The Abdus Salam
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2239-3

Workshop on Integrability and its Breaking in Strongly Correlated and Disordered Systems

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Quantum Quench in the Transverse Field Ising Chain

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Analytic Results for Quantum Quenches in the Transverse Field Ising Chain

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EPSRC Engineering and Physical Sciences Research Council

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Quantum systems out of equilibrium

Idea:

A. Consider a quantum many-particle system with Hamiltonian H

B. Prepare the system in a state $|\psi\rangle$ that is **not** an eigenstate.

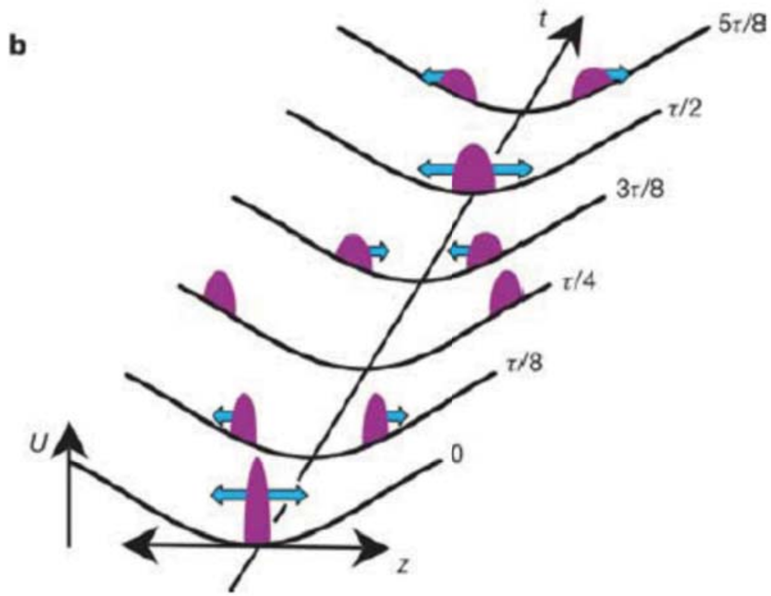
C. Time evolution $|\psi(t)\rangle = \exp(-iHt)\,|\psi\rangle$

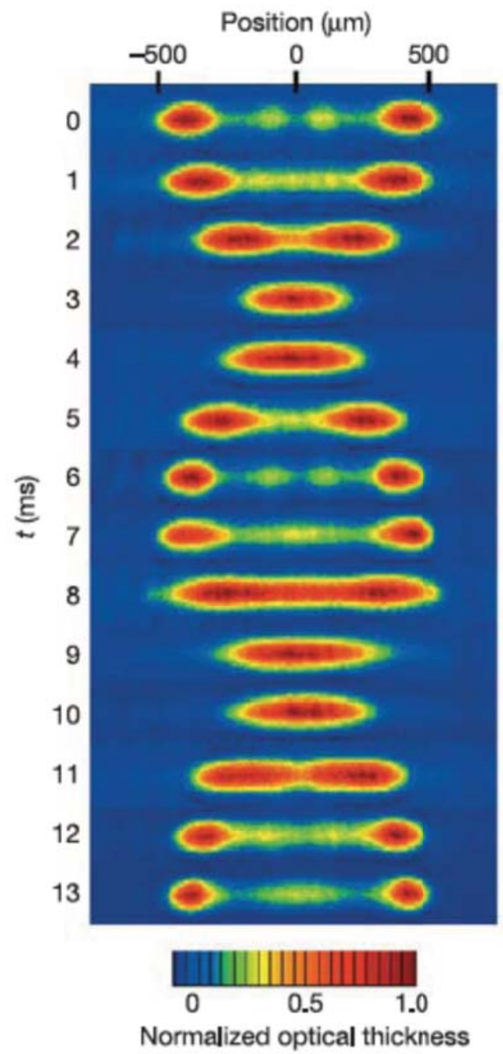
D. Study time evolution of local observables $\langle\psi(t)|O(x)|\psi(t)\rangle$ in the **thermodynamic limit**.

