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2239-9

Workshop on Integrability and its Breaking in Strongly Correlated and Disordered Systems

23 - 27 May 2011

Generalized Thermalization in an Integrable Lattice System

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Outline

1. Introduction
   - Experiments in the 1D regime
   - Generalized Gibbs ensemble (GGE)
   - Unitary evolution and generalized thermalization
2. Generalized Thermalization
   - Exact theoretical approach at integrability
   - Time evolution, time average, and diagonal ensemble
   - Generalized microcanonical ensemble (GME)
   - Eigenstate expectation values and ETH
3. Summary

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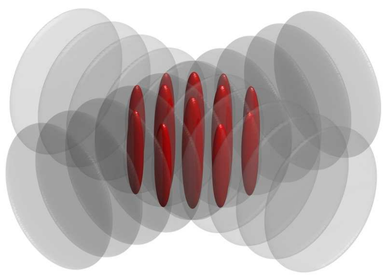
Experiments in the 1D regime



Girardeau '60

T. Kinoshita, T. Wenger, and D. S. Weiss, Science **305**, 1125 (2004).

T. Kinoshita, T. Wenger, and D. S. Weiss, Phys. Rev. Lett. **95**, 190406 (2005).

$\gamma_{eff} = \frac{m g_{1D}}{\hbar^2 \rho}$

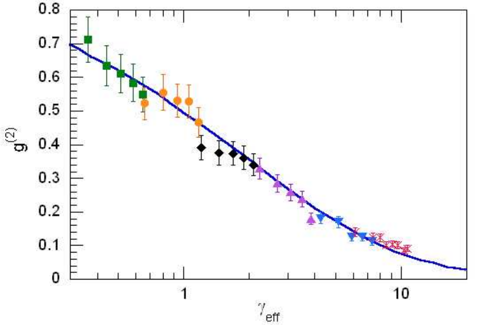
Effective one-dimensional δ potential

M. Olshanii, PRL **81**, 938 (1998).

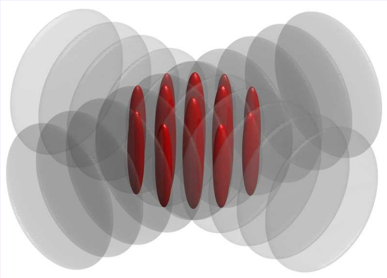
$$U_{1D}(x) = g_{1D}\delta(x)$$

where

$$g_{1D} = \frac{2\hbar a_s \omega_\perp}{1 - C a_s \sqrt{\frac{m\omega_\perp}{2\hbar}}}$$



Experiments in the 1D regime



Lieb, Schulz, and Mattis '61

B. Paredes *et al.*,

Nature (London) **429**, 277 (2004).

$n(x)$: Density distribution

$n(p)$: Momentum distribution

Effective one-dimensional δ potential

M. Olshanii, PRL **81**, 938 (1998).

$$U_{1D}(x) = g_{1D}\delta(x)$$

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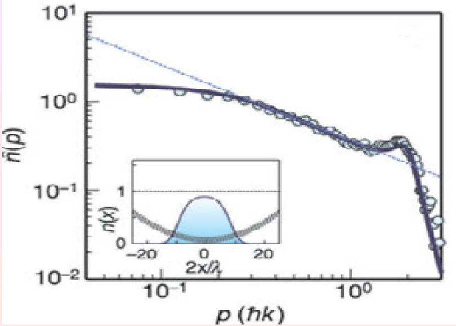
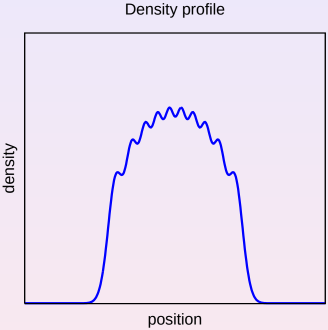
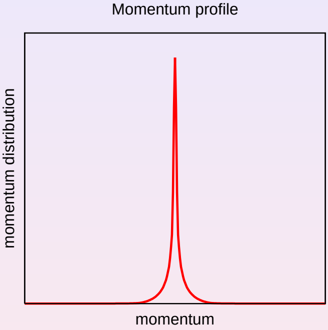


Absence of thermalization in 1D?



