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2239-11

Workshop on Integrability and its Breaking in Strongly Correlated and Disordered Systems

23 - 27 May 2011

Integrability and its Breaking in two problems from ultracold physics

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‘Integrability and its Breaking’ in two problems from ultracold atomic physics

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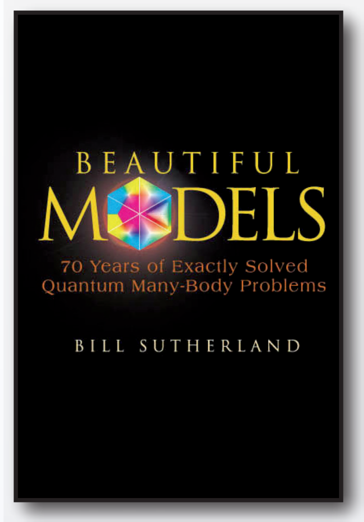
May 27th, 2011
Trieste Integrability Workshop

arXiv:1104.0655

What is integrability?

For a recent discussion see J.-S. Caux, J. Mossel, J. Stat. Mech. (2011)



Integrability $\equiv$ Non-diffractive scattering

- Scattering of two particles $p_1, p_2 \longrightarrow p_1', p_2'$ satisfies

$$p_1 + p_2 = p_1' + p_2'$$
$$\frac{p_1^2}{2m_1} + \frac{p_2^2}{2m_2} = \frac{p_1'^2}{2m_1} + \frac{p_2'^2}{2m_2}$$

- For equal masses two solutions are

$$p_1' = p_1,\ p_2' = p_2\ (retain) \text{ or } p_1' = p_2,\ p_2' = p_1\ (swap)$$

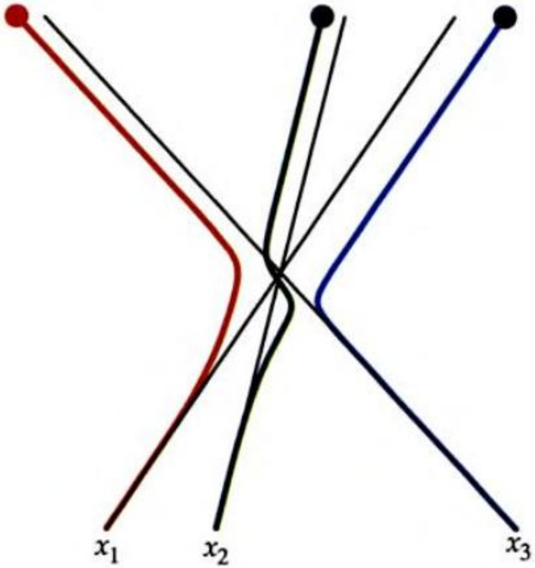
Integrability $\equiv$ Non-diffractive scattering

- Scattering of three particles (equal masses)

$$p_1 + p_2 + p_3 = p_1' + p_2' + p_3'$$
$$p_1^2 + p_2^2 + p_3^2 = p_1'^2 + p_2'^2 + p_3'^2$$

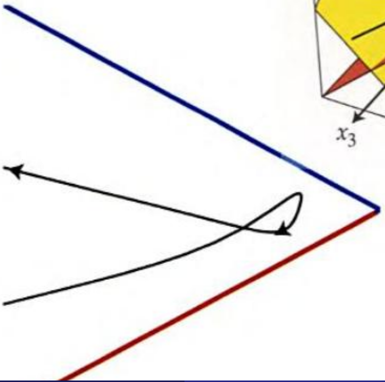
- Solution not unique!

