



**The Abdus Salam
International Centre for Theoretical Physics**



2240-Exercise

**Advanced School on Scaling Laws in Geophysics: Mechanical and
Thermal Processes in Geodynamics**

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Simple problems with channel flow

G. Houseman
Univ. of Leeds
UK

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Exercise: Channel Flow

In a plane-strain channel flow the stress balance equation:

$$\nabla \cdot \boldsymbol{\sigma} + \mathbf{f} = 0 \quad (1)$$

reduces to:

$$\frac{\partial \sigma_{xy}}{\partial x} + \frac{\partial p}{\partial y} = 0 \quad (2)$$

if we ignore gravity, assume 2D plane-strain, and consider a vertical conduit with side walls $x = \pm 1$.

1. Using
$$\sigma_{xy} = \eta \frac{\partial v}{\partial x} \quad (3)$$

assume that the viscosity and pressure gradient are constant, and integrate (2) to obtain an expression for $v(x)$. Determine the integration constants from boundary conditions that the velocity gradient is zero in the centre ($x = 0$), and the velocity is zero on the wall ($x = 1$). What is the velocity in the center of the channel ($x = 0$) if $\eta = 1$ and $\partial p / \partial y = 1$?

2. Using *sybil*, examine the numerical solution (*chan1*) for this test problem solved in a conduit which is 3 units high and 2 units wide (only the right half of it is shown, so $x = 0$ is the middle of the conduit) the right hand boundary is rigid, a pressure of +3 is applied to the base of the channel, and a pressure of zero is applied to the top boundary at $y = 3$ to obtain $\partial p / \partial y = 1$, gravity is set to zero and viscosity to 1 for this test problem.

- plot U_y and U_x
- plot the velocity arrows
- plot the shear strain rate (edxy)
- plot pressure (pres)
- plot the principal strain-rate orientations (Arrow → Strain → pstd)
- plot profiles of U_y vs x and edxy at half-height ($y = 1.5$)

How accurate is the numerical solution ? What is the cause of any inaccuracy ?

3. What if we have a pipe rather than a vertical layer ($x = 0$ is the axis of symmetry)? Use *sybil* to compare the solutions *chax1* (axisymmetric) and *chan1* (plane-strain). For axisymmetry, we interpret x as the radius r , and (2) is replaced by:

$$\frac{\partial(r \sigma_{ry})}{\partial r} + \frac{\partial(rp)}{\partial y} = 0 \quad (4)$$

Integrate (4) and compare result with answer to Q1. what is the velocity at the center of the pipe ?

4. Now suppose we have a non-Newtonian viscosity (strain-rate proportional to n th power of stress difference). We'll consider $n = 3$ and 5 , values that apply to rock deformation in some situations. We parameterize the viscosity using:

$$\eta = \frac{B}{2} \dot{E}^{(1-n)/n} \quad \text{where} \quad \dot{E} = \sqrt{\dot{\epsilon}_{ij} \dot{\epsilon}_{ij}} \quad (5)$$

The only component of strain-rate in the idealized problem is the vertical shear, so

$$\eta = 2^{-(n+1)/(2n)} B \left(\frac{\partial v}{\partial x} \right)^{(1-n)/n} \quad (6)$$

Substitute (6) into (2) (plane-strain) and integrate, what should the centre velocity be if $n = 3$ or $n = 5$? Compare with the *sybil* solutions (*chan3*, *chan5*). Note that the flow becomes more plug-like as n increases.

