The Abdus Salam International Centre for Theoretical Physics

2234-6

Meeting of Modern Science and School Physics: College for School Teachers of Physics in ICTP

27 April - 3 May, 2011

Method of virtual displacements

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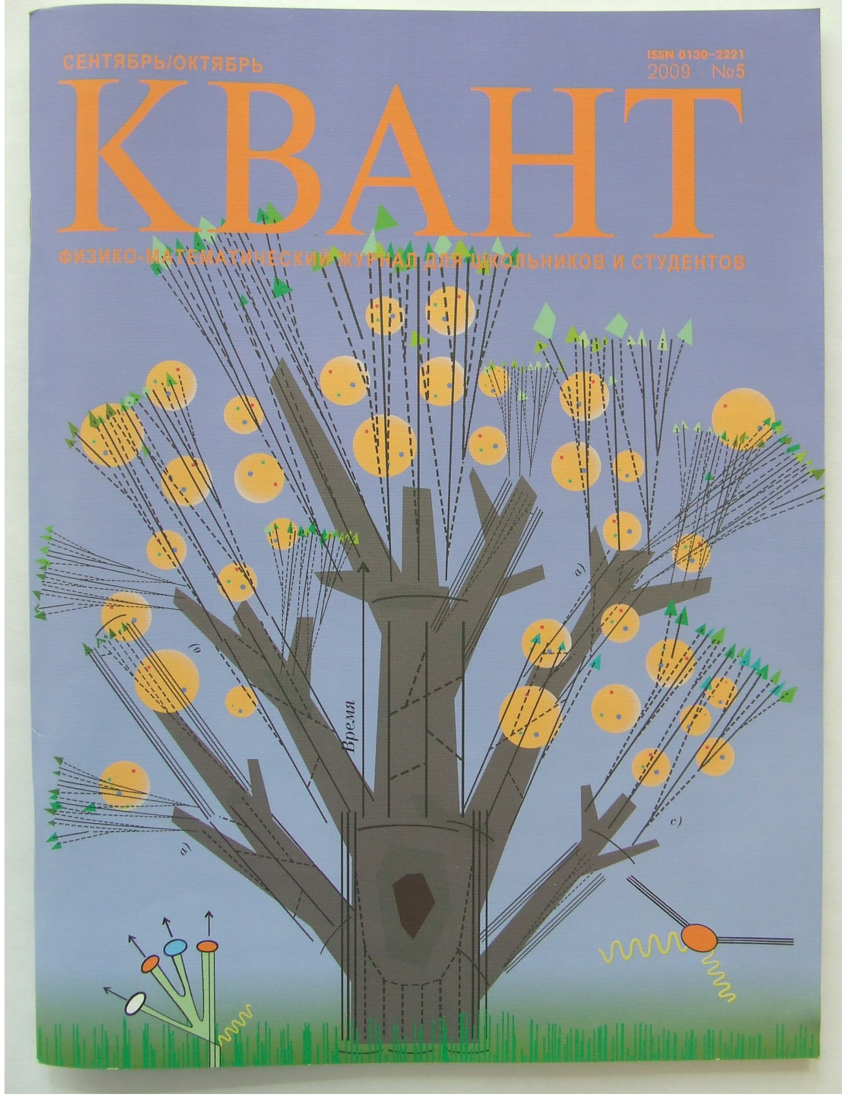
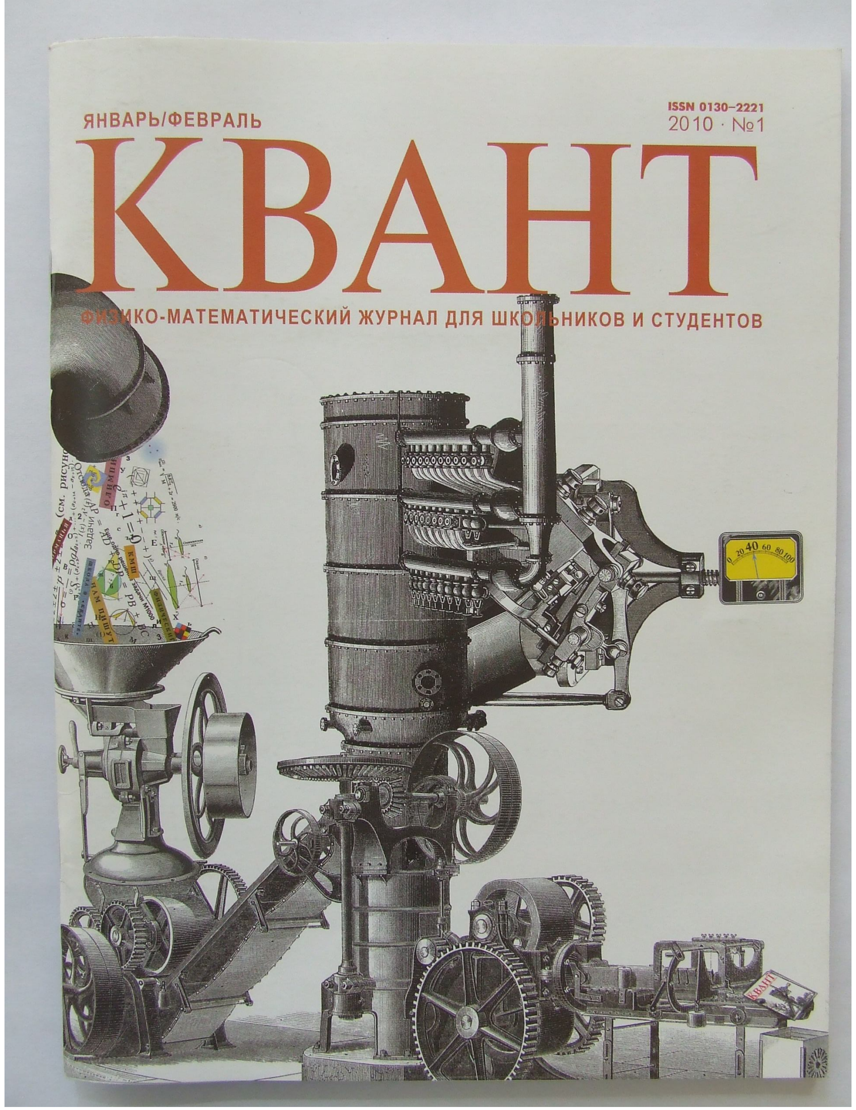
METHOD OF VIRTUAL DISPLACEMENTS