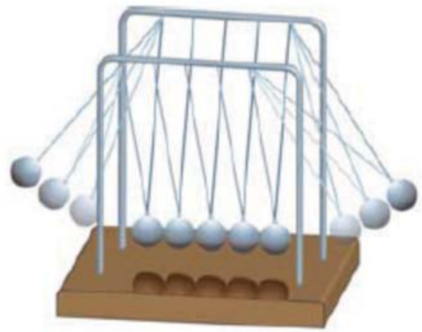
Experiments: "Quantum Newton's Cradle"

T. Kinoshita, T. Wenger and D.S. Weiss, Nature 440, 900 (2006)

40-250 ^{87}Rb atoms in a 1D optical trap

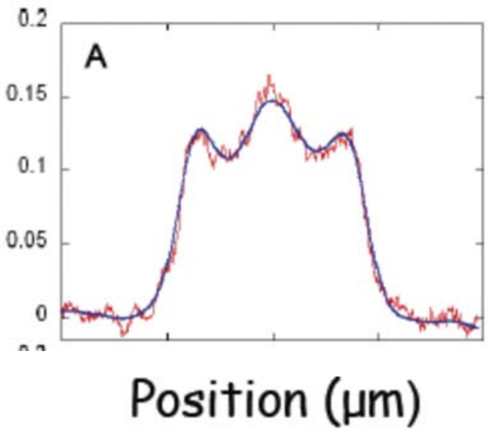






Essentially unitary time evolution.

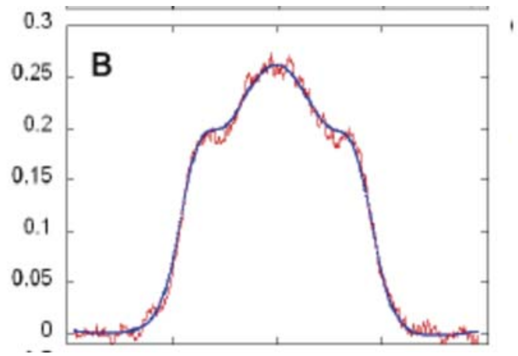
- 1D system does not "relax" in time.



γ=18

$\underline{\tau_{th}}$

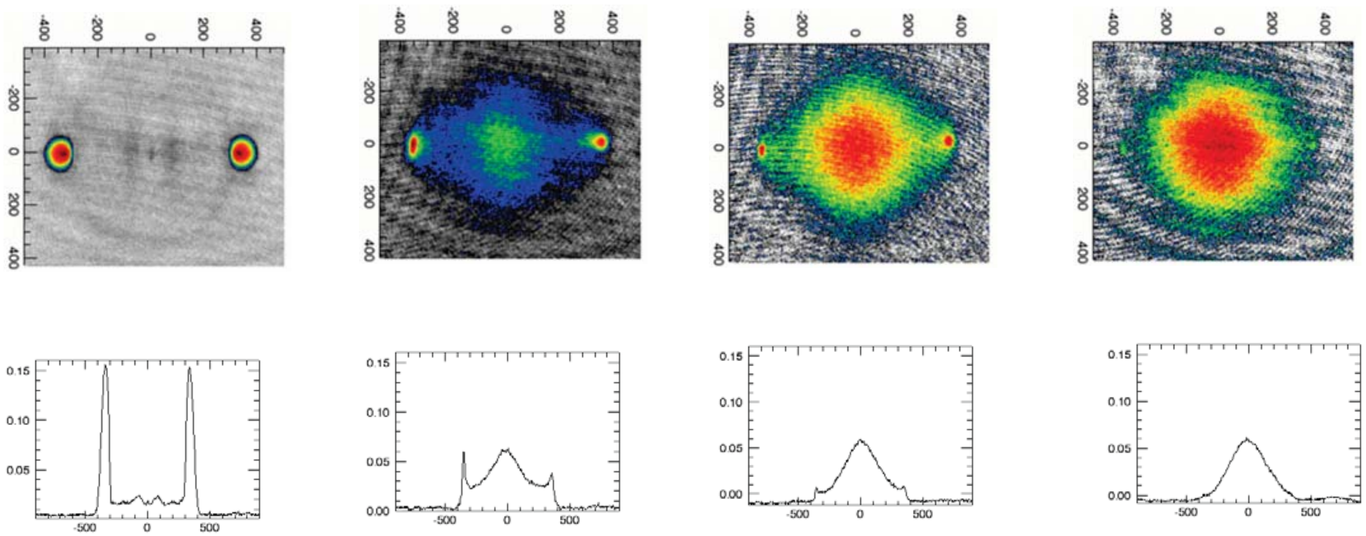
>390τ



γ=3.2

>1910τ

- 2D and **3D** systems do: Thermalization occurs in ~3 collisions.



