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Workshop on Integrability and its Breaking in Strongly Correlated and Disordered Systems

23 - 27 May 2011

Magnetic and percolative Ising universality classes. Results from integrable field theory

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Integrability and its Breaking in Strongly Correlated and Disordered Systems

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MAGNETIC AND PERCOLATIVE ISING UNIVERSALITY CLASSES. RESULTS FROM INTEGRABLE FIELD THEORY

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Integrability in (1+1)d quantum field theory (2d euclidean) :

- sharply defined (purely elastic scattering)

- contributes essentially to characterization of universality classes

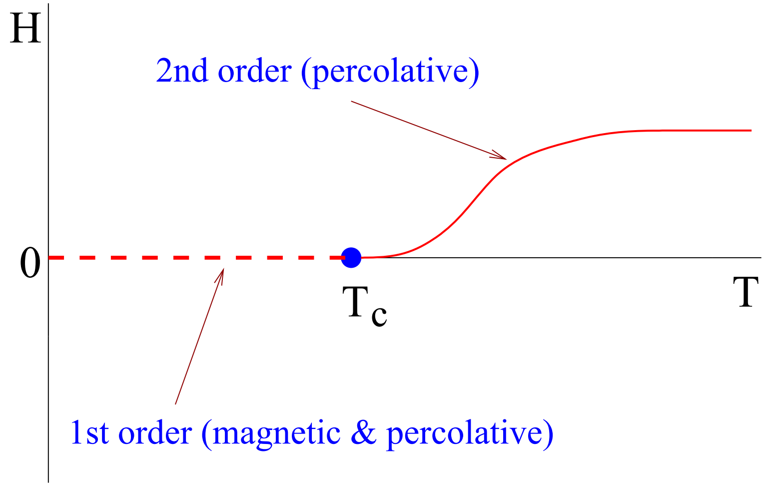
- relevant for a variety of physical situations (confinement, particle decay, magnetic as well as geometric transitions, ...)

Ising model

$$\mathcal{H} = -\frac{1}{T}\sum_{\langle i,j\rangle}\sigma_i\sigma_j - H\sum_i \sigma_i\,, \qquad \sigma_i = \pm 1$$

magnetic order parameter: $\langle\sigma_i\rangle$

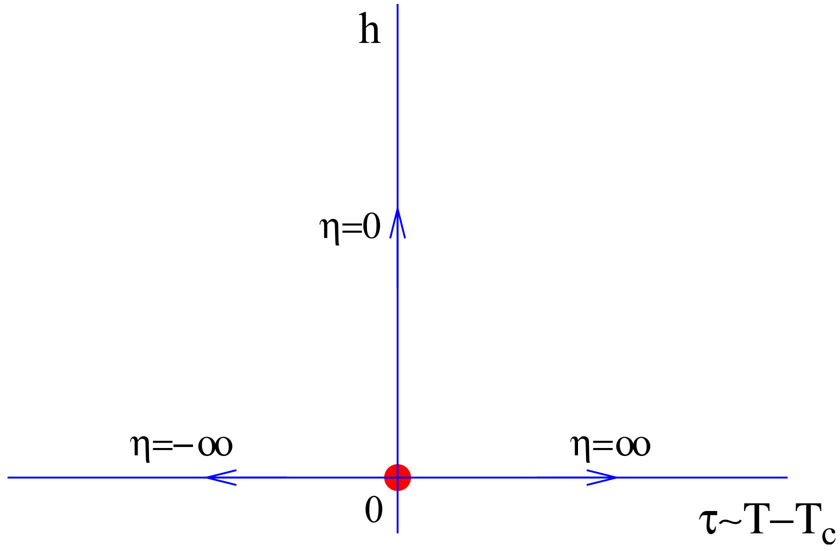
percolative order parameter: infinite cluster of $+$ spins



I'll discuss field theoretical description, role of integrability and quantitative results

Scaling Ising ferromagnet

$$\mathcal{A} = \mathcal{A}_{CFT} - \tau \int d^2x\, \varepsilon(x) - h \int d^2x\, \sigma(x)$$



$\tau \sim T - T_c$ $\qquad X_\varepsilon = 1 \quad \longrightarrow \quad \tau \sim m$

$h \sim H$ $\qquad X_\sigma = \frac{1}{8} \quad \longrightarrow \quad h \sim m^{15/8}$

$\eta = \dfrac{\tau}{|h|^{8/15}}$ labels RG trajectories

Integrable trajectories:

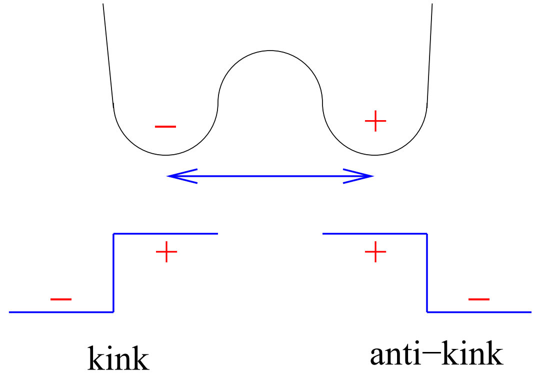
$\eta = \pm\infty$: free fermions, but spin sector non-trivial