Three body scattering



Three body scattering

Fig. 4.5



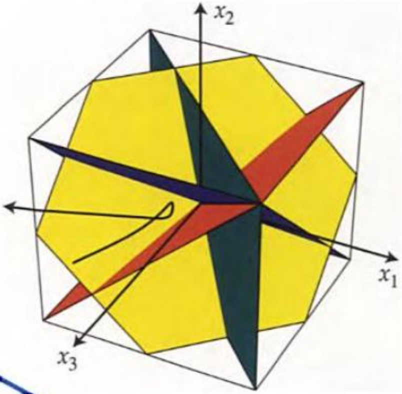
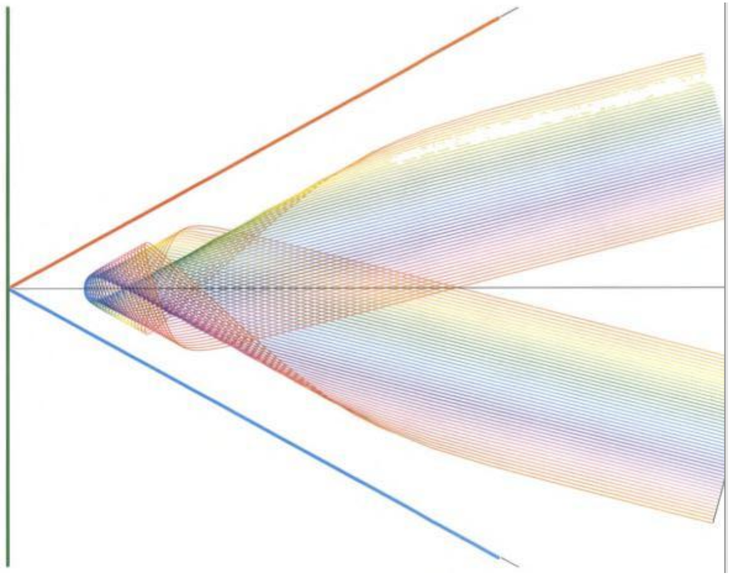
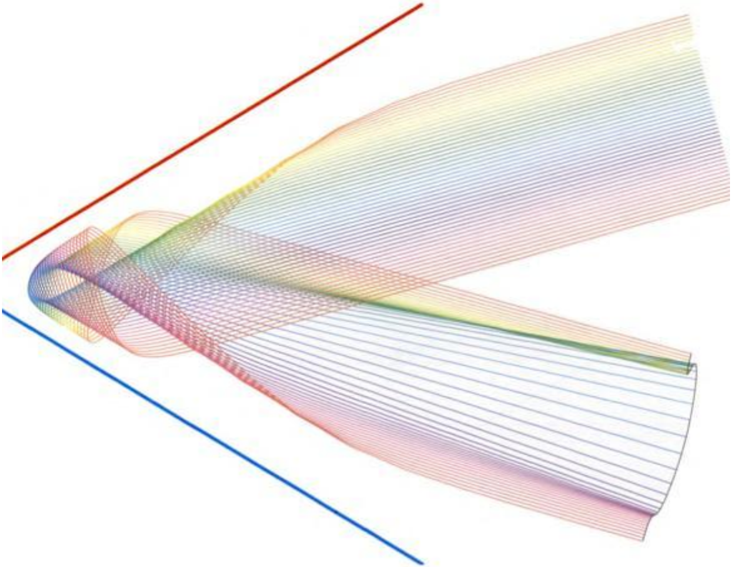


Fig. 4.6

Diffractive *vs.* Non-diffractive scattering



Diffractive *vs.* Non-diffractive scattering



Outline

1. Breaking integrability in a simple model
   - Integrability and transport
   - Consequences for mobility of an impurity in a 1D Fermi gas

2. Noise correlations in an expanding 1D Bose gas
   - The Hanbury Brown Twiss effect
   - Tracy–Widom form of the N-particle propagator
   - Effect of interactions on the Hanbury Brown and Twiss effect

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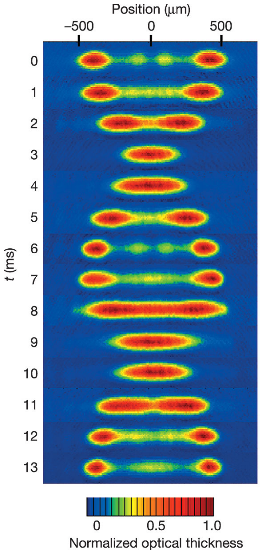
Integrability and transport



- Within Boltzmann picture $f(p, x, t)$ is not altered by 2-body collisions
- 3-body collisions *can* lead to relaxation (thermalization) if non-diffractive

Formalizing the relationship...

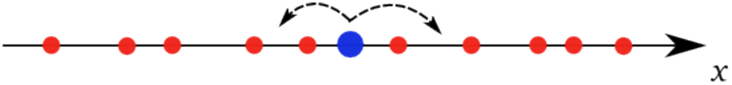
Integrability conjecture (Castella, Zotos, and Prelovšek, 1995)

A finite Drude weight at $T \neq 0$ is a generic property of integrable systems

Subtleties e.g. for Heisenberg model (Sirker, Pereira, and Affleck, 2009)

Breaking integrability: a simple model

- 'Impurity' of mass M in gas of fermions of mass m
- Interaction $H_{\text{int}} = V \sum_i \delta(x_i - X)$



- For $M = m$ problem integrable by Bethe ansatz (Yang–Gaudin, 1967)
- Solved by McGuire (1965) by 'baby' (nested) Bethe ansatz