0τ 2τ 4τ 9τ

"Quantum Newton's Cradle"

T. Kinoshita, T. Wenger and D.S. Weiss, Nature 440, 900 (2006)

Suggestion: the 1D case is special because the system is close to being integrable

Without trap:

$$\mathcal{H}_N = -\sum_{j=1}^{N} \frac{\partial^2}{\partial x_j^2} + 2c \sum_{N \geq j > k \geq 1} \delta(x_j - x_k).$$

Has infinite number of local higher conservation laws, solvable by Bethe Ansatz (Lieb+Liniger '63)

Question: is the nonequilibrium evolution of integrable models special and if so, how?

"Thermalization"

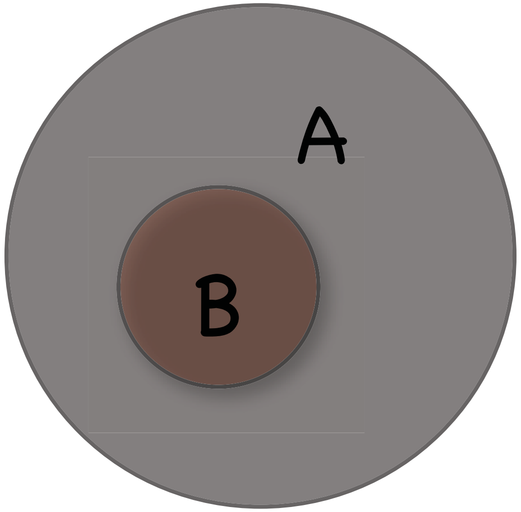
(Deutsch '91, Srednicki '94)

Belief: "generic" system "thermalize" at infinite times.

Density matrix: $\rho = |\psi\rangle\langle\psi|$

Reduced density matrix: $\rho_B = \mathrm{tr}_A\, |\psi\rangle\langle\psi|$

If A is infinite then $\rho_B = \exp(-\beta_{eff}\, H_B)/Z_B$



Rigol et al (2007): **Integrable systems do not thermalize.**

Let I_m be local integrals of motion $[I_m, I_n]=0$

Late time behaviour described by **generalized Gibbs ensemble:**

$$\rho_{gG}=\exp(-\Sigma\ \lambda_m\ I_m)/Z_{gG}$$

$$Z_{gG}=\mathrm{tr}\ \exp(-\Sigma\ \lambda_m\ I_m)$$

$$\lim_{t\to\infty}\ \langle\psi(t)|O(x)|\psi(t)\rangle\ =\mathrm{tr}[\rho_{gG}\ O(x)]$$

λ_m fixed by $\quad \mathrm{tr}[\rho_{gG}\ I_m]=\ \langle\psi(0)|\ I_m\ |\psi(0)\rangle$

Transverse Field Ising Chain

Simplest paradigm of a T=0 **Quantum Phase Transition**

Hamiltonian:

$$H = -J \sum_{j=1}^{L} \left[\sigma_j^z \sigma_{j+1}^z + h \sigma_j^x \right].$$

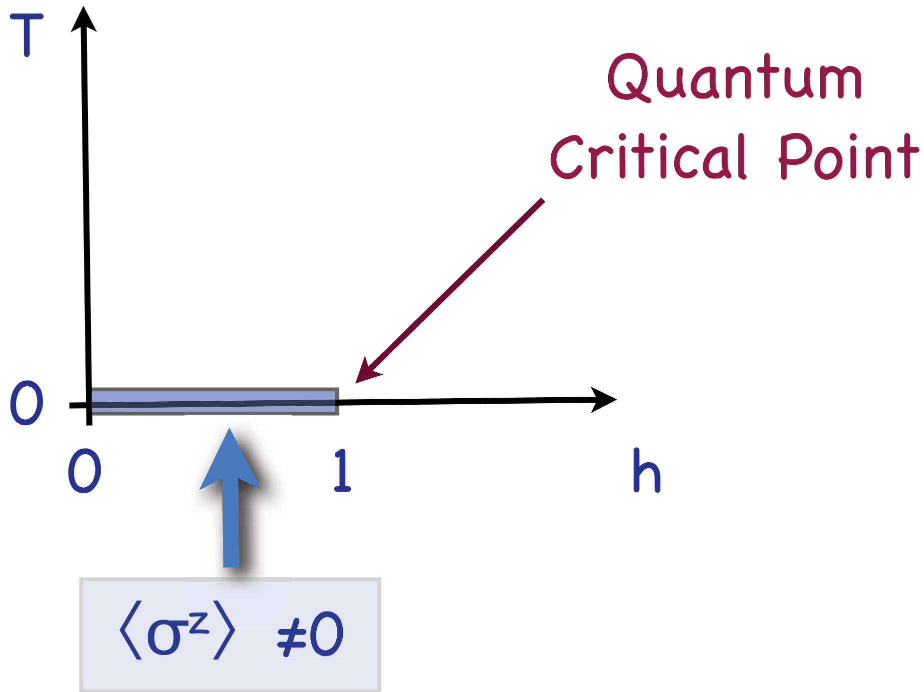
$\mathbb{Z}_2$ symmetry: rotation by π around x-axis.

