The Abdus Salam International Centre for Theoretical Physics

United Nations Educational, Scientific and Cultural Organization

IAEA International Atomic Energy Agency

2240-1

Advanced School on Scaling Laws in Geophysics: Mechanical and Thermal Processes in Geodynamics

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What are scaling laws, why are they important in geodynamics, and what problems in geophysics make them attractive.

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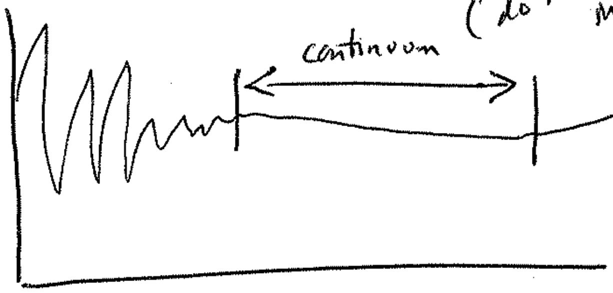
Continuum hypothesis

(do not describe individual atoms/molecules but their bulk properties)



Notation

a , $\underline{a}$, $\underline{\underline{a}}$

↑ scalar ↑ vector ↑ 2nd rank tensor

$\underline{U} = U_i$ (or $U_i \underline{e}_i$ ← unit vector in i direction)

vector calculus review

$$\underline{a} \cdot \underline{b} = a_i \overbrace{\underline{e}_i \cdot b_j \underline{e}_j}^{\delta_{ij}} \qquad \delta_{ij} = \begin{cases} 0 & i=j \\ 1 & i \neq j \end{cases}$$

$$= a_i b_i$$

$$= \underbrace{\sum_{i=1}^{3} a_i b_i}_{\text{Einstein summation convention}} = a_x b_x + a_y b_y + a_z b_z$$

$$\underline{\underline{I}} = \delta_{ij} \underline{e}_i \underline{e}_j$$

$$= \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$\underline{a} \wedge \underline{b} = a_i \underline{e}_i \wedge b_j \underline{e}_j$$

$$= a_i b_j \underline{e}_i \wedge \underline{e}_j$$

$$\underline{e}_i \wedge \underline{e}_j = \epsilon_{ijk} \underline{e}_k$$ ← new direction $\perp$ to other 2!

$$\epsilon_{ijk} = \begin{cases} 0 & \text{any 2 of } ijk \text{ equal} \\ +1 & \text{cyclic } 123 \quad 231 \quad 312 \\ -1 & \text{anticyclic } 213 \quad 321 \quad 132 \end{cases}$$

$$\nabla = \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z} \right) = \frac{\partial}{\partial x_i} e_i$$

$$\nabla \cdot \underline{v} = \frac{\partial v_x}{\partial x} + \frac{\partial v_y}{\partial y} + \frac{\partial v_z}{\partial z} = \frac{\partial}{\partial x_i} \overbrace{\underline{e}_i \cdot v_j \underline{e}_j} = \frac{\partial v_j}{\partial x_i} \delta_{ij} = \frac{\partial v_i}{\partial x_i}$$

$$= v_{i,i}$$

$$\nabla \wedge \underline{v} = \begin{vmatrix} \underline{e}_x & e_y & e_z \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ v_x & v_y & v_z \end{vmatrix} = \left(\frac{\partial v_z}{\partial y} - \frac{\partial v_y}{\partial z}, \frac{\partial v_z}{\partial x} - \frac{\partial v_x}{\partial z}, \frac{\partial v_y}{\partial x} - \frac{\partial v_x}{\partial y} \right)$$

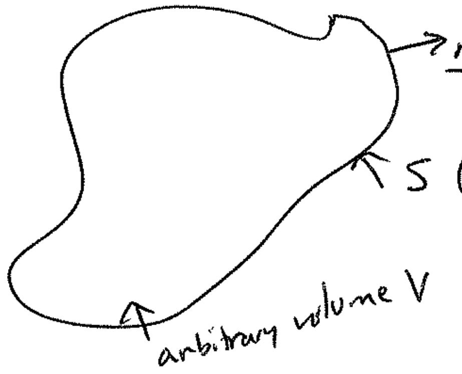
$$= \frac{\partial}{\partial x_i} \underline{e}_i \wedge v_j \underline{e}_j = \frac{\partial v_j}{\partial x_i} \epsilon_{ijk} \underline{e}_k = v_{j,i} \epsilon_{ijk} \underline{e}_k$$

$$\nabla^2 = \nabla \cdot \nabla$$

$$\nabla^2 a = \frac{\partial^2 a}{\partial x^2} + \frac{\partial^2 a}{\partial y^2} + \frac{\partial^2 a}{\partial z^2}$$

$$\nabla^2 \underline{v} = \frac{\partial}{\partial x_k} \overbrace{\underline{e}_k \cdot \frac{\partial v_j}{\partial x_i} \underline{e}_i}^{\delta_{ik}} e_j = \frac{\partial^2 v_j}{\partial x_i^2} \underline{e}_j$$