- Finite Drude weight at $T > 0$ (Castella, Zotos, and Prelovšek, 1995)

Equal masses

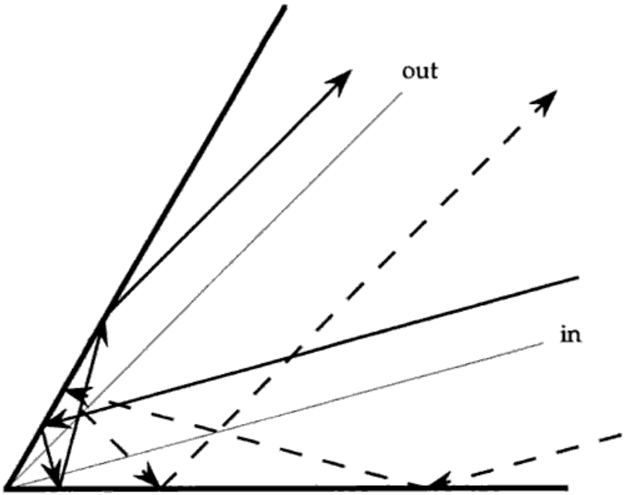


Fig. 7.1 Two rays scattering from a $\pi/3$ wedge of mirrors.

Unequal masses

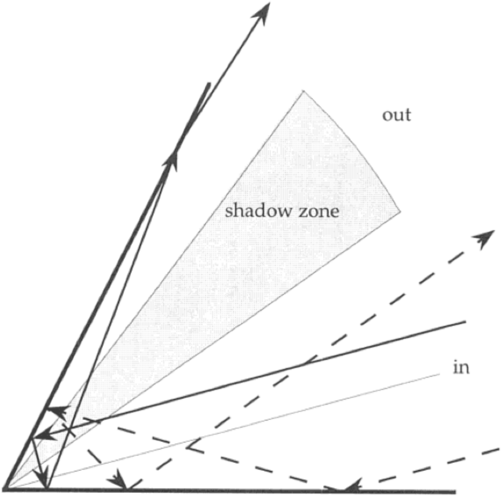
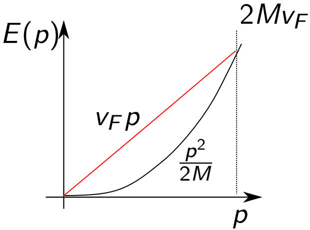


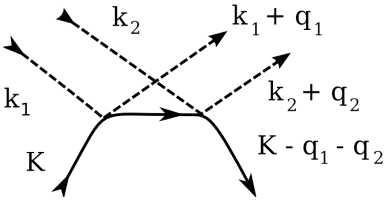
Fig. 7.2 Two rays scattering from a wedge of mirrors which meet at an angle $\theta = \pi/3 - \Delta\theta$ slightly less than $\pi/3$. Note that the two outgoing rays now diverge by an angle $6\Delta\theta$, giving a small wedge of darkness, and hence implying diffraction.

Kinematics at low temperatures

- Impurity + 1 particle processes frozen out for $k_B T < 2Mv_F$

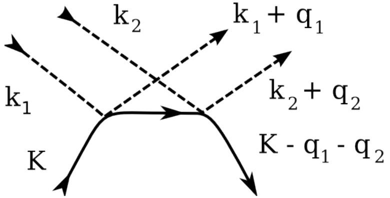


- Low temperature kinetics determined by *two-particle* processes



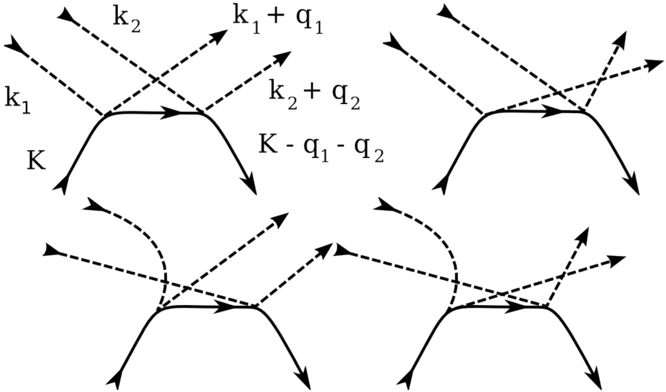
(Castro-Neto and Fisher, 1996)

Perturbative calculation – 3-body scattering



$$\mathcal{T}^{(2)} = i\left(\frac{V}{L}\right)^2 \frac{1}{\xi_{k_1+q_1} - \xi_{k_1} + \epsilon_{K-q_1} - \epsilon_K}$$