$$\sigma_j^\alpha \to -\sigma_j^\alpha , \quad \alpha = y, z.$$

Phase Diagram:

order parameter: $\langle \sigma_j^z \rangle$

$(\langle \sigma_j^x \rangle \neq 0$ always)



Transverse Field Ising Chain

Jordan-Wigner transformation to spinless fermions:

$$\sigma_j^x = 1 - 2c_j^\dagger c_j \;, \qquad \text{local}$$

$$\sigma_j^z = -\prod_{l<j}(1-2c_l^\dagger c_l)(c_j + c_j^\dagger). \qquad \text{nonlocal}$$

Fourier+Bogoliubov transformations:

$$c(k) = \frac{1}{\sqrt{L}}\sum_j c_j\, e^{-ikj}.$$

$$\begin{pmatrix} c(k) \\ c^\dagger(-k) \end{pmatrix} = R_h(k)\begin{pmatrix} \alpha_k \\ \alpha_{-k}^\dagger \end{pmatrix}$$

$$\rightarrow \quad H = \sum_k \epsilon_h(k)\left[\alpha_k^\dagger \alpha_k - \frac{1}{2}\right] \qquad \epsilon_h(k) = 2J\sqrt{1+h^2-2h\cos(k)}.$$

Ground State: $\alpha_k|0\rangle = 0.$

This will be our initial state

Quantum Quench h→h'

New Hamiltonian:

$$H(h') = \sum_k \epsilon_{h'}(k) \left[\beta_k^\dagger \beta_k - \frac{1}{2}\right]$$

Time evolution:

$$\beta_k(t) = e^{-i\epsilon_{h'}(k)t}\beta_k$$

New vs old Bogoliubov fermions:

$$\begin{pmatrix} \beta_k \\ \beta_{-k}^\dagger \end{pmatrix} = U(k) \begin{pmatrix} \alpha_k \\ \alpha_{-k}^\dagger \end{pmatrix} \qquad U(k) = R_{h'}^\dagger(k) R_h(k)$$

Time evolution of σˣ

(Barouch, McCoy & Dresden '70)

$$\sigma_j^x(t) = -\frac{1}{L}\sum_{k,p} e^{-i(k-p)j}\, (\alpha_k^\dagger, \alpha_{-k})\; S(k,p,t) \begin{pmatrix} \alpha_p \\ \alpha_{-p}^\dagger \end{pmatrix}$$

S(k,p,t) a known 2x2 matrix.

L→∞ ↓ straightforward calculation

$$\langle 0|e^{iH(h')t}\sigma_j^x e^{-iH(h')t}|0\rangle = \int_{-\pi}^{\pi}\frac{dk}{2\pi}\left[\cos(\theta_k)\cos^2(\epsilon_{h'}(k)t) + \cos(\theta_k - 2\theta'_k)\sin^2(\epsilon_{h'}(k)t)\right].$$

$$\xrightarrow[t\to\infty]{} \boxed{\int_{-\pi}^{\pi} \frac{dk}{4\pi} \left[\cos(\theta_k') \cos(\theta_k - \theta_k')\right] + \mathcal{O}(t^{-3/2}).}$$

Different from $\frac{1}{\mathcal{Z}} \mathrm{tr}\left[e^{-H(h')/T_{\mathrm{eff}}} \sigma_j^x\right]$ → **no thermalization.**

How about the Generalized Gibbs Ensemble?

Conserved Quantities: $I(k) = \beta_k^\dagger \beta_k$

Density Matrix: $\rho_{gG} = \frac{1}{\mathcal{Z}_{gG}} e^{-\sum_k \lambda_k I(k)}$

$$\boxed{\mathrm{tr}\left[\rho_{gG}\, \sigma_j^x\right] = -\int_{-\pi}^{\pi} \frac{dk}{2\pi} \left[\frac{2\cos(\theta_k')}{1+e^{\lambda_k}} - \cos(\theta_k')\right]}$$

agrees.

