimensional U(N) gauge theory is dual to the (D+1)-dimensional field theory of closed loops which become local and weakly coupled in the large N and large 't Hooft coupling limits