СЕНТЯБРЬ/ОКТЯБРЬ
ISSN 0130–2221
2009 · №5
КВАНТ
ФИЗИКО-МАТЕМАТИЧЕСКИЙ ЖУРНАЛ ДЛЯ ШКОЛЬНИКОВ И СТУДЕНТОВ
Время


ЯНВАРЬ/ФЕВРАЛЬ
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A.B. Migdal

1911 - 1991

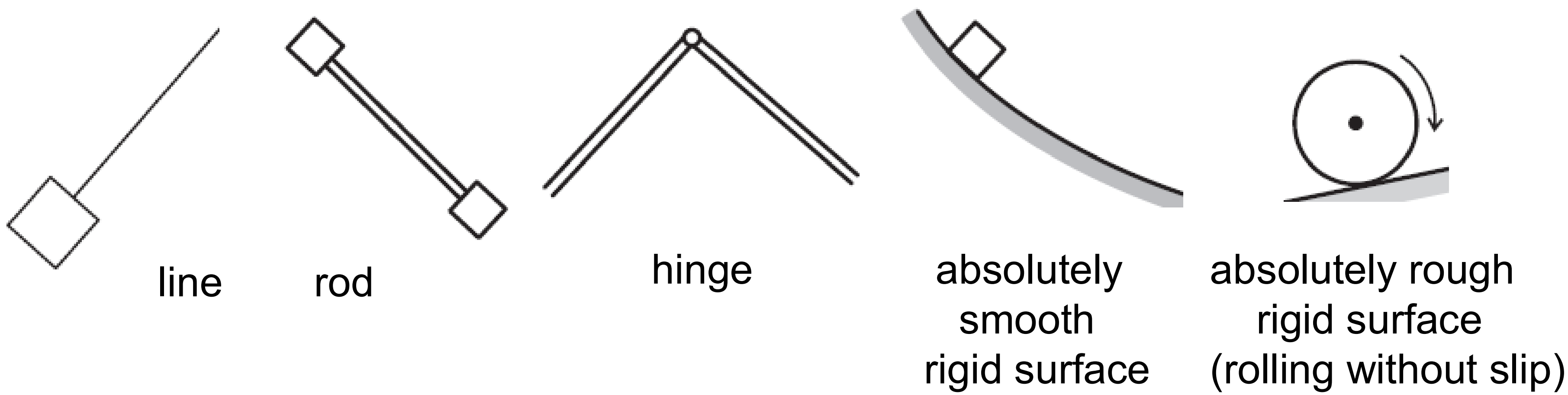
What are virtual displacements?

Any possible displacements, compatible with constraints.

For N point masses and s constraints there are $3N - s$ degrees of freedom, i.e. $3N - s$ independent virtual displacements.

What are ideal constraints?