⇒ **GGE works.**

The role of locality

(Rossini et al'10)
(Fioretto and Mussardo '10)

σ^x is quite **special** (non generic): it is **local** w.r.t. to the fermion excitations and couples only to 2-particle states.

σ^z is **non-local** (couples to states with arbitrary number of fermions) and it's difficult to say from numerical studies whether 2-point function **thermalizes** (particularly for small quenches).

Is it possible that certain operators integrable models thermalize and others don't?

Our work: 1 and 2-point functions of σ^z

Calculations are **difficult**. Developed two analytic methods based on (a) determinants (b) form factors.

Result 1: t=∞ behaviour for arbitrary h,h'

$$\lim_{t\to\infty} \langle 0|\sigma_j^z(t)\ \sigma_{j+\ell}^z(t)|0\rangle \sim \exp\left(-\ell/\xi\right)\ ,\ell \gg 1,$$

ξ a simple function of h,h':

$$\xi^{-1} = \begin{cases} \ln\left[x_+ + x_- + \sqrt{4x_+x_-}\right] & \text{if } h, h' < 1 \\ \ln\left(\min[h, h_1]\right) - \ln\left[x_+ + x_- + \sqrt{4x_+x_-}\right] & \text{if } h, h' > 1 \\ \ln\left[x_+ + x_-\right], & \text{else.} \end{cases}$$

$$x_\pm = \frac{1}{4}[\min(h', h'^{-1}) \pm 1][\min(h, h^{-1}) \pm 1]$$

$$h_1 = \frac{1 + h'h + \sqrt{(h'^2 - 1)(h^2 - 1)}}{h' + h}$$

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Compatible with **GGE,** but **not with thermalization!**

GGE amounts to introducing a **mode-dependent temperature**

$$T_{\rm eff}(k) = \frac{2}{\epsilon_h(k)}\text{arctanh}\left[\cos\Delta_k\right]$$

$$\cos\Delta_k = \frac{h'h - (h'+h)\cos k + 1}{\epsilon_h(k)\epsilon_{h'}(k)}.$$

$$\xi^{-1} = \int_0^\pi \frac{dk}{\pi}\xi^{-1}(k) = -\int_0^\pi \frac{dk}{\pi}\epsilon_h(k)\tanh\frac{\epsilon_h(k)}{2T_{\rm eff}(k)}.$$

c.f. thermal correlation length

$$\xi_T^{-1} = -\int_{-\pi}^{\pi}\frac{dk}{2\pi}\ln\left|\tanh\frac{\epsilon_h(k)}{2T}\right|.$$

Result 2: Time dependence for late times

Quenches within the ordered phase (h<1 to h'<1):

$$\langle 0|\sigma_j^z(t)|0\rangle \sim \exp\left(-t\int_0^\pi \frac{dk}{\pi}\epsilon'_h(k)\ln\left[\cos(\Delta_k)\right]\right)$$

$$\cos\Delta_k = \frac{h'h-(h'+h)\cos k+1}{\epsilon_h(k)\epsilon_{h'}(k)}.$$

(approaches **zero** although we remain in the ordered phase).

Mode-dependent decay rate:

$$\tau^{-1} = \int_0^\pi \frac{dk}{\pi}\tau^{-1}(k) = \int_0^\pi \frac{dk}{\pi}\epsilon_h(k)\ln\left[\cos\Delta_k\right].$$