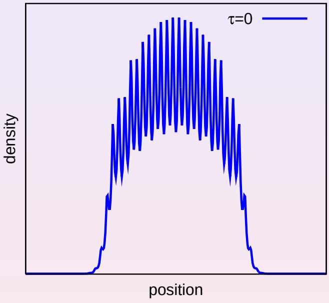


T. Kinoshita, T. Wenger, and D. S. Weiss, Nature **440**, 900 (2006).

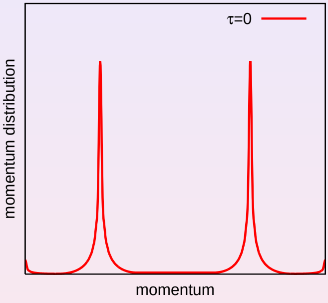
MR, A. Muramatsu, and M. Olshanii, Phys. Rev. A **74**, 053616 (2006).

Absence of thermalization in 1D?

Density profile



Momentum profile



T. Kinoshita, T. Wenger, and D. S. Weiss, Nature **440**, 900 (2006).

MR, A. Muramatsu, and M. Olshanii, Phys. Rev. A **74**, 053616 (2006).

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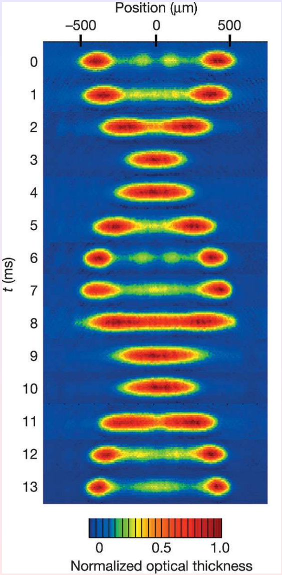
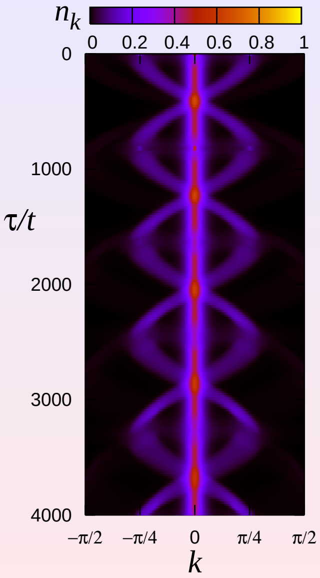
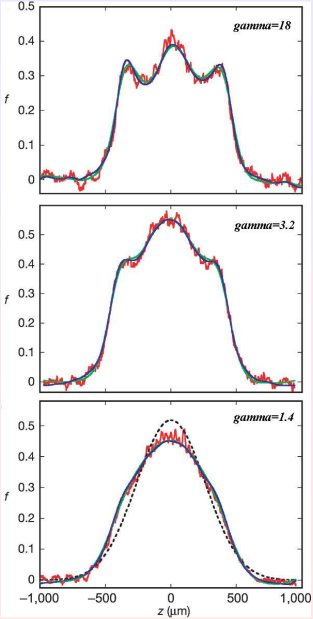
Experiment





Theory

Absence of thermalization in 1D?



$$\gamma = \frac{mg_{1D}}{\hbar^2 \rho}$$

g_{1D}: Interaction strength
ρ: One-dimensional density

If $\gamma \gg 1$ the system is in the strongly correlated Tonks-Girardeau regime

If $\gamma \ll 1$ the system is in the weakly interacting regime

Hofferberth, Lesanovsky, Fischer, Schumm, and Schmiedmayer, Nature **449**, 324 (2007).

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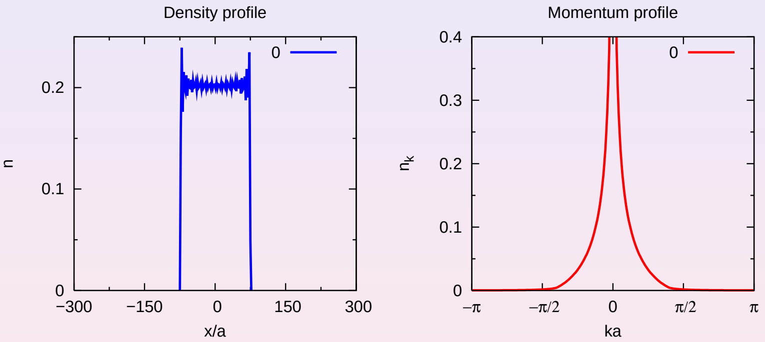
Relaxation dynamics in an integrable system



MR, V. Dunjko, V. Yurovsky, and M. Olshanii, PRL **98**, 050405 (2007).