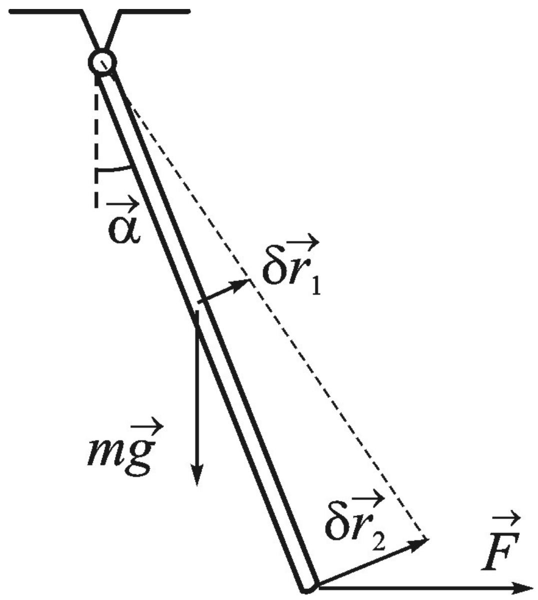
The constraints which don't lead to energy loss, i.e. the work of constraint reactions for any virtual displacements is equal to zero: $\sum \vec{R}_i \delta \vec{r}_i = 0$



Principle of virtual displacements

- For a mechanical system with ideal constraints to be in equilibrium, it is necessary and sufficient that the sum of elementary works of *active forces* $\vec{F}_i$ on any virtual displacement of system be zero
$$\sum \vec{F}_i \delta \vec{r}_i = 0$$
- If some forces are conservative, then their work is equal to $-dE_p$, so
$$\sum \vec{F}_i \delta \vec{r}_i = dE_p$$

Some mechanical problems
Statics 1



The rod of mass *m* is in equilibrium. Find *F.*

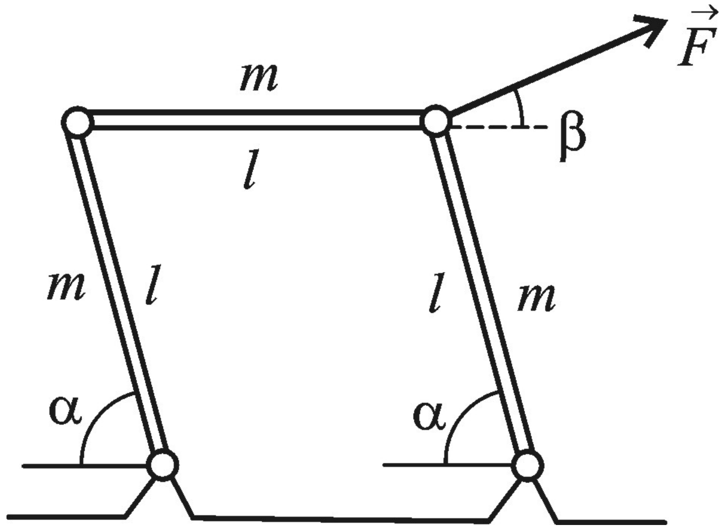
Virtual rotation for small angle. $\delta r_2 = 2\delta r_1$

$$\vec{F}\cdot\delta\vec{r}_2 + m\vec{g}\cdot\delta\vec{r}_1 = 0$$

$$F\cdot 2\delta r_1 \cos\alpha - mg\cdot\delta r_1 \sin\alpha = 0$$

$$\boxed{F = \frac{1}{2} mg\cdot\tan\alpha}$$

Statics 2
a more difficult problem



Find F in equilibrium.

Instead of 5 equations for

$$N_{1x}, N_{2x}, N_{1y}, N_{2y}, F$$

only one equation.

Virtual rotation at small angle.

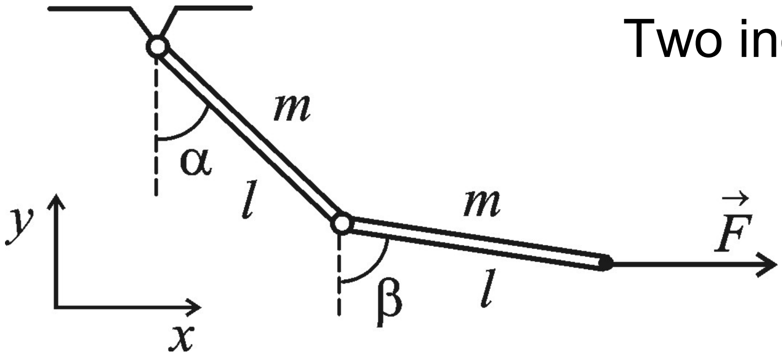
$$F\cos\beta\cdot(l\delta\alpha)\sin\alpha + F\sin\beta\cdot(l\delta\alpha)\cos\alpha -$$

$$-2mg\cdot\left(\frac{l}{2}\delta\alpha\right)\cos\alpha - mg\cdot(l\delta\alpha)\cos\alpha = 0$$

$$\boxed{F = \frac{2mg\cos\alpha}{\sin(\alpha+\beta)}}$$

Statics 3
two degrees of freedom

$\vec{F}$ is horizontal. Find α and β in equilibrium.