Decay rate and correlation length related by

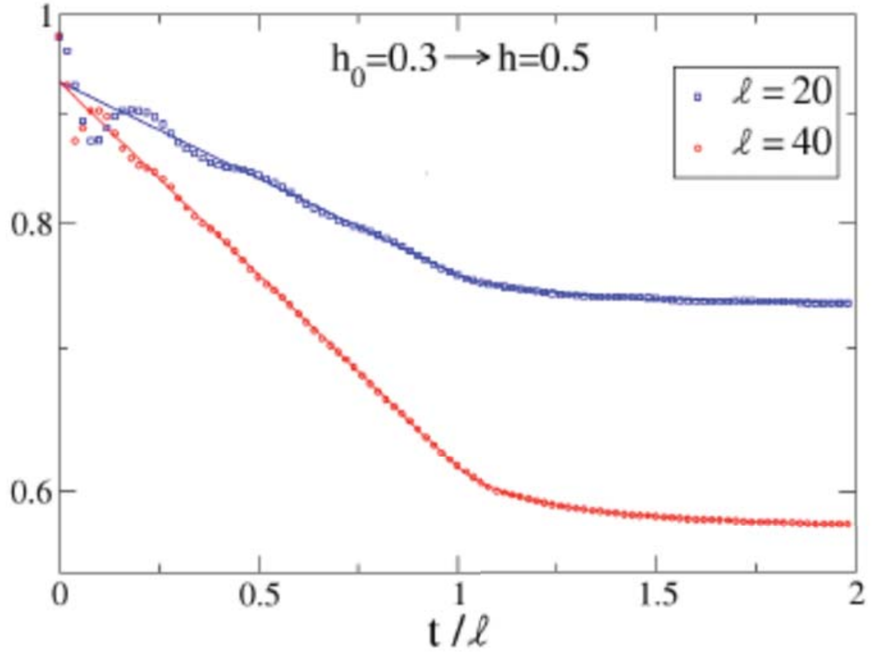
$$\xi(k) = \epsilon_h(k)\ \tau(k).$$

Can understand approach to stationary state in terms of GGE

Quenches within the ordered phase (h<1 to h'<1):

$$\langle 0|\sigma_j^z(t)\ \sigma_{j+\ell}^z(t)|0\rangle \sim \exp\left(-t\int_{2t\epsilon'_{h'}(k)<\ell}\frac{dk}{\pi}2\epsilon'_{h'}(k)\ln\left[\cos(\Delta_k)\right]-\ell\int_{2t\epsilon'_{h'}(k)>\ell}\frac{dk}{\pi}\ln\left[\cos(\Delta_k)\right]\right)$$

Asymptotics
vs
Numerics:



Approach I: Block-Toeplitz Determinants

Express $\sigma_j^z(t)$ in terms of the "old" Bogoliubov fermions α_k

$$\langle 0|\sigma_j^z(t)\ \sigma_{j+n}^z(t)|0\rangle \xrightarrow{\text{Wick's thm}} \mathrm{Pf}(T)$$

$$T_{ln} = \begin{pmatrix} f_{l-n} & -g_{n-l} \\ g_{l-n} & -f_{l-n} \end{pmatrix}$$ Block-Toeplitz matrix

$$\begin{aligned} f_l &= i\int_{-\pi}^{\pi}\frac{dk}{2\pi}e^{-ikl}\sin\left(\Delta_k\right)\ \sin\left(2\epsilon_h'(k)t\right) \\ g_l &= \int_{-\pi}^{\pi}\frac{dk}{2\pi}e^{-ik(l-1)}\left[\cos\left(\Delta_k\right)+i\sin\left(\Delta_k\right)\cos\left(2\epsilon_h'(k)t\right)\right] \end{aligned}$$

Approach I: Block-Toeplitz Determinants

Express $\sigma_j^z(t)$ in terms of the "old" Bogoliubov fermions α_k

$$\langle 0|\sigma_j^z(t)\ \sigma_{j+n}^z(t)|0\rangle \xrightarrow{\text{Wick's thm}} \mathrm{Pf}(T)$$

$$\Big\downarrow \ \text{Multi-dim stationary phase approx.}$$

$$\exp\left(-t \int_{2t\epsilon'_{h'}(k)<n} \frac{dk}{\pi} 2\epsilon'_{h'}(k) \ln\left[\cos(\Delta_k)\right] - n \int_{2t\epsilon'_{h'}(k)>n} \frac{dk}{\pi} \ln\left[\cos(\Delta_k)\right]\right)$$

Approach II: "Form-Factor" Sums

Consider a quench within the ordered phase h,h'<1

1. Go to large, finite volume L

2. initial state = one of the two ground states

$$|0\rangle = \frac{1}{\sqrt{2}}\left[|0\rangle_{\rm R} \pm |0\rangle_{\rm NS}\right] \qquad \alpha_q|0\rangle_{\rm R} = 0 \qquad q_m = \frac{2\pi}{L}m\,, \quad m = -\frac{L}{2},\ldots,\frac{L}{2}-1$$

periodic bc's on fermions

antiperiodic bc's on fermions

3. Express this in terms of the new Bogoliubov fermions

$$|0\rangle_{\rm NS} = \exp\left(i\sum_{p>0} K(q)\beta^\dagger_q\beta^\dagger_{-q}\right)|0'\rangle_{\rm NS}\,,$$

$$|0\rangle_{\rm R} = \exp\left(i\sum_{q>0} K(q)\beta^\dagger_q\beta^\dagger_{-q}\right)|0'\rangle_{\rm R}\,.$$