Conservation of mass



mass flux at each point on S

$$\underbrace{\rho}_{\substack{\text{not}\\ \text{necessarily}\\ \text{constant}}}\ \underbrace{\underline{u}\cdot\underline{n}}_{\text{mass flux}}$$

$$\underbrace{\int_S \rho\underline{u}\cdot\underline{n}\,ds}_{\substack{\text{net flow of}\\ \text{mass out of } V}} = \underset{\text{why?}}{-}\int_V \underbrace{\frac{\partial\rho}{\partial t}\,dV}_{\text{change of mass in } V}$$

Use divergence theorem

$$\int_S \rho\underline{u}\cdot\underline{n}\,ds = \int_V \nabla\cdot(\rho\underline{u})\,dV$$

$$\int_V \left[\frac{\partial\rho}{\partial t} + \nabla\cdot(\rho\underline{u})\right] dV = 0$$

since V is arbitrary

$$\left.\frac{\partial\rho}{\partial t} + \nabla\cdot\rho u = 0\ \right] \text{ continuity equation}$$

if $\rho =$ constant

$$\underbrace{\nabla \cdot \underline{v} = 0}_{\text{incompressible fluid}} \qquad \left(\frac{\partial v_x}{\partial x} + \frac{\partial v_y}{\partial y} + \frac{\partial v_z}{\partial z} = 0 \right)$$

Material derivation - derivative moving with fluid $\underline{v}$

$$\frac{D\phi}{Dt} = \frac{\partial \phi}{\partial x} + v \cdot \nabla \phi$$

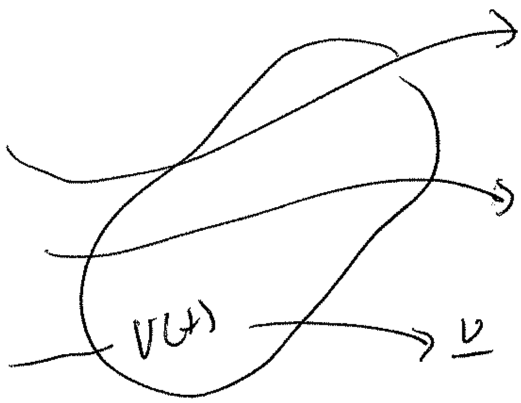
ϕ varies in $\underline{x}, t$

$$d\phi = \left.\frac{\partial \phi}{\partial t}\right|_x dt + \left.\frac{\partial \phi}{\partial x}\right|_t dx$$

$$\frac{d\phi}{d\cancel{t}} = \frac{\partial \phi}{dt} \cdot \frac{dt}{d\cancel{t}} + \frac{\partial \phi}{\partial x} \cdot \underbrace{\frac{dx}{dt}}_{v}$$

generalize

$$\frac{d\phi}{dt} = \frac{\partial \phi}{\partial t} + \underline{v} \cdot \nabla \phi$$



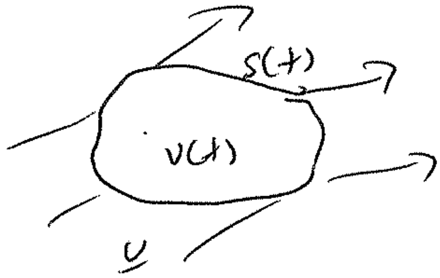
Reynolds transport theorem

$$\frac{D}{Dt}\left[\int_{V(t)} \phi \, dV\right] = \int_{V(t)} \left[\frac{\partial \phi}{\partial t} + \nabla\cdot(\underline{u}\,\phi)\right] dV$$

revisit mass conservation

$$\frac{D}{Dt}\left[\int_{V(t)} \rho \, dV\right] = 0$$

$$= \int_{V(t)} \underbrace{\left[\frac{\partial \rho}{\partial t} + \nabla\cdot(\rho\,\underline{u})\right]}_{=0} dV$$



Conservation of momentum

$$m\underline{a} = \underline{f}$$

time-rate-of-change of momentum = sum of forces acting on body] Newton's 2nd law $\underline{F} = m\underline{a}$

$$\frac{D}{Dt}\left\{\int_{V(t)} \rho\,\underline{u}\, dV\right\} = \underbrace{\int_{V(t)} \rho\,\underline{g}\, dV}_{\text{body forces (gravity)}} + \int_{S(t)} \underline{t}\, dS$$