Generalized Gibbs ensemble (GGE)

Thermal equilibrium

$$\hat{\rho} = Z^{-1} \exp\left[-\left(\hat{H} - \mu \hat{N}\right)/k_B T\right]$$

$$Z = \mathrm{Tr}\left\{\exp\left[-\left(\hat{H} - \mu \hat{N}\right)/k_B T\right]\right\}$$

$$E = \mathrm{Tr}\left\{\hat{H}\hat{\rho}\right\}, \quad N = \mathrm{Tr}\left\{\hat{N}\hat{\rho}\right\}$$

MR, PRA **72**, 063607 (2005).

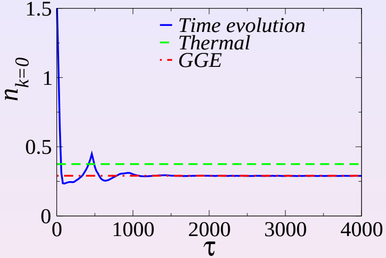
Generalized Gibbs ensemble

$$\hat{\rho}_c = Z_c^{-1} \exp\left[-\sum_m \lambda_m \hat{I}_m\right]$$

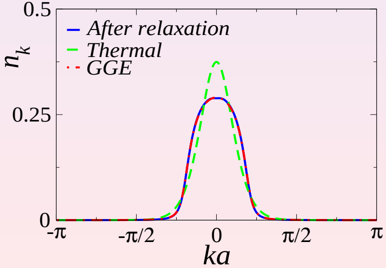
$$Z_c = \mathrm{Tr}\left\{\exp\left[-\sum_m \lambda_m \hat{I}_m\right]\right\}$$

$$\langle \hat{I}_m \rangle_{\tau=0} = \mathrm{Tr}\left\{\hat{I}_m \hat{\rho}_c\right\}$$

Evolution of $n_{k=0}$



n_k after relaxation



Statistical description after relaxation

Integrals of motion

(underlying noninteracting fermions)

$$\hat{H}_F \hat{\gamma}_m^{f\dagger}|0\rangle = E_m \hat{\gamma}_m^{f\dagger}|0\rangle$$
$$\left\{\hat{I}_m^f\right\} = \{\hat{\gamma}_m^{f\dagger}\hat{\gamma}_m^f\}$$

Lagrange multipliers

$$\lambda_m = \ln\left[\frac{1 - \langle\hat{I}_m\rangle_{\tau=0}}{\langle\hat{I}_m\rangle_{\tau=0}}\right]$$

Other examples in:

- M. A. Cazalilla, PRL **97**, 156403 (2006).
- P. Calabrese and J. Cardy, J. Stat. Mech.: Theory Exp., P06008 (2007).
- M. Cramer *et al.*, PRL **100**, 030602 (2008).
- T. Barthel and U. Schollwöck, PRL **100**, 100601 (2008).
- M. Eckstein and M. Kollar, PRL **100**, 120404 (2008).
- M. Kollar and M. Eckstein, PRA **78**, 013626 (2008).
- A. Flesch *et al.*, PRA **78**, 033608 (2008).
- A. Iucci and M. A. Cazalilla, PRA **80**, 063619 (2009).
- D. Fioretto and G. Mussardo, NJP **12**, 055015 (2010).
- J. Mossel and J.-S. Caux, NJP **12**, 055028 (2010).