Perturbative calculation – 3-body scattering



$$\mathcal{T}^{(2)} = i\left(\frac{V}{L}\right)^2 \left[\frac{1}{\xi_{k_1+q_1} - \xi_{k_1} + \epsilon_{K-q_1} - \epsilon_K} - \frac{1}{\xi_{k_2+q_2} - \xi_{k_1} + \epsilon_{K-k_2-q_2+k_1} - \epsilon_K} + \frac{1}{\xi_{k_2+q_2} - \xi_{k_2} + \epsilon_{K-q_2} - \epsilon_K} - \frac{1}{\xi_{k_1+q_1} - \xi_{k_2} + \epsilon_{K-k_1-q_1+k_2} - \epsilon_K}\right].$$

Implications for impurity mobility

- Momentum relaxation rate of the impurity

$$\tau_{\mathrm{mom}}^{-1} = \frac{2\pi}{\hbar M T} \sum_{k_1,k_2,q_1,q_2} (q_1+q_2)^2 |\mathcal{T}^{(2)}|^2 \delta(E_i - E_f) \\ \times n_{k_1} n_{k_2} \left(1 - n_{k_1+q_1}\right)\left(1 - n_{k_2+q_2}\right)$$

vanishes as T^4, leading to a low temperature mobility

$$\mu = \tau_{\mathrm{mom}}/M \propto T^{-4}$$

- Matrix element vanishes for $M = m \longrightarrow$ *ballistic motion*
- Calculation to leading order in $M - m$ for any V using exact solution.

The view from LuttingerLand

PRL **102,** 070402 (2009) PHYSICAL REVIEW LETTERS week ending 20 FEBRUARY 2009

Bloch Oscillations in a One-Dimensional Spinor Gas

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(Received 21 November 2008; revised manuscript received 19 January 2009; published 18 February 2009)

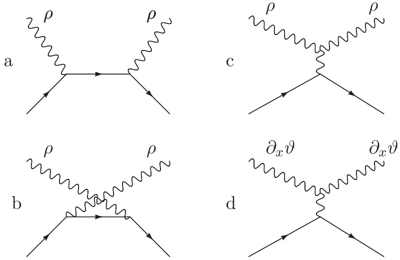

FIG. 2. Second-order diagrams contributing to two-phonon amplitudes. Spin excitation is represented by a full line, while phonons by wavy lines. Diagrams (a),(b) and (c) contribute to Γ_ρ, Eq. (12), while diagram (d) represents Γ_ϑ, Eq. (13).

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'Traditional' uses of integrability

$$H = -\frac{1}{2}\sum_{i=1}^{N}\frac{\partial^2}{\partial x_i^2} + c\sum_{i<j}\delta(x_i - x_j).$$

Lieb–Liniger gas (1963)

Two conceptual steps...

1. The 'Bethe ansatz'

$$\Psi_N(\mathbf{x}) = \sum_P a_P \exp\left(i\sum_{j=1}^{N} k_{P(j)}x_j\right)$$

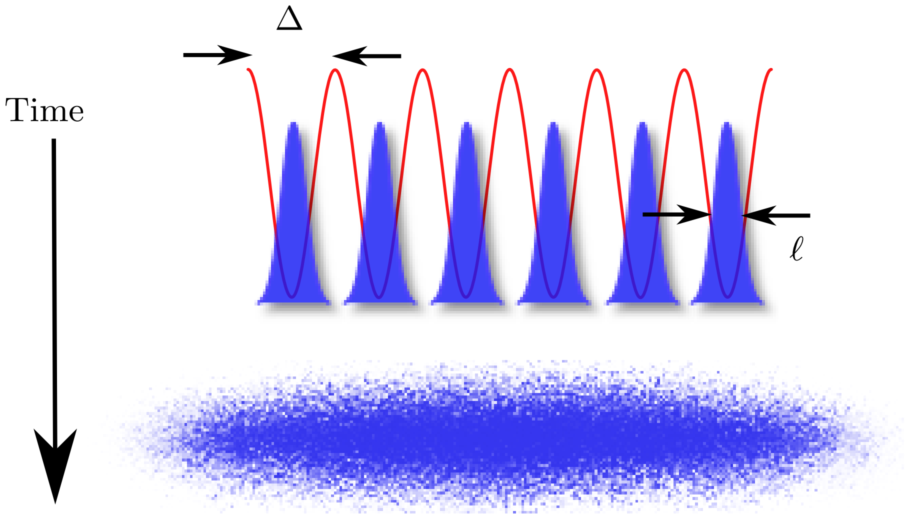
$$x_1 \leq x_2 \leq \cdots \leq x_N$$

2. Impose boundary conditions

$$k_j L = 2\pi n_j - 2\arctan\left(\frac{k_j - k_i}{c}\right)$$

Thermodynamic limit, form factors...