For 2D channel flow:

$$\frac{\partial \sigma_{xy}}{\partial x} + \frac{\partial p}{\partial y} = 0$$

$$\sigma_{xy} = \eta \frac{\partial v}{\partial x}$$

Therefore

$$\eta \frac{d^2 v}{dx^2} = - \frac{dp}{dy}$$

Therefore

$$\frac{d^2 v}{dx^2} = - \frac{1}{\eta} \frac{dp}{dy} = -C$$

$$\frac{dv}{dx} = -Cx \quad \text{where the integration constant is zero by zero shear at } x = 0.$$

Therefore

$$v = -\frac{1}{2}Cx^2 + C_1$$

and the boundary condition at $x = a$ gives

$$v = \frac{1}{2}C(a^2 - x^2)$$

With $a = 1$ and $C = 1$ in the test case,

$$v = \frac{1}{2}(1 - x^2)$$

At $x = 0$, v should be equal to 0.5

For axisymmetry:

$$\frac{\partial(r\sigma_{ry})}{\partial r} + \frac{\partial(rp)}{\partial y} = 0$$

or

$$\frac{d(r\sigma_{ry})}{dr} = -r \frac{dp}{dy}$$

Integrating:

$$r\sigma_{ry} = -\frac{1}{2}r^2 \frac{dp}{dy}$$

therefore, assuming as before that stress is zero at $r = 0$

$$\sigma_{ry} = \eta \frac{dv}{dr} = -\frac{1}{2} r \frac{dp}{dy}$$

Therefore

$$\frac{dv}{dr} = -\frac{1}{2} \frac{r}{\eta} \frac{dp}{dy}$$

therefore

$$v = -\frac{1}{4\eta} r^2 \frac{dp}{dy} = -\frac{1}{4} r^2 C + C_1$$

and using the boundary condition

$$v = \frac{C}{4} (a^2 - r^2)$$

with $a = C = 1$, at $r = 0$, $v = 0.25$.

Now the non-Newtonian problem, with 2D plane-strain

$$\frac{\partial \sigma_{xy}}{\partial x} + \frac{\partial p}{\partial y} = 0$$

$$\eta = 2^{(1-3n)/(2n)} B \left(\frac{\partial v}{\partial x} \right)^{(n-1)/n}$$

$$\sigma_{xy} = \eta \frac{\partial v}{\partial x}$$

The first integration is as before:

$$\eta \frac{dv}{dx} = -\frac{dp}{dy} x$$

Therefore:

$$2^{(1-3n)/(2n)} B \left(\frac{dv}{dx} \right)^{(n-1)/n} \frac{dv}{dx} = -\frac{dp}{dy} x$$

$$\left(\frac{dv}{dx} \right)^{(2n-1)/n} = -2^{(3n-1)/(2n)} \frac{1}{B} \frac{dp}{dy} x$$

Raising both sides to the power

$$\frac{dv}{dx} = -2^{(3n-1)/(4n-2)} \left[\frac{1}{B} \frac{dp}{dy} \right]^{\frac{n}{2n-1}} x^{\frac{n}{2n-1}}$$

and integrating

$$v = -2^{(3n-1)/(4n-2)} \frac{2n-1}{(3n-1)} \left[\frac{1}{B} \frac{dp}{dy} \right]^{\frac{n}{2n-1}} x^{\frac{3n-1}{2n-1}} + C_1$$

When $x = 1$, $v = 0$, so

$$v = 2^{(3n-1)/(4n-2)} \frac{2n-1}{(3n-1)} \left[\frac{1}{B} \frac{d p}{d y} \right]^{(2n-1)} \left[a^{\frac{3n-1}{(2n-1)}} - x^{\frac{3n-1}{(2n-1)}} \right]$$

When $C = 1$, $B = 1$, $a = 1$, then

$$v = 2^{(3n-1)/(4n-2)} \frac{2n-1}{(3n-1)} \left[1 - x^{\frac{3n-1}{(2n-1)}} \right]$$

when $n = 3$,

$$v = 2^{4/5} \frac{5}{8} \left[1 - x^{8/5} \right]$$

In the middle of the layer v should be = 1.088188

when $n = 5$,

$$v = 2^{7/9} \frac{9}{14} \left[1 - x^{14/9} \right]$$

In the middle of the layer v should be = 1.10217