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Exact results from quantum mechanics

If the initial state is not an eigenstate of $\widehat{H}$

$$|\psi_0\rangle \neq |\alpha\rangle \quad \text{where} \quad \widehat{H}|\alpha\rangle = E_\alpha|\alpha\rangle \quad \text{and} \quad E_0 = \langle\psi_0|\widehat{H}|\psi_0\rangle,$$

then a generic observable O will evolve in time following

$$O(\tau) \equiv \langle\psi(\tau)|\widehat{O}|\psi(\tau)\rangle \quad \text{where} \quad |\psi(\tau)\rangle = e^{-i\widehat{H}\tau}|\psi_0\rangle.$$

What is it that we call generalized thermalization?

$$\overline{O(\tau)} = O(I_1, \ldots, I_L).$$

One can rewrite

$$O(\tau) = \sum_{\alpha',\alpha} C^\star_{\alpha'} C_\alpha e^{i(E_{\alpha'} - E_\alpha)\tau} O_{\alpha'\alpha} \quad \text{where} \quad |\psi_0\rangle = \sum_\alpha C_\alpha |\alpha\rangle,$$

and taking the infinite time average (diagonal ensemble?)

$$\overline{O(\tau)} = \lim_{\tau\to\infty} \frac{1}{\tau}\int_0^\tau d\tau' \langle\Psi(\tau')|\hat{O}|\Psi(\tau')\rangle \stackrel{?}{=} \sum_\alpha |C_\alpha|^2 O_{\alpha\alpha} \equiv \langle\hat{O}\rangle_{\text{diag}},$$

which depends on the initial conditions through $C_\alpha = \langle\alpha|\psi_0\rangle$.

Outline

Bose-Fermi mapping

Hard-core boson Hamiltonian in an external potential

$$\hat{H} = -J \sum_i \left(\hat{b}_i^\dagger \hat{b}_{i+1} + \text{H.c.} \right) + \sum_i \mu_i \, \hat{n}_i$$

Constraints on the bosonic operators

$$\hat{b}_i^{\dagger 2} = \hat{b}_i^2 = 0$$

$$\Downarrow$$

Map to spins and then to fermions (Jordan-Wigner transformation)

$$\sigma_i^+ = \hat{f}_i^\dagger \prod_{\beta=1}^{i-1} e^{-i\pi \hat{f}_\beta^\dagger \hat{f}_\beta}, \quad \sigma_i^- = \prod_{\beta=1}^{i-1} e^{i\pi \hat{f}_\beta^\dagger \hat{f}_\beta} \hat{f}_i$$

$$\Downarrow$$

Non-interacting fermion Hamiltonian

$$\hat{H}_F = -J \sum_i \left(\hat{f}_i^\dagger \hat{f}_{i+1} + \text{H.c.} \right) + \sum_i \mu_i \, \hat{n}_i^f$$

One-particle density matrix

One-particle Green's function

$$G_{ij} = \langle \Psi_{HCB} | \sigma_i^- \sigma_j^+ | \Psi_{HCB} \rangle = \langle \Psi_F | \prod_{\beta=1}^{i-1} e^{i\pi \hat{f}_\beta^\dagger \hat{f}_\beta} \hat{f}_i \hat{f}_j^\dagger \prod_{\gamma=1}^{j-1} e^{-i\pi \hat{f}_\gamma^\dagger \hat{f}_\gamma} | \Psi_F \rangle$$

$$\Downarrow$$

Time evolution

$$|\Psi_F(\tau)\rangle = e^{-i\hat{H}_F \tau/\hbar} |\Psi_F^I\rangle = \prod_{\delta=1}^{N} \sum_{\sigma=1}^{L} P_{\sigma\delta}(\tau) \hat{f}_\sigma^\dagger \, |0\rangle$$

$$\Downarrow$$

Exact Green's function

$$G_{ij}(\tau) = \det \left[\left(\mathbf{P}^l(\tau) \right)^\dagger \mathbf{P}^r(\tau) \right]$$

Computation time $\sim L^2(L^2 N + L N^2 + N^3) \rightarrow$ study large systems

3000 lattice sites, 300 particles

MR and A. Muramatsu, PRL **93**, 230404 (2004); PRL **94**, 240403 (2005).