The setup



Ready availability of noise correlations in ultracold physics

PHYSICAL REVIEW A **70**, 013603 (2004)

Probing many-body states of ultracold atoms via noise correlations

Ehud Altman, Eugene Demler, and Mikhail D. Lukin
Physics Department, Harvard University, Cambridge, Massachusetts 02138, USA
(Received 10 June 2003; published 6 July 2004)

We propose to utilize density-density correlations in the image of an expanding gas cloud to probe complex many-body states of trapped ultracold atoms. In particular, we show how this technique can be used to detect superfluidity of fermionic gases and to study spin correlations of multicomponent atoms in optical lattices. The feasibility of the method is investigated by analysis of the relevant signal to noise ratio including experimental imperfections.

Hanbury Brown and Twiss

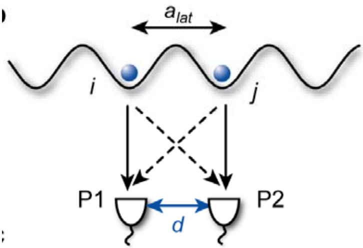
A TEST OF A NEW TYPE OF STELLAR INTERFEROMETER ON SIRIUS

By R. HANBURY BROWN
Jodrell Bank Experimental Station, University of Manchester

AND

DR. R. Q. TWISS
Services Electronics Research Laboratory, Baldock

Hanbury Brown and Twiss



The two-particle wavefunction is

$$\Psi_2(x_1, x_2; t) = \frac{1}{\sqrt{2}} \left[\varphi_i(x_1; t)\varphi_j(x_2; t) \pm \varphi_i(x_2; t)\varphi_j(x_1; t)\right]$$

Hanbury Brown and Twiss

For separation Δ between harmonic wells

$$\varphi_\alpha(y) = \frac{1}{(\pi\ell^2)^{1/4}} \exp\left[-\frac{(y-\alpha\Delta)^2}{2\ell^2}\right]$$

$$\varphi_\alpha(x; t \gg \ell^2) \to \sqrt{\frac{\ell}{i\sqrt{\pi}t}} \exp\left[\left(\frac{i}{2t} - \frac{\ell^2}{2t^2}\right)(x-\alpha\Delta)^2\right].$$

Probability density is then

$$|\Psi_2(x_1, x_2; t)|^2 \to \frac{\ell^2}{\pi t^2} e^{-\ell^2\left(\xi_1^2+\xi_2^2\right)} \left[1 \pm \cos\left([\xi_1 - \xi_2]\,\Delta\right)\right]$$

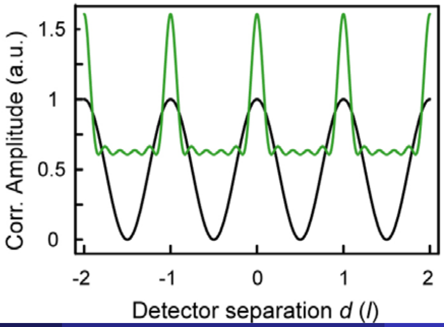
$\xi_{1,2} \equiv x_{1,2}/t$ and $e^{-\Delta^2/4\ell^2}$ neglected

Hanbury Brown and Twiss

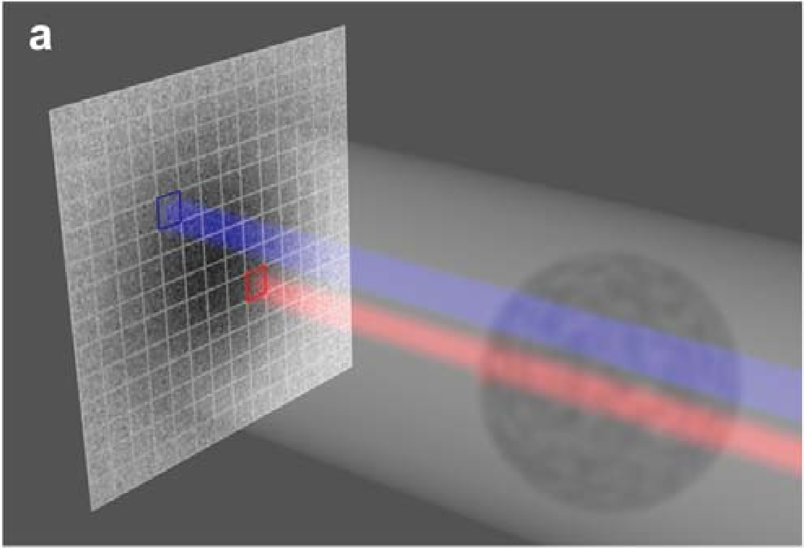
Array of N particles $\longrightarrow$ higher harmonics arising from pairs of particles separated by multiples of Δ

$$\mathcal{C}(x_1, x_2; t) \equiv \int dx_3 \cdots dx_N |\Psi_N(x_1, x_2, \ldots, x_N; t)|^2.$$