$\eta = 0$: not solved on the lattice

Evolution of the mass spectrum

- $\eta = -\infty$

The excitations are free kinks



- $\eta \to -\infty$

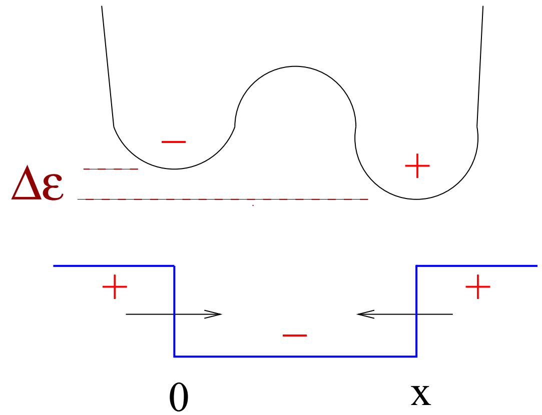
$\Delta\mathcal{E} \simeq 2h \langle \sigma \rangle$

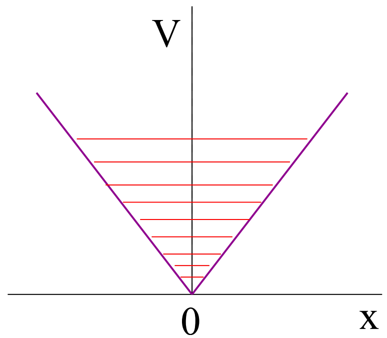
The potential $\quad V(x) \simeq \Delta\mathcal{E}\,|x|$

confines the kinks into "mesons" A_n with masses

$$m_n = 2m_{kink} + \frac{(\Delta\mathcal{E})^{2/3} z_n}{m_{kink}^{1/3}}, \qquad h \to 0$$

$$\mathrm{Ai}(-z_n) = 0 \qquad z_n > 0 \qquad n = 1, 2, \ldots$$





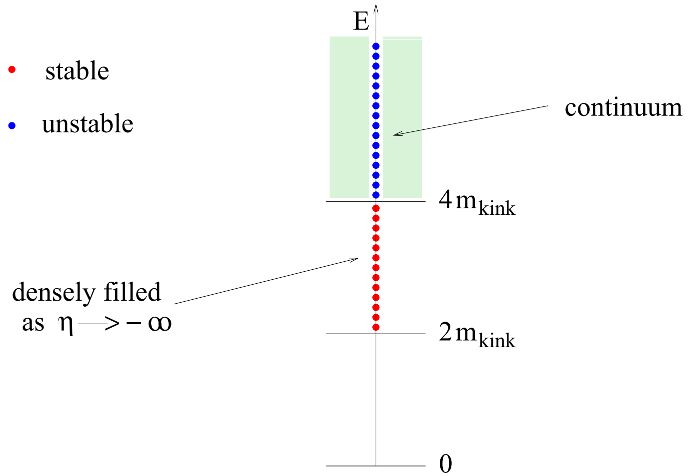
– McCoy, Wu, '78 (lattice)

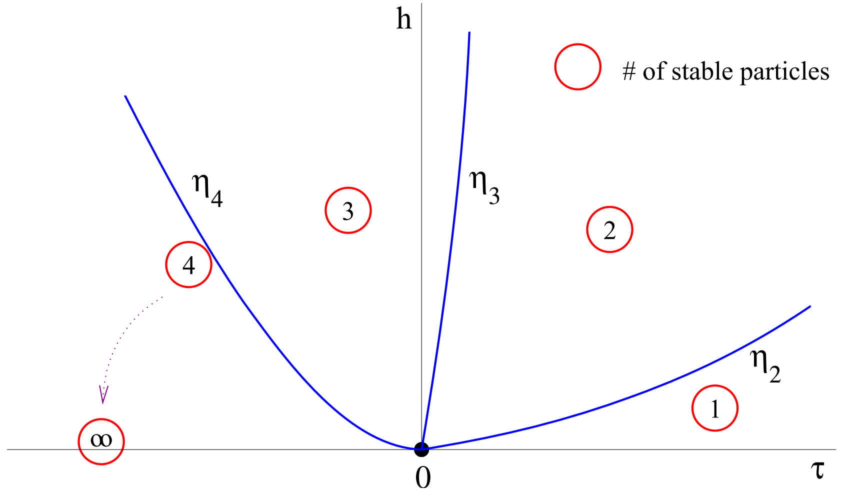
– Fonseca, A.Zamolodchikov, '01,'06 (relativistic corrections)

– Rutkevich, '05, '09 (large n)

The number of stable particles decreases as η increases from $-\infty$, until only A_1 is stable as $\eta \to +\infty$

$\forall n > 1 \quad \exists\ \eta_n \quad$ for which $\quad m_n = 2m_1 \qquad$ particle A_n is unstable for $\eta \geq \eta_n$





- Confirmed by numerical studies:

– GD, Mussardo, Simonetti, '96 (TCSA)

– Fonseca, A.Zamolodchikov, '01, '06 (TFFSA)

Notice $\eta_4 < 0 < \eta_3$

Integrability at $\eta = 0$ (A.Zamolodchikov, '88)