Two independent angle displacements: $\delta\alpha$ and $\delta\beta$.

$$-mg\cdot\delta y_1 - mg\cdot\delta y_2 + F\cdot\delta x_3 = 0$$

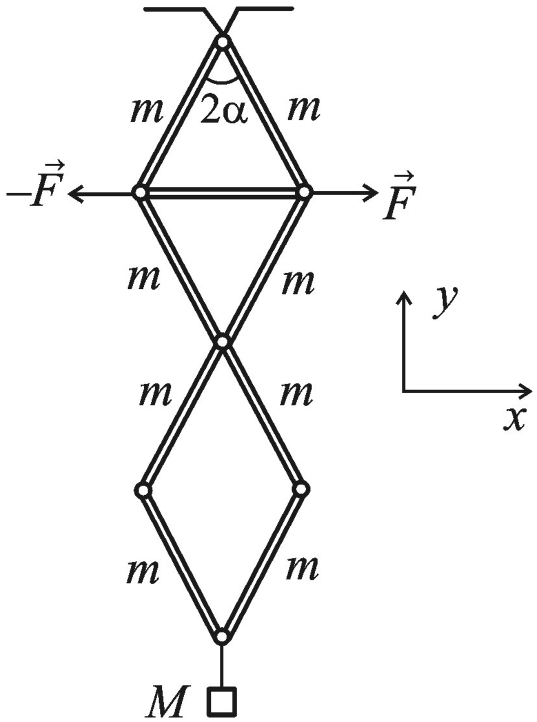
$$\delta y_1 = \frac{l}{2}\delta\alpha\sin\alpha,$$

$$\delta y_2 = l\delta\alpha\sin\alpha + \frac{l}{2}\delta\beta\sin\beta,$$

$$\delta x_3 = l\delta\alpha\cos\alpha + l\delta\beta\cos\beta.$$

$$\delta\alpha\left(Fl\cos\alpha - mg\frac{l}{2}\sin\alpha - mgl\sin\alpha\right) + \delta\beta\left(Fl\cos\beta - mg\frac{l}{2}\sin\beta\right) = 0,$$

$$\boxed{\tan\alpha = \frac{2F}{3mg}} \qquad \boxed{\tan\beta = \frac{2F}{mg}}$$



Statics 4
no degrees of freedom

Find the reaction *F* of horizontal rod.

Substitute the rod by two forces. There is now one degree of freedom.

Virtually increase α by δα.

$$2F\cdot\delta x-2mg\cdot\delta y_1-2mg\cdot\delta y_2-2mg\cdot\delta y_3-2mg\cdot\delta y_4-Mg\cdot\delta y_5=0$$

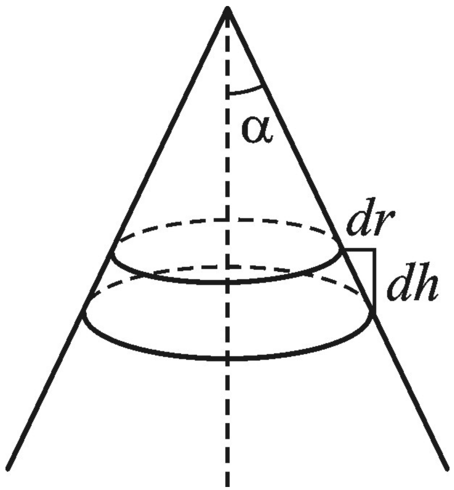
$$\delta x=\left(l\delta\alpha\right)\cos\alpha,\quad \delta y_1=\left(\frac{l}{2}\delta\alpha\right)\sin\alpha,\quad \delta y_2=3\delta y_1,\quad \delta y_3=5\delta y_1,$$

$$\delta y_4=7\delta y_1,\quad \delta y_5=8\delta y_1.$$

$$\boxed{F=\left(8m+2M\right)g\cdot\tan\alpha}$$

Statics 5
no degrees of freedom

Rope of mass *m* on a smooth cone. Find tension *T*.