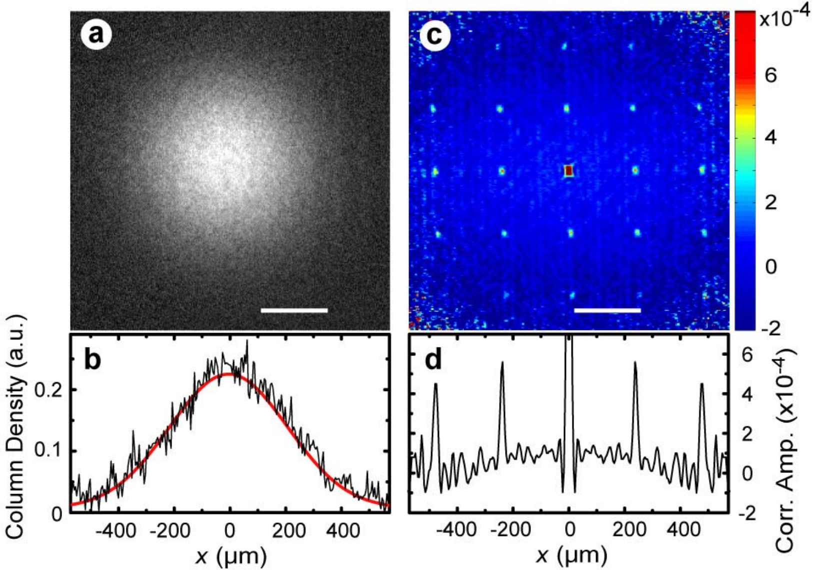
$$\to \frac{\ell^2}{\pi t^2} e^{-\ell^2(\xi_1^2+\xi_2^2)} \left[1 \pm \frac{2\pi}{N} \sum_{n=-\infty}^{\infty} \delta(\Delta[\xi_1 - \xi_2] - 2\pi n)\right]$$



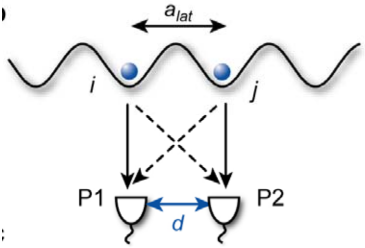
Expansion of Mott state (Fölling *et al.*, 2005)



Expansion of Mott state (Fölling *et al.*, 2005)



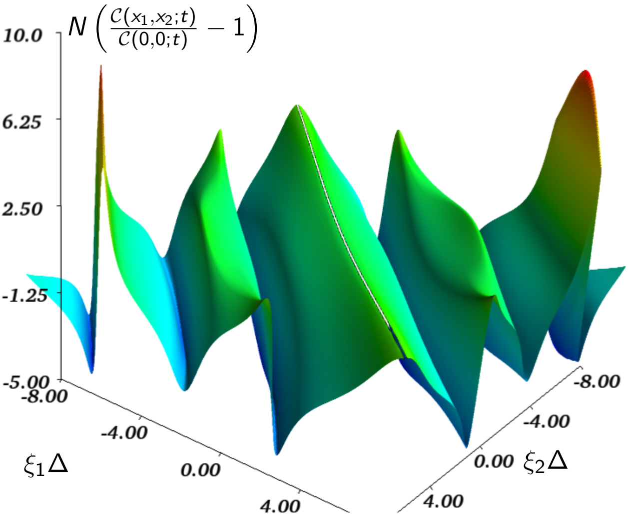
1D case: interactions must be important!