"E_8" mass spectrum: 8 stable particles $A_1, \ldots, A_8$ with masses

$$m_1 \sim h^{8/15}$$

$$m_2 = 2m_1 \cos\tfrac{\pi}{5} = (1.61803..)\, m_1$$

$$m_3 = 2m_1 \cos\tfrac{\pi}{30} = (1.98904..)\, m_1$$

$$m_4 = 2m_2 \cos\tfrac{7\pi}{30} = (2.40486..)\, m_1$$

$$m_5 = 2m_2 \cos\tfrac{2\pi}{15} = (2.95629..)\, m_1$$

$$m_6 = 2m_2 \cos\tfrac{\pi}{30} = (3.21834..)\, m_1$$

$$m_7 = 4m_2 \cos\tfrac{\pi}{5} \cos\tfrac{7\pi}{30} = (3.89115..)\, m_1$$

$$m_8 = 4m_2 \cos\tfrac{\pi}{5} \cos\tfrac{2\pi}{15} = (4.78338..)\, m_1$$

Scattering amplitudes:

$$S_{11}(\theta) = \frac{\tanh\frac{1}{2}(\theta + \frac{2}{3}i\pi)}{\tanh\frac{1}{2}(\theta - \frac{2}{3}i\pi)} \frac{\tanh\frac{1}{2}(\theta + \frac{2}{5}i\pi)}{\tanh\frac{1}{2}(\theta - \frac{2}{5}i\pi)} \frac{\tanh\frac{1}{2}(\theta + \frac{1}{15}i\pi)}{\tanh\frac{1}{2}(\theta - \frac{1}{15}i\pi)}$$

and similarly for the other $S_{ab}(\theta)$ $\quad a, b = 1, \ldots, 8$

$s_{ab} = m_a^2 + m_b^2 + 2m_a m_b \cosh\theta$ (center of mass energy)^2

Form factors at $\eta = 0$ (GD, Mussardo, '95; GD, Simonetti, '96)

1-particle:

$$F_a^{\Phi} = \frac{\langle 0|\Phi(0)|A_a\rangle}{\langle\Phi\rangle}$$

	σ	ε
F_1	$-0.640902..$	$-3.706584..$
F_2	$0.338674..$	$3.422288..$
F_3	$-0.186628..$	$-2.384334..$
F_4	$0.142771..$	$2.268406..$
F_5	$0.060326..$	$1.213383..$
F_6	$-0.043389..$	$-0.961764..$
F_7	$0.016425..$	$0.452303..$
F_8	$-0.003036..$	$-0.105848..$

- Caselle, Hasenbusch, '00 (transfer matrix):

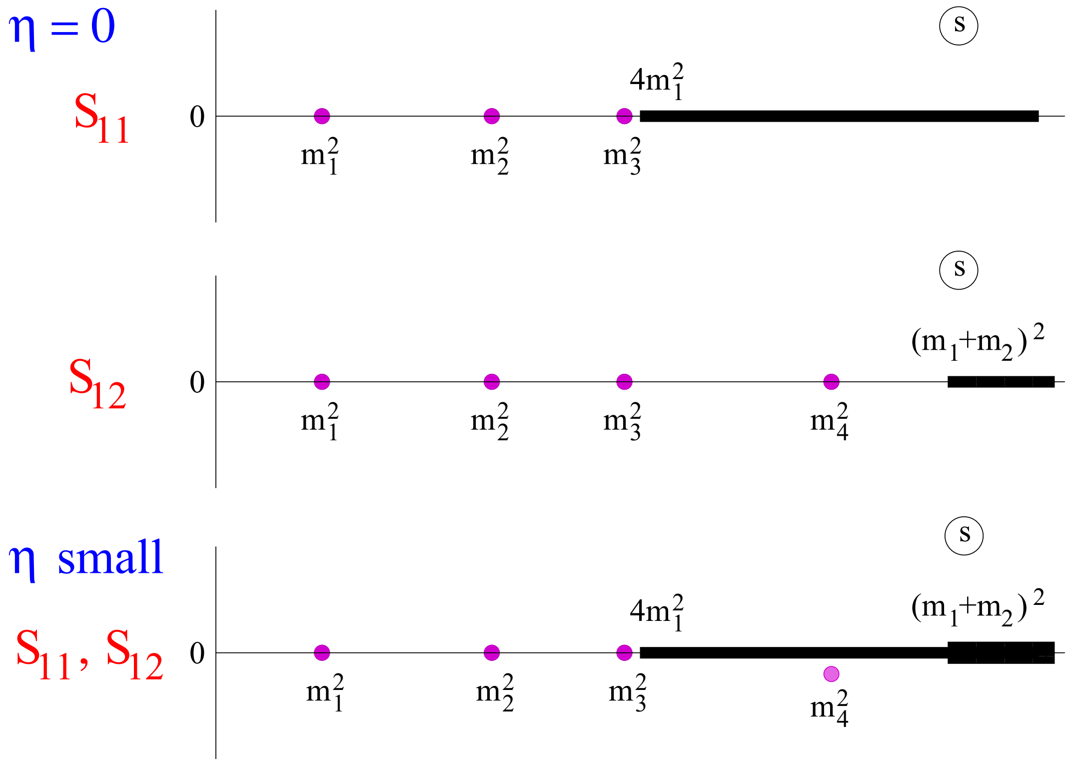
	σ	ε
F_1	$-0.6408(3)$	$-3.707(7)$

Form factors determine: correlation functions via spectral sum

corrections around integrable trajectories

Integrability breaking and particle decay (GD, Mussardo, Simonetti, '96; GD, Grinza, Mussardo, '06)

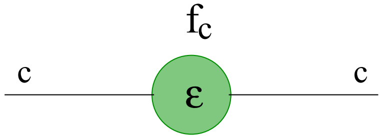
Five stable particles above threshold at $\eta = 0$; fine as long as the theory is integrable