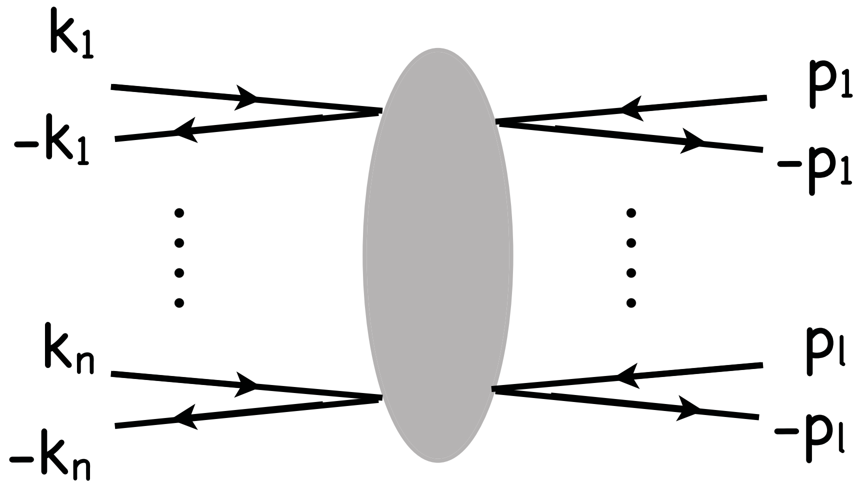
$$K(q) = \tan\left[\frac{\theta_{h'}(q)-\theta_h(q)}{2}\right]$$

4. Lehmann representation in terms of new Bogoliubov fermions

$$
{}_{\mathrm{NS}}\langle 0|\sigma_m^z(t)|0\rangle_{\mathrm{R}} = \sum_{l,n=0}^{\infty} \frac{1}{n!\,l!} \sum_{\substack{k_1,\dots,k_n \\ p_1,\dots,p_l}} \left[\prod_{j=1}^{n} K(k_j)\right] \left[\prod_{i=1}^{l} K(p_i)\right]
$$

$$
{}_{\mathrm{NS}}\langle -k_1, k_1, \dots, -k_n, k_n|\sigma_m^z(t)|p_1, -p_1, \dots, p_l, -p_l\rangle_{\mathrm{R}}
$$

$$
= \sum_{l,n=0}^{\infty} \frac{1}{n!\,l!}
$$



Idea: Consider K(q) as expansion parameter:

$$
n(q) = \frac{\langle 0|\beta_q^\dagger \beta_q|0\rangle}{\langle 0|0\rangle} = \frac{K^2(q)}{1+K^2(q)}
$$

density of excitations

n(q) small ⇔ K(q) uniformly small in q

4. Lehmann representation in terms of new Bogoliubov fermions

$$
{}_{\mathrm{NS}}\langle 0|\sigma_m^z(t)|0\rangle_{\mathrm{R}} = \sum_{l,n=0}^{\infty} \frac{1}{n!\, l!} \sum_{\substack{k_1,\dots,k_n \\ p_1,\dots,p_l}} \left[\prod_{j=1}^{n} K(k_j)\right] \left[\prod_{i=1}^{l} K(p_i)\right]
$$

$$
{}_{\mathrm{NS}}\langle -k_1, k_1, \dots, -k_n, k_n|\sigma_m^z(t)|p_1, -p_1, \dots, p_l, -p_l\rangle_{\mathrm{R}}
$$

$$
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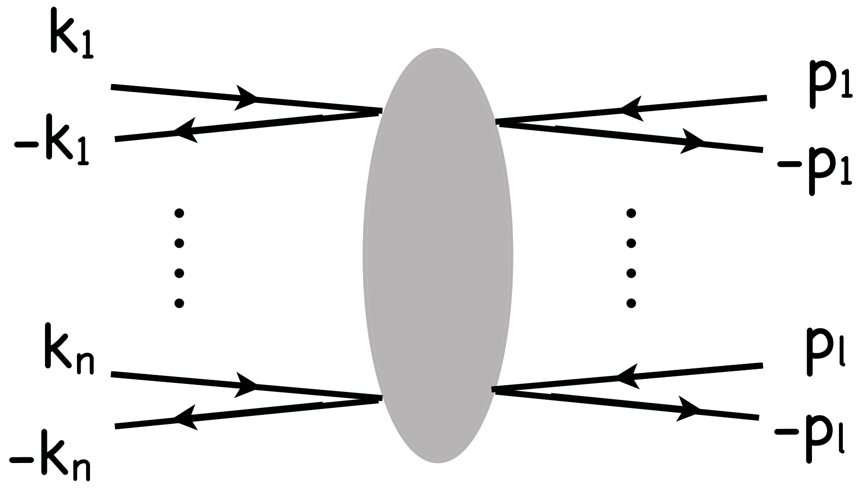
$$