Trajectories must cross

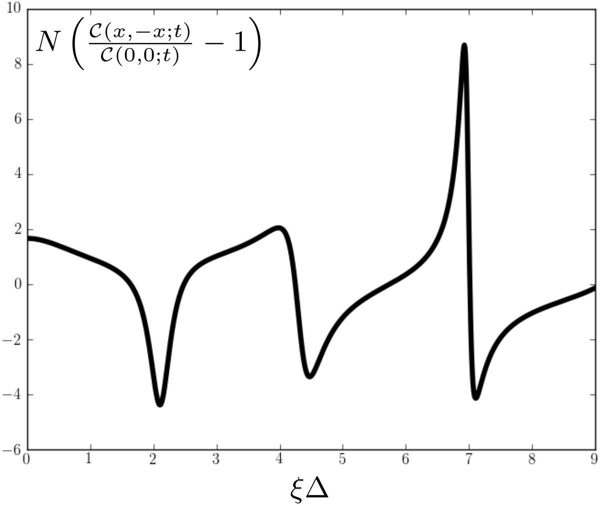
- For infinite repulsion have mapping to free fermions
- Crossover from *bosonic* to *fermionic* HBT effect

Correlations in the crossover



$c\Delta = 2,\ \ell/\Delta = 0.2$

Correlations in the crossover



$c\Delta = 2$, $\ell/\Delta = 0.2$. Slice with $x_1 = -x_2$

Correlations in the crossover

- Series of *Fano lineshapes*

$$\frac{\left[q_n\Gamma_n/2+(\varepsilon-\eta_n)\right]^2}{\Gamma_n^2/4+(\varepsilon-\eta_n)^2}$$

$\varepsilon=\Delta\left(\xi_1-\xi_2\right)-2\pi n$ is deviation from the n^{th} peak.

- The asymmetry parameter q_n is

$$\arg S(2\pi n/\Delta)=\frac{2q_n}{q_n^2-1}.$$

expressed in terms of two particle scattering matrix

$$S(k)=-\frac{c-ik}{c+ik},$$

- Evolution from $q_n=\infty$ for free bosons (resonance lineshape) to $q_n\to 0$ as $c\to\infty$ (antiresonance).

Tracy–Widom propagator (2008)

$$\mathcal{G}_N(\mathbf{x}|\mathbf{y};t) = \sum_{\sigma\in\mathcal{S}_N} \int \cdots \int A_\sigma \prod_{j=1}^{N} e^{ik_{\sigma(j)}(x_j - y_{\sigma(j)})} e^{-\frac{it}{2}\sum_j k_j^2} \frac{dk_1}{2\pi} \cdots \frac{dk_N}{2\pi}$$

$$A_\sigma = \prod \left\{ S(k_{\sigma(\alpha)} - k_{\sigma(\beta)}) : x_\alpha < x_\beta \text{ but } y_{\sigma(\alpha)} > y_{\sigma(\beta)} \right\}.$$

Convolve with initial (product) state

$$\Psi_N(\mathbf{x};t) = \frac{1}{\sqrt{N!}} \int \mathcal{G}_N(\mathbf{x},\mathbf{y};t) \prod_j \varphi_j(y_j) d\mathbf{y}$$

Still have to

1. Do the integrals
2. Sum over permutations

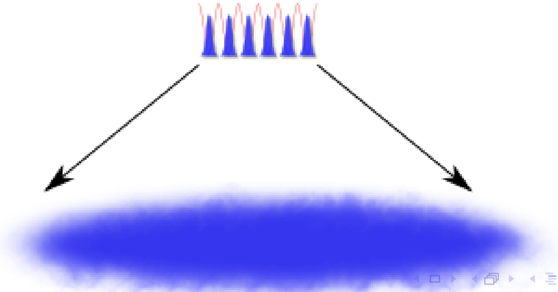
Going to the 'far field'

Stationary phase approximation at long times

$$\mathcal{G}_N(\mathbf{x}|\mathbf{y};t) \to \left(\frac{1}{2\pi i t}\right)^{N/2} \sum_{\sigma\in\mathcal{S}_N} A'_\sigma \prod_{j=1}^{N} e^{i\left(\frac{t}{2}\xi_j^2 - \xi_j y_{\sigma(j)}\right)}$$

$\xi_j = x_j/t$ and A'_σ denotes

$$A'_\sigma = \prod \left\{ S(\xi_\alpha - \xi_\beta) : x_\alpha < x_\beta \text{ but } y_{\sigma(\alpha)} > y_{\sigma(\beta)} \right\}.$$