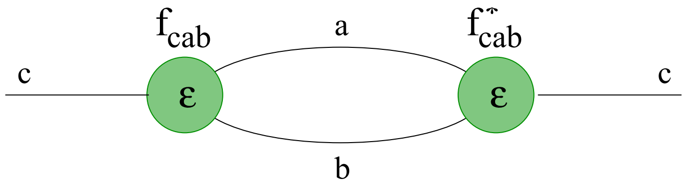
Integrability breaking opens inelastic channels and forces the decay of particles above threshold

Mass corrections for η small :

$\delta\,\mathrm{Re}\,m_c$:



$\mathrm{Im}\,m_c$:



$$f_1 = -17.8933..$$
$$f_2 = -24.9467..$$
$$f_3 = -53.6799..$$
$$f_4 = -49.3206..$$

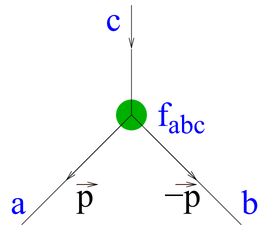
$$|f_{411}| = 36.73044..$$
$$|f_{511}| = 19.16275..$$
$$|f_{512}| = 11.2183..$$

– Branching ratio for A_5: decays at 47% into $A_1 A_1$ and at 53% into $A_1 A_2$

– Universal lifetime ratio: $\lim_{\eta\to 0} \frac{t_4}{t_5} = 0.23326..$ $\quad t_5 > t_4$!

$$\Gamma_{c\to ab} \propto g^2\,|f_{abc}|^2\,\Phi_d$$

$$\Phi_d \sim \int \frac{d^{d-1}\vec{p}_a}{p_a^0}\,\frac{d^{d-1}\vec{p}_b}{p_b^0}\,\delta^d(p_a+p_b-p_c) \sim |\vec{p}|^{d-3}/m_c$$ phase space



• Pozsgay, Takacs, '06 (TCSA) : $|f_{411}| = 36.5(3)$, $\quad |f_{511}| = 19.5(9)$

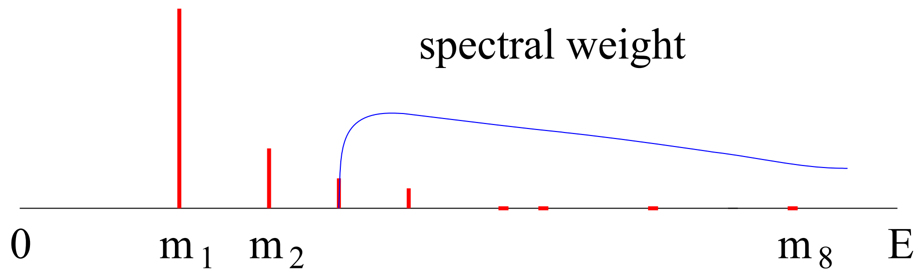
Experimental evidences in $CoNb_2O_6$ (Coldea et al, '10)

Near-isolated chains of Co^{2+} ions in the quantum Ising universality class

Neutron scattering shows in particular:

- $\eta \to -\infty$: first 5 meson masses

- $\eta \sim 0$: $m_2/m_1 \sim 1.6$



Classical magnetic observables :

Magnetization	$M = \langle \sigma \rangle$
Specific heat	$C = \int d^d x \, \langle \varepsilon(x) \varepsilon(0) \rangle$
Susceptibility	$\chi = \int d^d x \, \langle \sigma(x) \sigma(0) \rangle$
Correlation length	$\langle \sigma(x) \sigma(0) \rangle \sim e^{-\lvert x \rvert / \xi}, \qquad \lvert x \rvert \to \infty$

Example : $\quad \chi \simeq \Gamma_{\pm} \, \lvert T - T_c \rvert^{-\gamma} \, , \qquad T \to T_c^{\pm}$

$$\gamma = \frac{d - 2X_\sigma}{d - X_\varepsilon} \qquad \text{and} \qquad \frac{\Gamma_+}{\Gamma_-} \qquad \text{are universal}$$

Amplitude ratios of the magnetic Ising universality class:

Thermal ratios Wu, MacCoy, Tracy, Barouch, '76 (lattice)

$$\boxed{\begin{array}{l} A_+/A_- = 1 \\ \Gamma_+/\Gamma_- = 37.6936520.. \\ \xi_0^+/\xi_0^- = 2 \\ R_C = 0.318569391.. \\ R_\xi^+ = 1/\sqrt{2} \end{array}}$$

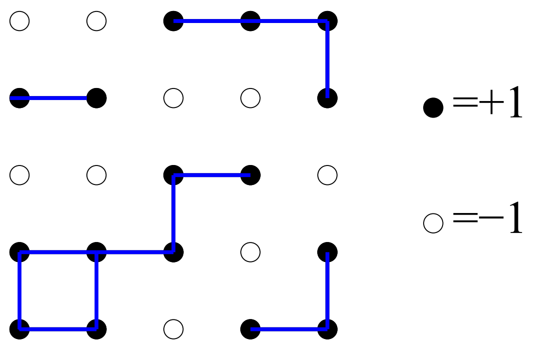
Thermal-Magnetic ratios GD, '98 (integrable QFT)