Virtually increase the rope length by $\delta l = 2\pi\delta r.$

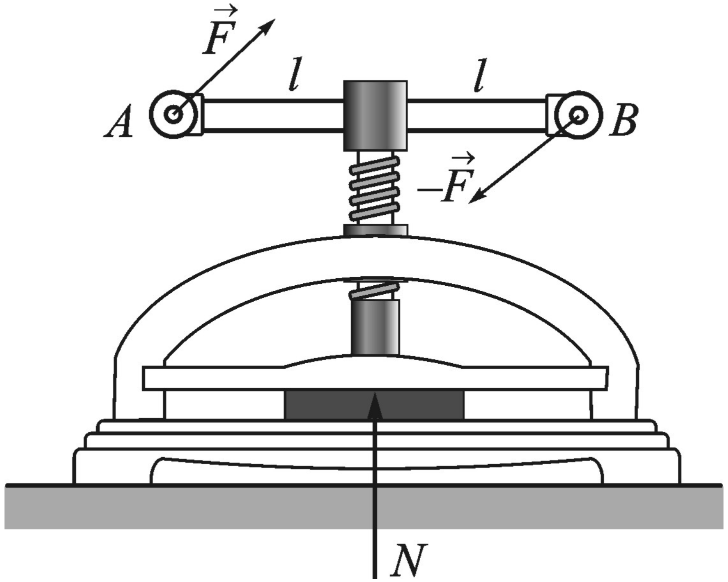
It will descend at $\delta h = \delta r \cot\alpha.$

$$T \cdot \delta l - mg \cdot \delta h = 0;$$

$$T \cdot 2\pi\delta r - mg \cdot \delta r \cot\alpha = 0;$$

$$\boxed{T = \frac{mg\cot\alpha}{2\pi}}$$

Statics 6
An ideal press



Press with step *h*. Find the active force *N*.

$$2F \cdot l\delta\alpha - N \cdot \delta y = 0;$$

$$\delta y = h\frac{\delta\alpha}{2\pi}$$

$$\boxed{N = 4\pi\frac{l}{h}F}$$

Application to dynamic problems (d'Alembert-Lagrange principle)

For every mass in a system with ideal constraints

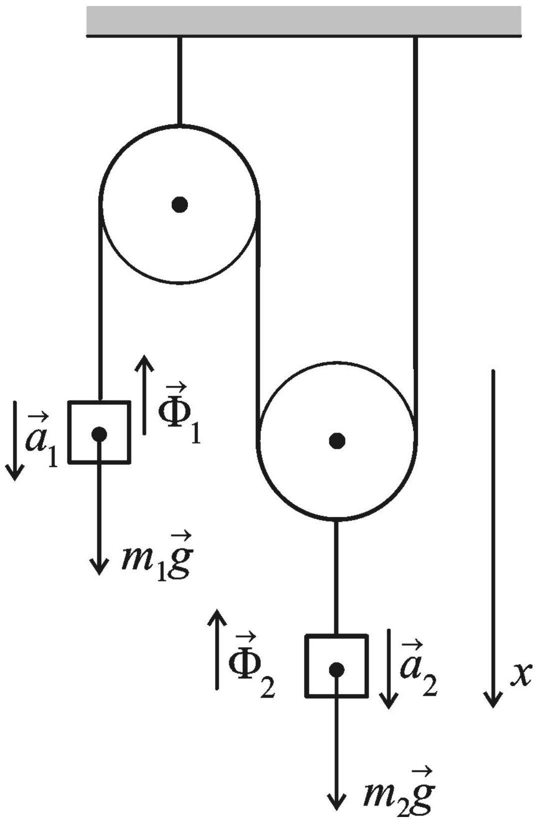
$$\vec{F}_i + \vec{R}_i = m_i \vec{a}_i \quad \Rightarrow \quad \vec{F}_i + \vec{R}_i + \vec{\Phi}_i = 0$$

($\vec{\Phi}_i = -m_i \vec{a}_i$ –

forces of inertia)

$$\Downarrow$$

$$\boxed{\sum_i \left(\vec{F}_i + \vec{\Phi}_i \right) \cdot \delta \vec{r}_i = 0}$$



Dynamics 1

Find a_1, a_2

$$x_1 + 2x_2 = \text{const} \quad \Rightarrow$$

$$\delta x_1 + 2\delta x_2 = 0 \quad \Rightarrow$$

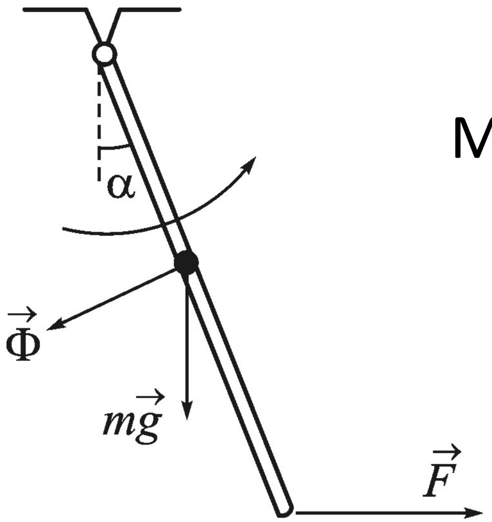
$$a_1 + 2a_2 = 0$$

$$m_1 g \cdot \delta x_1 - m_1 a_1 \cdot \delta x_1 + $$
$$+ m_2 g \cdot \delta x_2 - m_2 a_2 \cdot \delta x_2 = 0$$