Outline

Hamiltonian and numerical calculations

Hard-core boson Hamiltonian

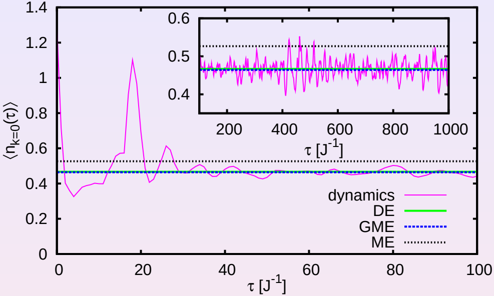
$$H = -J \sum_{i=1}^{L-1} \left(\hat{b}_i^\dagger \hat{b}_{i+1} + \text{H.c.} \right) + V(\tau) \sum_{i=1}^{L} (i - L/2)^2 \hat{n}_i$$

A. C. Cassidy, C. W. Clark, and MR, Phys. Rev. Lett. **106**, 140405 (2011).

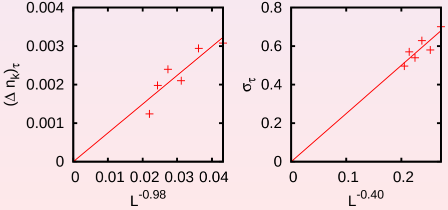
Numerical calculations

- $V(\tau < 0) = V_0$, $V(\tau \geq 0) = 0$, and $n = N/L = 0.2$.
- The initial state $|\Psi_0\rangle$ is the ground state for $\tau < 0$.
- All the many-body eigenstates are generated from single particle states ($L = 50$ and $N = 10 \Rightarrow 10^{10}$ states).
- For the microcanonical and canonical ensembles the usual weights are used.
- For the diagonal ensemble $C_\alpha = \det \left[\mathbf{P}_\alpha^\dagger \mathbf{P}_0 \right]$.

Evolution of $\langle \hat{n}_{k=0} \rangle$



Scaling



Error

$$(\Delta n_k)_\tau = \frac{\sum_k |\langle \hat{n}_k \rangle_{\text{DE}} - \overline{n_k}|}{\sum_k \langle \hat{n}_k \rangle_{\text{DE}}}$$

where

$$\overline{n_k} = \frac{1}{\tau_2 - \tau_1} \int_{\tau_1}^{\tau_2} d\tau \langle \hat{n}_k(\tau) \rangle$$

Fluctuations

$$\sigma_\tau = \sum_k \sqrt{\overline{n_k^2} - \overline{n_k}^2}$$

Also in: Kollar and Eckstein, PRA **78**, 013626 (2008).

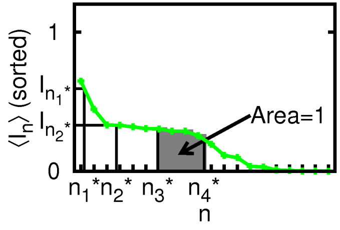
Outline

Generalized microcanonical ensemble (GME)

- The GGE is a grand-canonical ensemble (finite size effects). Computationally expensive to study microscopically
- GME idea, assign equal weight to all eigenstates whose values of the conserved quantities are close to a target distribution
- GME ingredients
  1. Ordered distribution of the conserved quantities in the initial state I_n
  2. Target distribution of the conserved quantities $\{I^*_{n^*_i} = 1\}$ in the initial state
  3. Distance between $\{n^\alpha_i\}$ in the many-body eigenstates and $\{n^*_i\}$

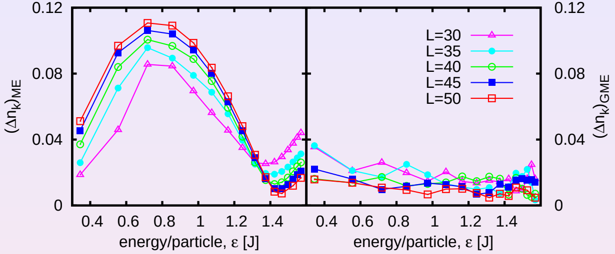
$$\delta^\alpha = \left[\frac{1}{N}\sum_{i=1}^{N} I_{n^*_i}(n^\alpha_i - n^*_i)^2\right]^{1/2}$$