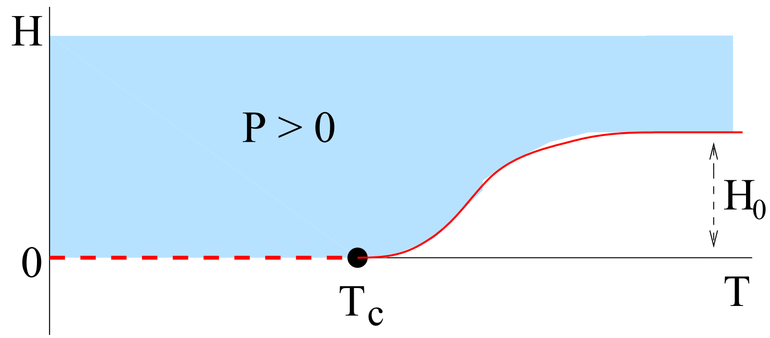
$$\boxed{\begin{array}{l} R_\chi = 6.77828502.. \\ Q_2 = 3.23513834.. \\ R_A = 0.0469985240.. \end{array}}$$

- Caselle, Hasenbusch, '00 (transfer matrix): $R_\chi = 6.7782(8)\,, \qquad Q_2 = 3.233(4)$

Ising percolation

$$-\mathcal{H}_{Ising} = \frac{1}{T}\sum_{\langle ij\rangle}\sigma_i\sigma_j + H\sum_i \sigma_i\,, \qquad \sigma_i = \pm 1$$





$P \equiv$ probability that a site belongs to an infinite cluster of + spins

spontaneous magnetization implies $P > 0$ (Coniglio et al '77)

$(T, H) = (\infty, H_0)$: random percolation fixed point, $\frac{e^{H_0}}{e^{H_0} + e^{-H_0}} = p_c$

$(T, H) = (T_c, 0)$: Ising percolation fixed point

Fortuin-Kasteleyn trick: couple to auxiliary Potts spins

$$\mathcal{H}_q = \mathcal{H}_{Ising} - J \sum_{\langle ij \rangle} t_i t_j \left(\delta_{s_i,s_j} - 1\right), \qquad t_i = \frac{1}{2}(\sigma_i + 1) = 0, 1, \qquad s_i = 1, \ldots, q$$

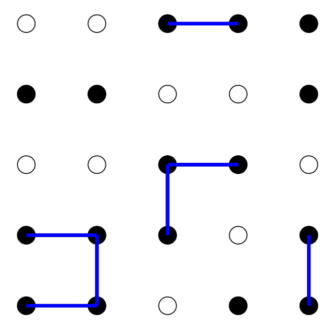
$$\begin{aligned} Z_q &= \sum_{\{t_i\}} \sum_{\{s_i\}} e^{-\mathcal{H}_q} = \\ &= \sum_{\{t_i\}} e^{-\mathcal{H}_{Ising}} q^{\#\, vacancies} \sum_{\{bonds\, between\, +\, spins\}} q^{\#\, clusters}\, p^{\#\, bonds}\, (1-p)^{\#\, absent\, bonds} \end{aligned}$$

$p \equiv 1 - e^{-J}$ bond occupation probability

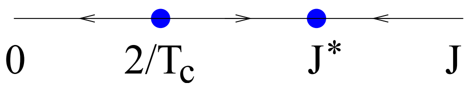
$q \to 1$ eliminates $q^{\#\, vacancies} q^{\#\, clusters}$

$P = \partial_q \,(\text{Potts magnetization})|_{q=1}$

spin clusters for $p = 1$



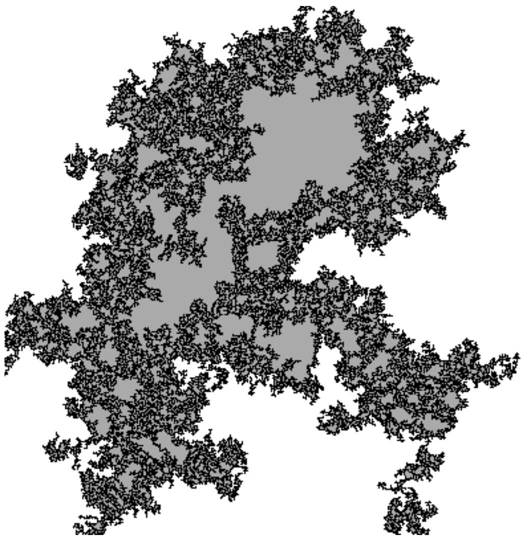
RG analysis at $T_c, H = 0$ (Coniglio, Klein, '80)



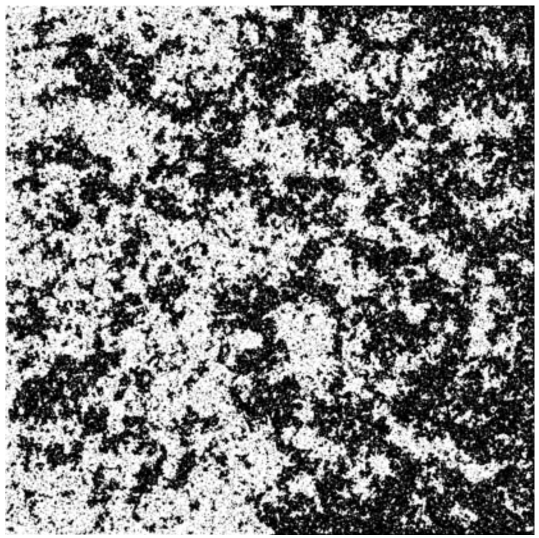
$J = J^*$: fixed point for spin clusters

$J = 2/T_c$: fixed point for FK clusters (or droplets) cluster size $\sim$ (linear extension)^D $\qquad D = d - X_s$ fractal dimension

$$\text{CFT}: \qquad D = \begin{cases} 91/48 = 1.89.. & \text{random percolation clusters} \\ 187/96 = 1.94.. & \text{Ising spin clusters} \\ 15/8 = 1.87.. & \text{Ising FK clusters} \end{cases}$$