$$-m_1 g \cdot 2\delta x_2 - m_1 2a_2 \cdot 2\delta x_2 + m_2 g \cdot \delta x_2 - m_2 a_2 \cdot \delta x_2 = 0$$

$$\boxed{a_2 = \frac{m_2 - 2m_1}{m_2 + 4m_1} g, \qquad a_1 = -2a_2}$$

Dynamics 2



Massless rod of length *L* with point mass

in its centre. Find ε.

Small angular displacement.

$$\Phi = ma_\tau = m\left(\frac{L}{2}\varepsilon\right);$$

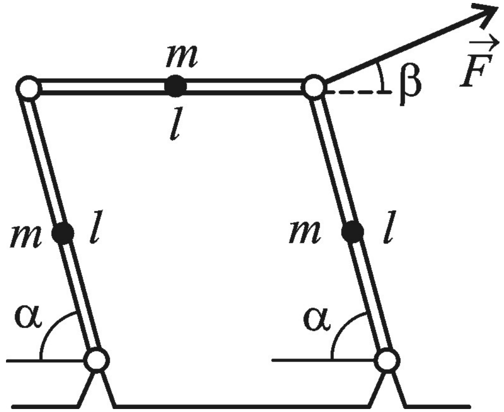
$$F\left(L\delta\alpha\right)\cos\alpha - mg\left(\frac{L}{2}\delta\alpha\right)\sin\alpha - \Phi\left(\frac{L}{2}\delta\alpha\right) = 0,$$

$$\boxed{\varepsilon = \frac{4F\cos\alpha - 2mg\sin\alpha}{mL}}$$

<u>A question</u>: what about the centrifugal force?

<u>The answer</u>: its work on virtual displacement is equal to zero.

Dynamics 3
more difficult problem



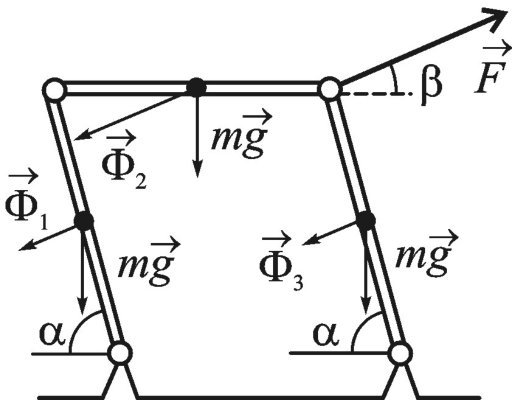
Three massless rods with three point masses. Find ε.

$$\Phi_1 = \Phi_3 = m\frac{l}{2}\varepsilon, \qquad \Phi_2 = ml\varepsilon.$$

Virtual rotation at small angle.

$$F(l\delta\alpha)\sin(\alpha+\beta) - 2mg(l\delta\alpha)\cos\alpha -$$

$$-2\left(m\frac{l}{2}\varepsilon\right)\left(\frac{l}{2}\delta\alpha\right) - (ml\varepsilon)(l\delta\alpha) = 0;$$