Target distribution

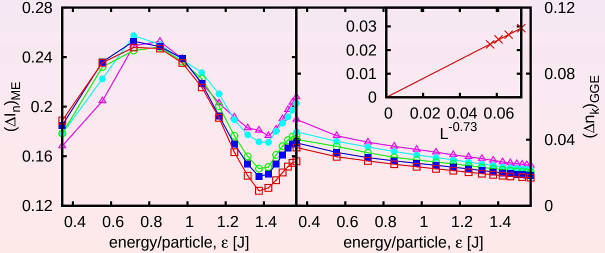


Finite size scaling

Results for Δn_k in the microcanonical ensembles



Integrals of motion and the GGE



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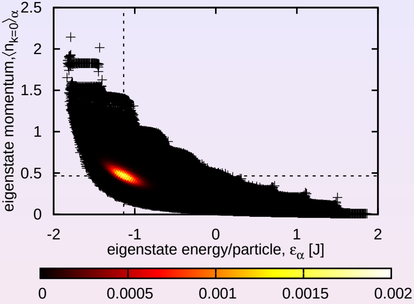
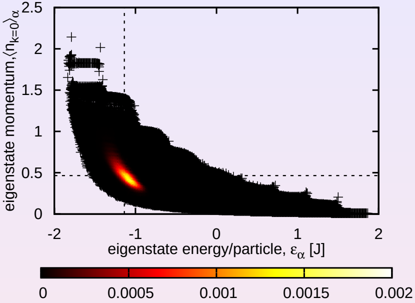
Eigenstate expectation values and scaling

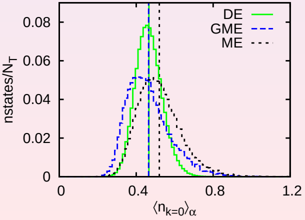
Weights in the diagonal ensemble



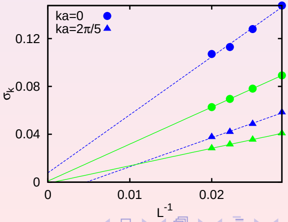
Weights in the GME



Histogram



Scaling of σ_k



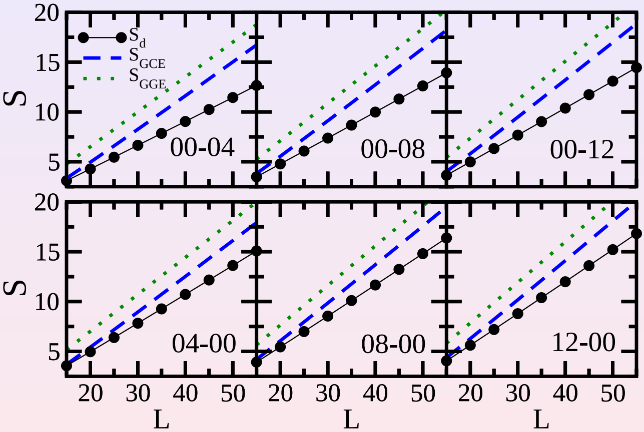
Diagonal Entropy

Quantum quenches in period-two superlattices

Definition of S_d

$$S_d = -\sum_n \rho_{nn} \ln(\rho_{nn})$$

A. Polkovnikov,
Ann. Phys. **326**, 486 (2011).



L. Santos, A. Polkovnikov,
and MR, arXiv:1103.0557.

We always find $S_{GGE} \sim S_{GCE} > S_d$, i.e.,
the generalized ensembles miss correlations present in the initial state

Eigenstate thermalization hypothesis

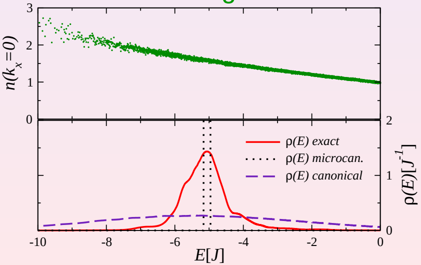
Eigenstate thermalization hypothesis (ETH)

[Deutsch, PRA **43** 2046 (1991); Srednicki, PRE **50**, 888 (1994); Rigol, Dunjko, and Olshanii, Nature **452**, 854 (2008).]