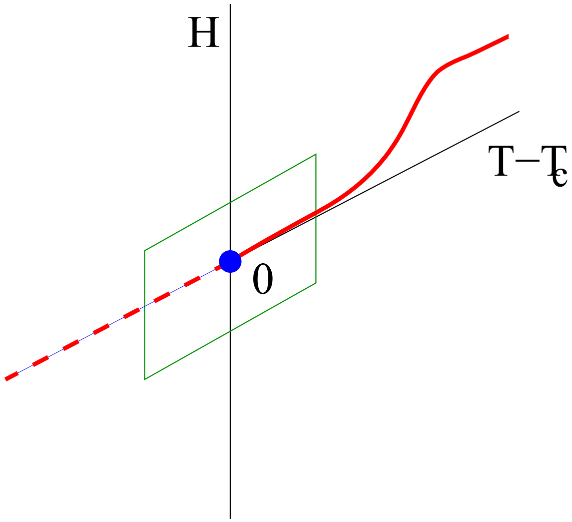


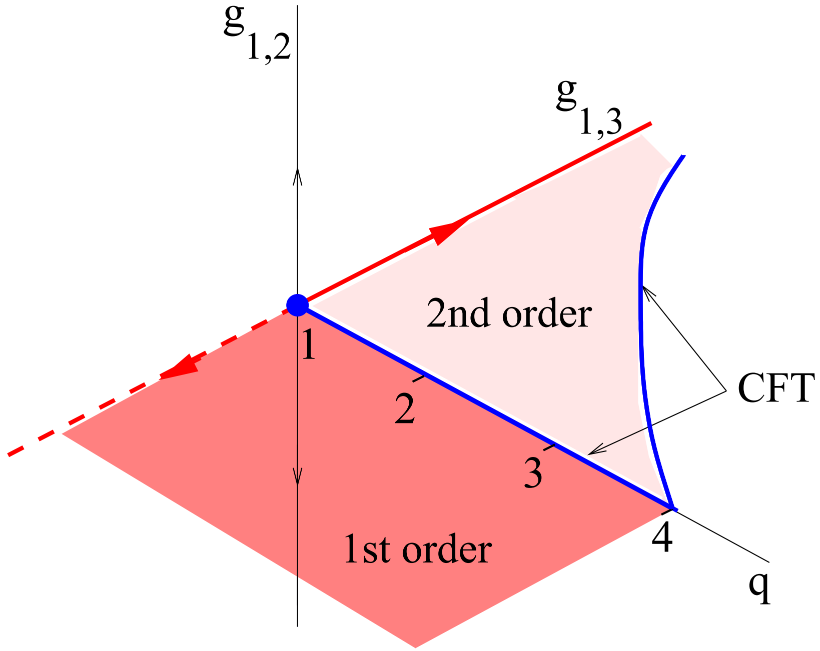
random



Ising spin

Field theory of Ising spin clusters (GD, '09)





1,2 and 1,3 CFT perturbations are integrable

Particles on 2nd order surface :

- $q = 1 + \epsilon$: ϵ massless, 1 massive with lifetime $\sim 1/\epsilon$ $\Longrightarrow$ $\xi_{perc} = \infty$
- $q = 1$: 0 massless, one stable massive $\Longrightarrow$ $\xi_{magn} < \infty$

Canonical percolative observables :

order parameter	$P \simeq B(p - p_c)^{\beta}$
mean cluster size	$S \simeq \Gamma^{\pm}\lvert p - p_c\rvert^{-\gamma}$
correlation length	$\xi \simeq f^{\pm}\lvert p - p_c\rvert^{-\nu}$
mean cluster number	$\frac{\langle N_c \rangle}{N} \simeq A^{\pm}\lvert p - p_c\rvert^{2-\alpha}$

Universal amplitude ratios for Ising clusters (GD, Viti, '10)

	spin clusters	FK clusters
Γ_a/Γ_b^+	not defined	40.3
$f_{2nd,a}/f_a$	"	0.99959..
f_a/f_b^+	"	2
$f_a/\hat{f}_a$	"	1
$A_{k,a}/A_{k,b}^+; k=0,-1$	"	1
Γ_b^+/Γ_b^-	-	1
f_b^+/f_b^-	1/2	1
$f_{2nd,b}^-/f_b^-$	0.6799	0.61
$f_{2nd,b}^+/f_{2nd,b}^-$	-	1
$f_b^+/\hat{f}_b^\pm$	1/2	1
U_b	24.72	15.2
$A_{k,b}^+/A_{k,b}^-; k=0,-1$	1	1
$A_{0,b}^\pm/A_{-1,b}^\pm$	$-\gamma-\ln\pi=-1.7219..$	$-\gamma-\ln\pi=-1.7219..$
r_b	$\frac{3\sqrt{3}(\gamma+\ln\pi)}{64\pi^2}=0.014165..$	$-\frac{\gamma+\ln\pi}{12\pi^2}=-0.014539..$
f_c^+/f_c^-	1/2	-
$f_{2nd,c}^-/f_c^-$	1.002	-
$f_c^+/\hat{f}_c^\pm$	$\sin\frac{\pi}{5}=0.58778..$	-
$A_{k,c}^+/A_{k,c}^-; k=0,1$	1	-
$A_{0,c}^\pm/A_{1,c}^\pm$	$-0.42883..$	-
r_c	$-3.7624..\times10^{-3}$	-