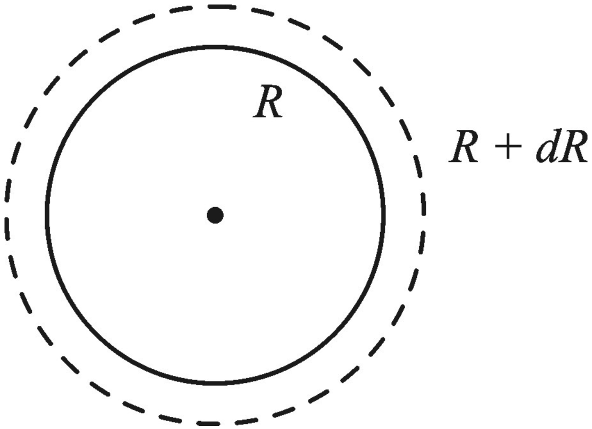


$$\boxed{\varepsilon = \frac{F\sin(\alpha+\beta) - 2mg\cos\alpha}{1,5ml}}$$

Thermodynamics

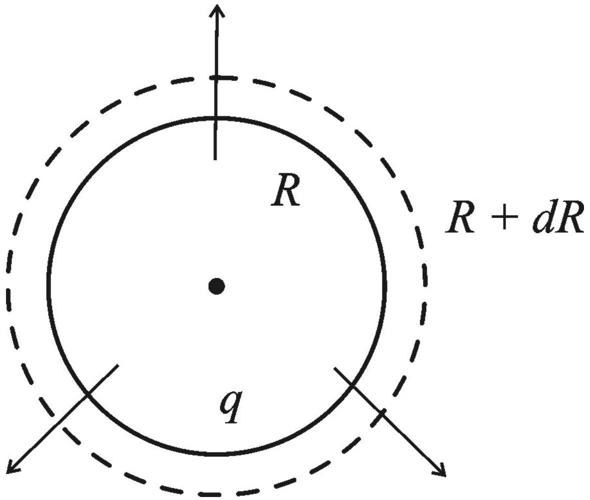
Find additional pressure inside the spherical soap bubble.

Increase the bubble radius by δR. The work of additional pressure is $\Delta p \cdot 4\pi R^2 \cdot \delta R$, the increase of surface energy is $2\sigma \cdot \delta(4\pi R^2) = 16\sigma\pi R\delta R$ (two surfaces). So $\Delta p \cdot 4\pi R^2 \cdot \delta R = 16\sigma\pi R\delta R$, or



$$\boxed{\Delta p = \frac{4\sigma}{R}}$$

Electrostatics 1



Sphere of radius *R* with charge *q*. Find the force *f* due to unit surface ($f = \Delta F/\Delta s$).

$$W_{\mathrm{el}} = \frac{q^2}{8\pi\varepsilon_0 R}$$

Increase radius by δR. $\delta A_{\mathrm{ext}} = \delta W_{\mathrm{el}}$.

$$f_{\mathrm{ext}} \cdot 4\pi R^2 \cdot \delta R = \delta\left(\frac{q^2}{8\pi\varepsilon_0 R}\right)$$

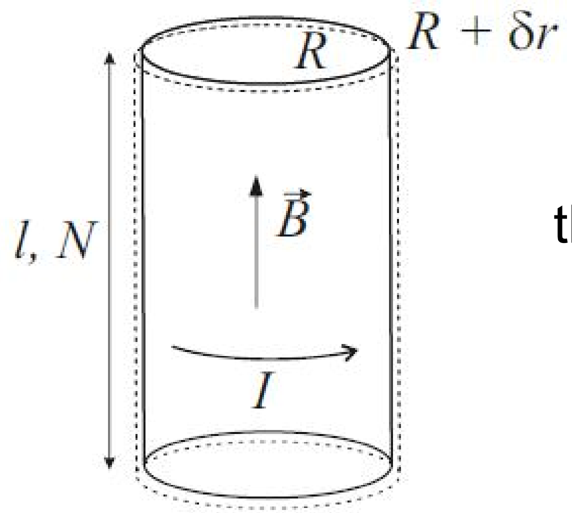
$$\boxed{f_{\mathrm{ext}} = -\frac{q^2}{32\pi^2\varepsilon_0 R^4} = -\frac{\varepsilon_0 E^2}{2} \quad \left(E = \frac{q}{4\pi\varepsilon_0 R^2}\right)}$$

(f_{ext} is directed inward, f_{el} – outward)

Note:

$$\boxed{f_{\mathrm{ext}} S\delta R = -w\delta V = -wS\delta R \quad (w = \frac{\varepsilon_0 E^2}{2})}$$

Difficulty with magnetism



If again $\delta A_{\text{ext}} = \delta W = +w S \delta R > 0$

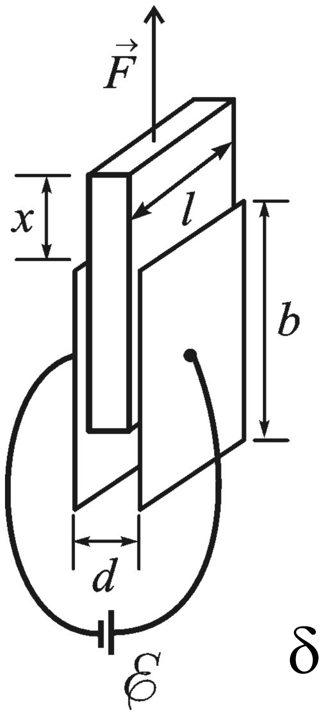
then $f_{\text{ext}} = w = \dfrac{B^2}{2\mu_0}$ is directed outward,

and f_{magn} is directed inward!??

The value is valid, but sign is wrong! f_{magn} is directed outward!

Why???
Where is mistake?

Electrostatics 2



Find the force at which a dielectric (ε) slab is pulled into the gap between the plates of parallel-plate capacitor. Capacitor voltage is *U*.