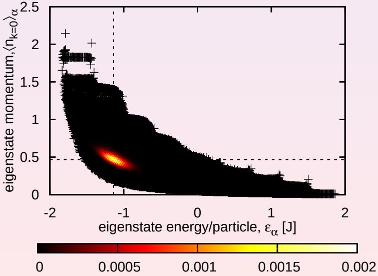
- The expectation value $\langle\alpha|\widehat{O}|\alpha\rangle$ of a few-body observable $\widehat{O}$ in an eigenstate of the Hamiltonian $|\alpha\rangle$, with energy E_α, of a many-body system equals the thermal average of $\widehat{O}$ at the mean energy E_α:

$$\langle\alpha|\widehat{O}|\alpha\rangle = \langle\widehat{O}\rangle_{\text{microcan.}}(E_\alpha)$$

Nonintegrable



Integrable



Summary

- Few-body observables in integrable systems undergo relaxation dynamics
  - ★ Recurrences occur but most of the time observables will be identical to the prediction of the diagonal ensemble
- After relaxation, observables can be described by generalized statistical ensembles (GGE and GME), which take into account the constraints imposed by the integrals of motion
  - ★ The number of constraints increases polynomially with system size while the Hilbert space increases exponentially with system size
- The validity of the “updated” ensembles has its origin in a generalized view of ETH, namely, the overwhelming majority of eigenstates of the Hamiltonian with similar integrals of motion have identical few-body observables
  - ★ Typicality and thermodynamics for isolated integrable systems?

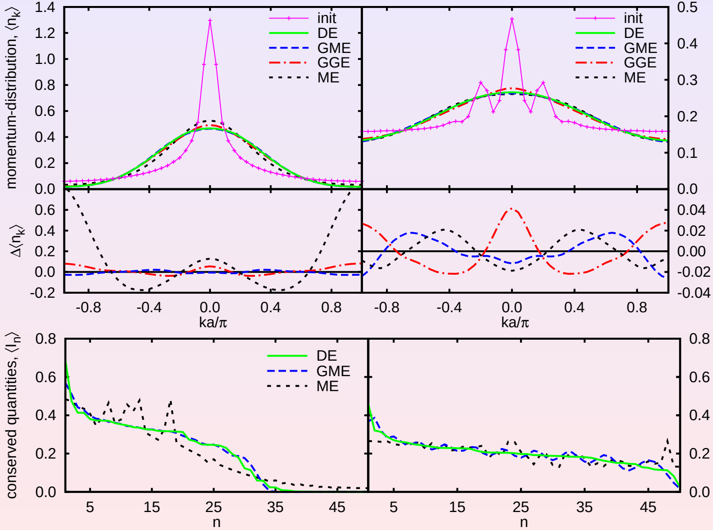
Collaborators

- Amy C. Cassidy (NIST)
- Charles W. Clark (NIST)
- Vanja Dunjko (U Mass Boston)
- Alejandro Muramatsu (U Stuttgart)
- Maxim Olshanii (U Mass Boston)
- Anatoli Polkovnikov (Boston U)
- Lea Santos (Yeshiva U)
- Vladimir Yurovsky (Tel Aviv U)

Supported by:

Statistical description after relaxation

Results for n_k and the integrals of motion



Additivity of the conserved quantities

Integrals of motion

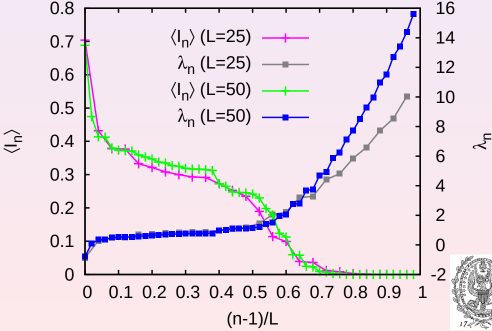
(underlying noninteracting fermions)

$$\hat{H}_F \hat{\gamma}_n^{f\dagger}|0\rangle = E_n \hat{\gamma}_n^{f\dagger}|0\rangle$$
$$\left\{\hat{I}_n^f\right\} = \{\hat{\gamma}_n^{f\dagger}\hat{\gamma}_n^f\}$$

Lagrange multipliers

$$\lambda_n = \ln\left[\frac{1-\langle\hat{I}_n\rangle_{\tau=0}}{\langle\hat{I}_n\rangle_{\tau=0}}\right]$$

$\langle\hat{I}_n\rangle_{\tau=0}$ and λ_n



$\langle\hat{I}_n\rangle$'s are not additive in a strict sense (their number increases with system size), but they can be thought as additive in a coarsed grained sense.

λ_n's are smooth in $\langle\hat{I}_n\rangle$'s