$\gamma=0.5772..$ Euler-Mascheroni constant

Universal amplitude ratios for random percolation (GD, Viti, Cardy, '10)

	Field Theory	Lattice
A^-/A^+	1	1^a
f^-/f^+	2	-
f^-_{2nd}/f^-	1.001	-
f^-_{2nd}/f^+_{2nd}	3.73	4.0 ± 0.5^c
$4B^2(f^-_{2nd})^2/\Gamma^-$	2.22	2.23 ± 0.10^d
$(-80/27\, A^-)^{1/2} f^-_{2nd}$	0.926	$\approx 0.93^{a+b}$
Γ^-/Γ^+	160.2	162.5 ± 2^e

[a] Domb, Pearce, '76

[b] Aharony, Stauffer, '97

[c] Corsten, Jan, Jerrard, '89

[d] Daboul, Aharony, Stauffer, '00

[e] Jensen, Ziff, '06

30 years of efforts by the lattice community, Γ^-/Γ^+ most elusive

Numerical results for Γ^-/Γ^+ in random percolation (from Ziff, '11)

year	author	system, method	Γ^-/Γ^+
1976	Sykes, Gaunt, Glen	lattice, series (12-20 order)	1.3-2.0
1976	Stauffer	lattice, series analysis	≈ 100
1978	Nakanishi, Stanley	lattice, MC	25(10)
1978	Wolff, Stauffer	lattice, series, fit to gaussian	180(36)
1979	Hoshen et al	lattice, MC	196(40)
1980	Nakanishi, Stanley	lattice, MC (reanalyze)	219(25)
1981	Gawlinsky, Stanley	overlapping disks, MC	50(26)
1985	Rushton, Family, Herrmann	additive polymerization, MC	140(45)
1987	Meir	lattice, series	210(10)
1987	Kim, Herrmann, Landau	continuum model, MC	14(10)
1987	Nakanishi	AB percolation, MC	139(24)
1988	Balberg	widthless sticks, MC	≈ 3
1988	Ottavi	approx. theory (gaussian fit)	193.9
1989	Corsten, Jan, Jerrard	lattice, MC	75(+40, -25)
1990	S. B. Lee, Torquato	penetrable conc. shell	1050(32)
1990	S. B. Lee	disks, MC	192(20)
1991	Hund	random contour model, MC	≈ 200
1993	Zhang, De'Bell	Penrose quasi-lattice, series	310(60)
1995	Conway, Guttmann	lattice, series (26-33 order)	$45(+20, -10)$
1996	S. B. Lee	penetrable conc. shell, disks	175(50)
1997	S. B. Lee, Jeon	kinetic gelation, MC	170(20)
2006	Jensen, Ziff	lattice, MC, series	162.5(2.0)

Insight into CFT of percolative fixed points (GD, Viti, '10)

Percolative fixed points are not standard CFT's (auxiliary fields not in the Kac's table). Little is known for bulk correlations apart from exponents

$P_n(x_1,\dots,x_n) \;\equiv\;$ probability $x_1,\dots,x_n$ in the same cluster

$$R = \frac{P_3(x_1,x_2,x_3)}{\sqrt{P_2(x_1,x_2)P_2(x_1,x_3)P_2(x_2,x_3)}}$$ universal constant at a fixed point

$$\begin{cases} P_2(x_1,x_2) = \text{lim}_{q\to 1}\,\langle\phi(x_1)\phi(x_2)\rangle \\ \\ P_3(x_1,x_2,x_3) = \text{lim}_{q\to 1}\,\sqrt{2}\,\langle\phi(x_1)\phi(x_2)\phi(x_3)\rangle \end{cases}$$

$\phi = \frac{\mu+\bar\mu}{\sqrt{2}}$, $\mu =$ Potts kink field

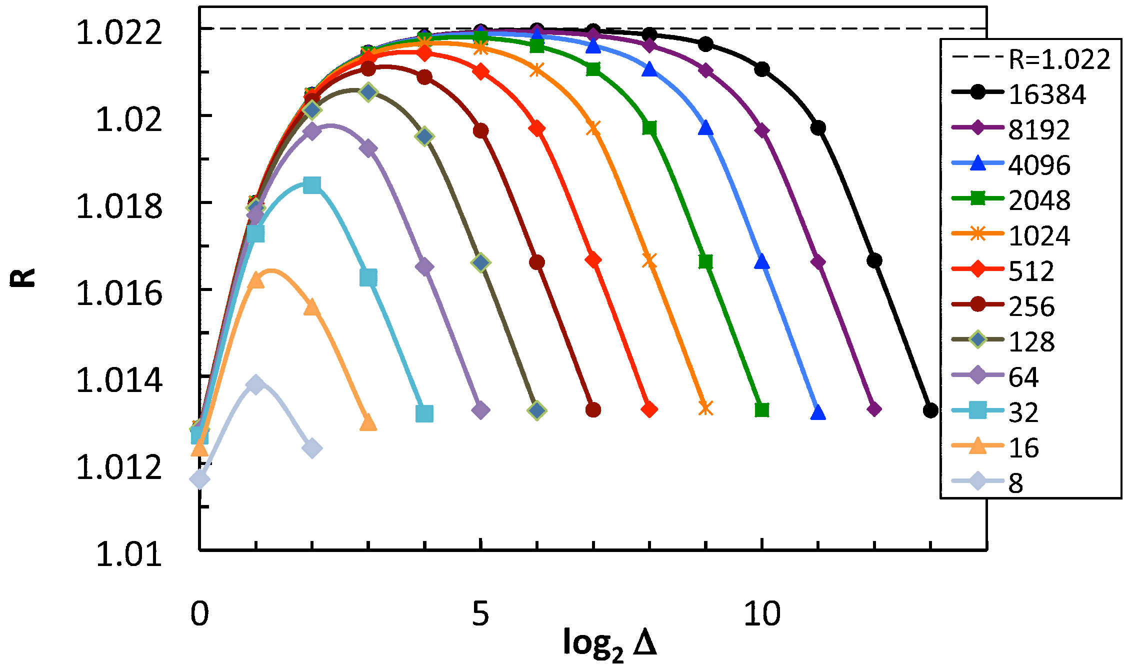
$$R = \sqrt{2}\lim_{X_\phi\to X_\mu} C_{\phi\phi\phi}$$