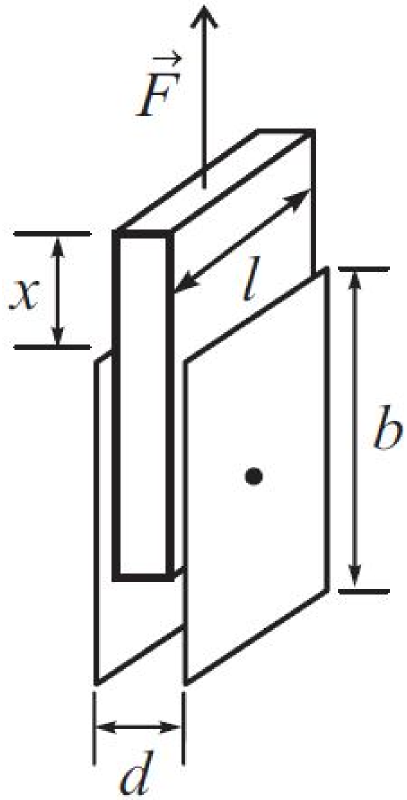
Connect the plates with a source ($\mathcal{E} = U$). Then $E = U/d$ = const.
Virtually increase x by δx.

$$\delta W = \delta x l d \left(\frac{\varepsilon_0 E^2}{2} - \frac{\varepsilon_0 \varepsilon E^2}{2} \right) < 0 \qquad \left(\text{or} \quad \delta W = \frac{\delta C U^2}{2} \right)$$

But $Fdx > 0$! Why? $\qquad \delta W = F\delta x + \delta A_{\text{source}}$

$$\delta A_{\text{source}} = \mathcal{E}\delta q = \mathcal{E}\left(\delta C \mathcal{E}\right) = 2\delta W \quad \Rightarrow F\delta x = -\delta W > 0$$

$$\boxed{F = \frac{\varepsilon_0 (\varepsilon - 1) l U^2}{2d}}$$



Electrostatics 2

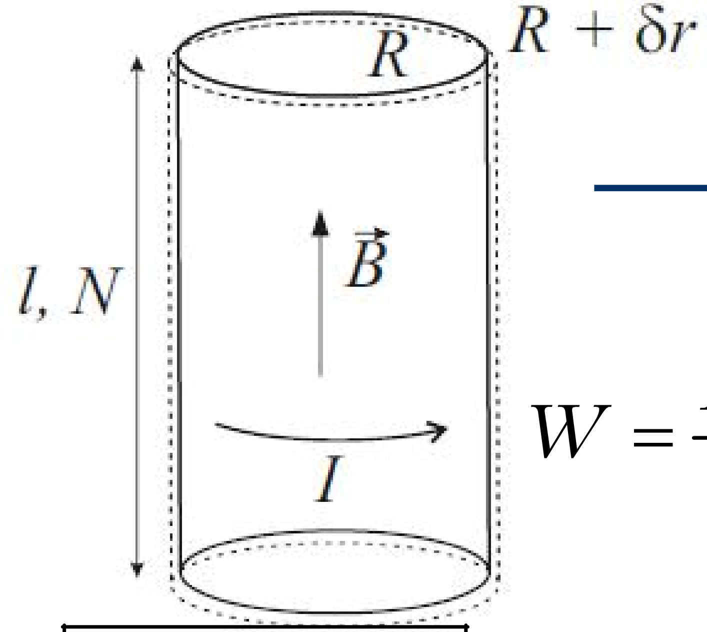
$$\left\{ C = \frac{\varepsilon_0 (xl)}{d} + \frac{\varepsilon_0 \varepsilon (b-x) l}{d} \right\} \qquad \delta C = \frac{\varepsilon_0 (1-\varepsilon) l \delta x}{d} < 0$$

We can disconnect capacitor from source.

Then $q = CU$ = const,

$$W = \frac{q^2}{2C}$$

$$F\delta x = \delta W = -\frac{q^2}{2C^2}\delta C = -\frac{\delta C \cdot U^2}{2} > 0$$



Back to magnetism

$$\delta W = \delta A_{\text{ext}} + \delta A_{\text{source}}$$

$$W = \frac{LI^2}{2} = \frac{B^2}{2\mu_0} Sl \qquad \delta W = \frac{\delta L \cdot I^2}{2}$$

$$\boxed{\begin{array}{l} B = \mu_0 \dfrac{N}{l} I \\[2ex] L = \mu_0 \dfrac{N^2}{l^2} Sl \end{array}}$$

$$\delta A_{\text{source}} = \int \left(-\mathcal{E}_{\text{ind}}\right) I dt =$$

$$= \int \frac{d\left(LI\right)}{dt} I dt = \delta L \cdot I^2 = 2\delta W$$

$$\delta A_{\text{ext}} = -\delta W$$

f_{ext} is pointed inward, f_{magn} is pointed outward

Thank you