1. Dominant contributions from even orders $\mathbf{K^{2n}}$
2. Leading contributions **at order** $\mathbf{K^{2n}}$ from terms with $n=l$ and $\{k_1,\dots,k_n\} = \{p_1,\dots,p_n\}$

sum these to all orders $\Rightarrow$

$$\frac{\langle 0|\sigma^z_m(t)|0\rangle}{\langle 0|0\rangle} \propto \exp\left(-t\int_0^\pi \frac{dk}{\pi}\left[K^2(k)+\mathcal{O}(K^6)\right]|2\epsilon'(k)|\right)$$

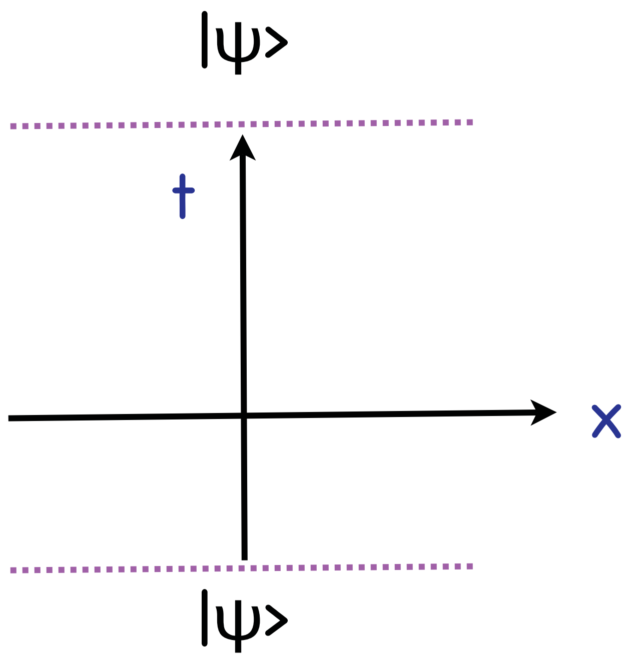
- **Low density expansion** of the full answer.
- Works well everywhere except very close to QCP.
- 2-point function treated analogously.

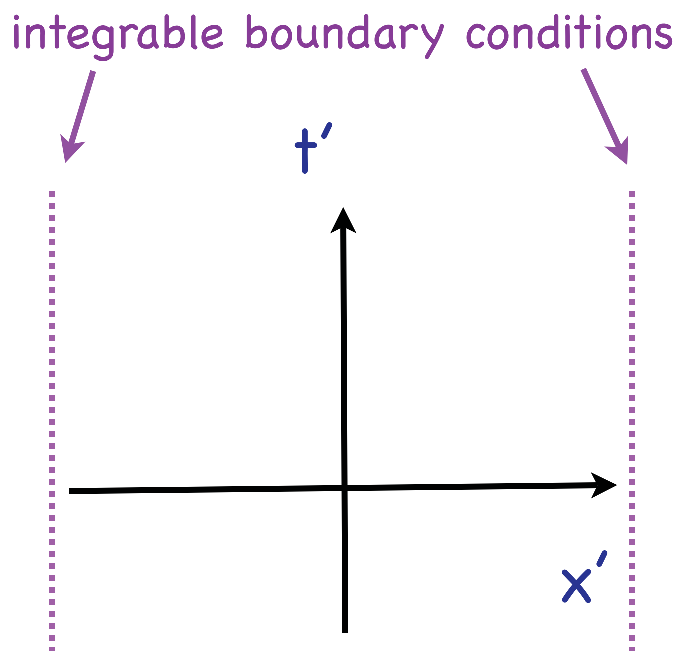
Conclusions

1. Nonequilibrium evolution in integrable models appears to be special.

2. Nonlocality does not save the day. ☹

3. "Form factor" approach generalizes to "integrable" quenches
   $\Rightarrow$ mass quench in sine-Gordon (at reflectionless points)





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3. "Form factor" approach generalizes to "integrable" quenches
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4. What happens for more general initial states (e.g. break translation invariance) ? $\Rightarrow$ Ising chain.