$C_{\phi\phi\phi}$ from "analytic continuation" of minimal models (Al.Zam, '05)

R	fixed point
1.0220..	random percolation
1.3767..	Ising spin clusters
1.0524..	Ising FK clusters

Monte Carlo determination for random percolation (Ziff, Simmons, Kleban, '10)

Equilateral triangle of side Δ on $L \times L$ lattice with periodic b.c. at p_c

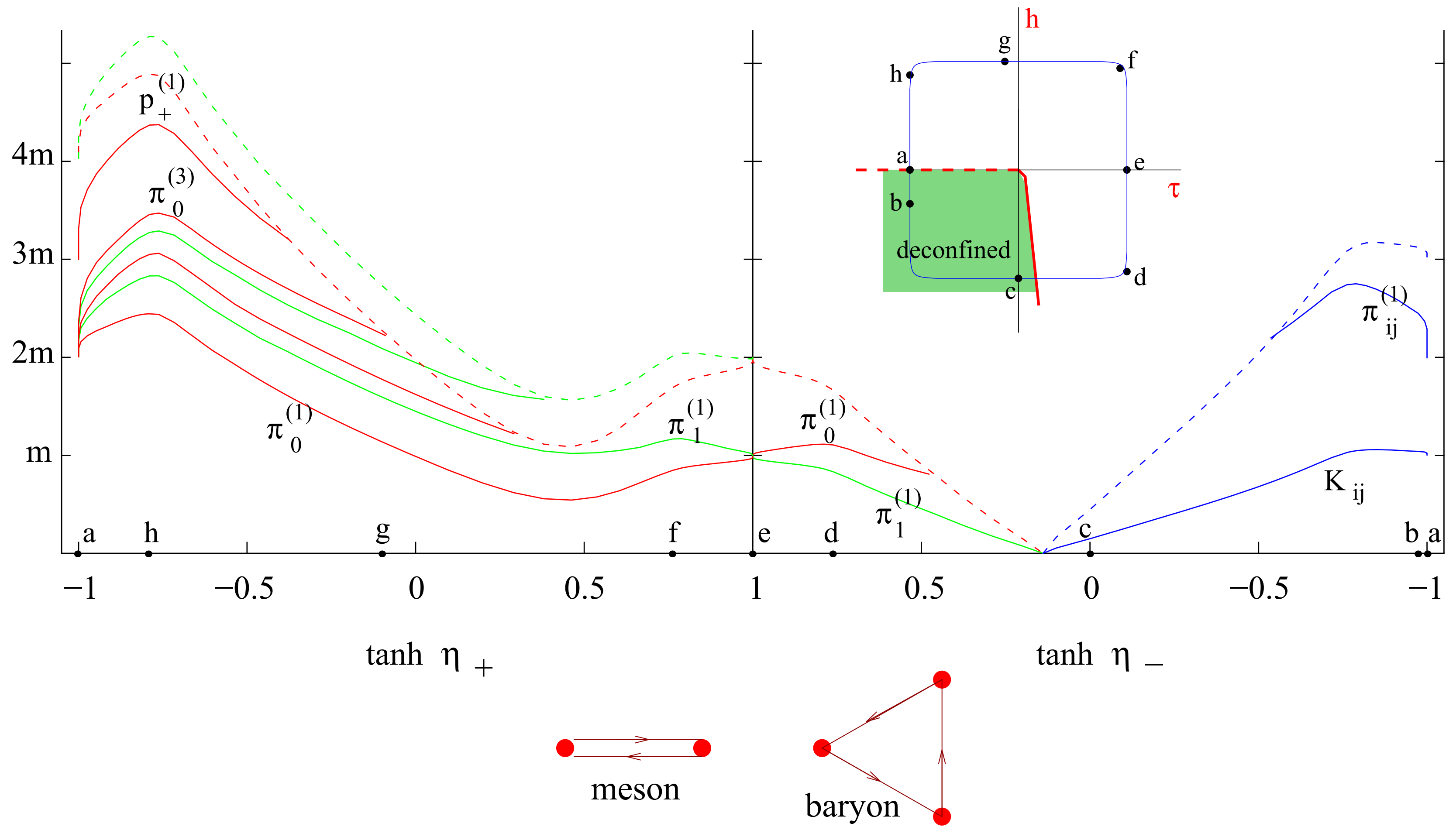


Confinement in the scaling p-state Potts model

(GD, Grinza, '08; Lepori, Toth, GD, '09)

$$\mathcal{H}_{Potts} = -\frac{1}{T}\sum_{\langle ij\rangle}\delta_{s_i,s_j} - H\sum_i \delta_{s_i,p}\,, \qquad s_i = 1,\ldots,p$$

$p = 3:$



- Study of weakly confined mesons in Rutkevich, '10