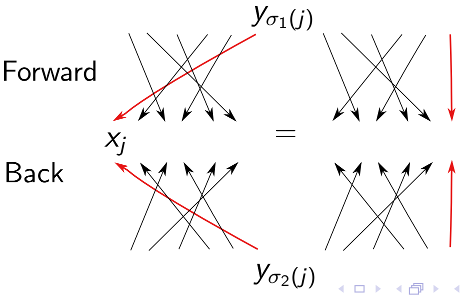


- Require the 'forward and back' propagator

$$\mathcal{G}_N(\mathbf{x}|\mathbf{y};t)\mathcal{G}^*_N(\mathbf{x}|\tilde{\mathbf{y}};t) \to \left(\frac{1}{2\pi t}\right)^N \sum_{\sigma_1,\sigma_2\in\mathcal{S}_N} A'_{\sigma_1}A'^*_{\sigma_2}\prod_j e^{-i\xi_j\left(y_{\sigma_1(j)}-\tilde{y}_{\sigma_2(j)}\right)}.$$

$$A'_{\sigma_1}A'^*_{\sigma_2} = \prod\left\{S(\xi_\alpha-\xi_\beta) : \sigma_1(\alpha)>\sigma_1(\beta) \text{ but } \sigma_2(\alpha)<\sigma_2(\beta)\right\}$$

- Form of the product does not depend upon the ordering of the $\{x_j\}$.



The Golden Rule

How to calculate the integrals

$$\mathcal{C}(x_1, x_2; t) \equiv \int dx_3 \cdots dx_N |\Psi_N(x_1, x_2, \ldots, x_N; t)|^2$$

Use the Golden Rule!

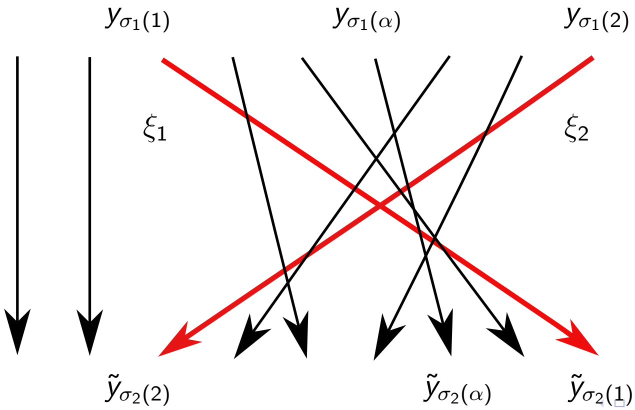
The Golden Rule

The Golden Rule

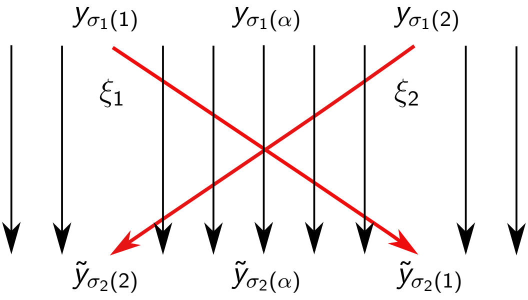
‘Never impose on others what you would not choose for yourself.’

Golden Rule (from analyticity of S matrix)

For those x_j we integrate over:
A particle moving to the left (right) must be overtaken by another particle moving to the left (right)



Usual HBT trajectories

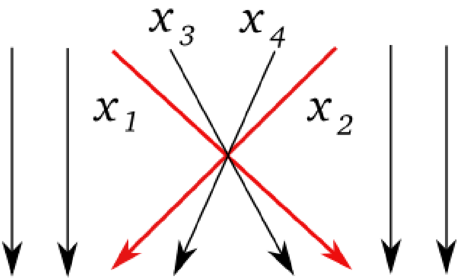


Integrate over $\{x_\alpha : \alpha \neq 1, 2\}$, convolve with Gaussians and sum series

$$\mathcal{C}(x_1, x_2 : t) \to \frac{\ell^2}{\pi t^2} e^{-\ell^2(\xi_1^2+\xi_2^2)} \left[1 + \frac{2}{N} \mathrm{Re}\left(\frac{S(\xi_2-\xi_1)e^{i\Delta(\xi_1-\xi_2)}}{1-e^{i\Delta(\xi_1-\xi_2)}\zeta(\xi_1,\xi_2)}\right)\right]$$

Fano with width $\Gamma_n = 2(1 - \mathrm{Re}\,\zeta) > 0$

What about 'non HBT' trajectories?



- Small in $e^{-2c\Delta}$, where $c\Delta = \gamma$ of equivalent LL gas
- In the same way, show that the power of $e^{-c\Delta}$ is at least twice the total number of moves to the right (or to the left).

Conclusions

1. Weak breaking of integrability can lead to anomalously large transport coefficients, calculable in a simple model
2. Integrability is useful simply at the level of the *scattering* problem in certain natural situations in ultracold physics
   - Example of HBT effect in interacting 1D Bose gas
   - Crossover from *bosonic* to *fermionic* HBT with increasing interaction