↑ surface forces

let $\underline{t} = \underline{n}\cdot\underline{\underline{T}}$

$\underline{\underline{T}} = T_{ij}$ ← direction of surface

↑ direction of force

LHS use RTT

$$\int_{V(t)} \left\{ \frac{\partial}{\partial t}(\rho \underline{v}) + \nabla \cdot (\rho \underline{v}\, \underline{v}) \right] dV$$

$$\int_{S(t)} (\underline{n} \cdot \underline{\underline{T}})\, dS = \int_{V(t)} \nabla \cdot \underline{\underline{T}}\, dV$$

since V is arbitrary

$$\underbrace{\frac{\partial(\rho \underline{v})}{\partial t} + \nabla \cdot (\rho \underline{v}\, \underline{v})}_{\substack{\text{acceleration} \\ \text{inertial Forces}}} = \rho \underline{g} + \nabla \cdot \underline{\underline{T}}$$

$$\nabla \cdot (\underline{v}\, \underline{v}) = \underline{v}(\nabla \cdot \underline{v}) + \underline{v} \cdot \nabla \underline{v}$$

if $\rho =$ constant

$$\left.\begin{aligned} \rho\left(\frac{\partial \underline{v}}{\partial t} + \underline{v} \cdot \nabla \underline{v}\right) &= \rho \underline{g} + \nabla \cdot \underline{\underline{T}} \\ \nabla \cdot \underline{v} &= 0 \end{aligned}\right\} \text{Cauchy equation of motion}$$

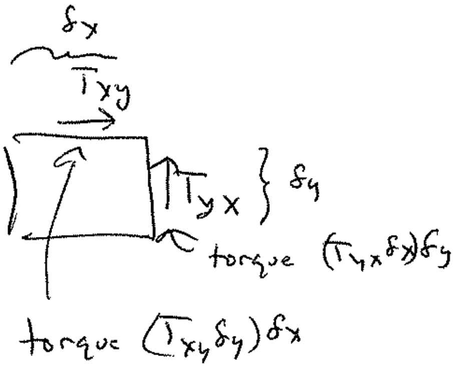
Are we ready to solve problems?

\# unknowns: $\underline{\underline{T}}$ has 9 $\quad \underline{v}$ has 3 $\Big\}$ 12 total

\# equations: 4

$\boxed{\text{oops}}$

Conserve angular momentum



$T_{xy} = T_{yx}$ so there is no net torque

$\underline{\underline{T}}$ is symmetric

$(\underline{\underline{T}} = \underline{\underline{T}}^T$ or $T_{ij} = T_{ji})$

$\underline{\underline{T}}$ has 6 indep components

only hope ...

$\underbrace{\text{express } \underline{\underline{T}} \text{ in terms of } \underline{u}}_{\text{constitutive relationship}}$

Constitutive relationships

$$\underbrace{\underline{\underline{T}} = \underbrace{-p\,\underline{\underline{I}}}_{\substack{\text{isotropic part}\\ \text{(pressure)}}} + \underbrace{\underline{\underline{\tau}}}_{\substack{\text{deviatoric}\\ \text{part}}}}_{\text{general}}(\underline{u}, \nabla\underline{u}, \nabla^2\underline{u}, \int \underline{u}\,dt, \dots)$$

assume $\underline{\underline{\tau}} = \underline{\underline{\tau}}(\nabla\underline{u})$

but $\underline{\underline{\tau}}$ is symmetric so

$$\nabla\underline{u} = \overbrace{\underbrace{\tfrac{1}{2}(\nabla\underline{u} + \nabla\underline{u}^T)}_{\substack{\underline{\underline{E}}\\ \text{rate-of-strain}\\ \text{tensor}}}}^{\text{symmetric}} + \overbrace{\underbrace{\tfrac{1}{2}(\nabla\underline{u} - \nabla\underline{u}^T)}_{\substack{\underline{\underline{\Omega}}\\ \text{vorticity}\\ \text{tensor}}}}^{\text{antisymmetric}}$$

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Assume $\underline{\underline{\tau}} \propto \underline{\underline{E}}$ (Newtonian)

$= \underline{\underline{\underline{\underline{C}}}} : \underline{\underline{E}}$

$\uparrow$ 81 components, only 2 independent ones for an isotropic fluid

$$\underline{\underline{T}} = (-p + \lambda \nabla \cdot \underline{u}) \underline{\underline{I}} + 2\mu \underline{\underline{E}}$$

λ: bulk viscosity

μ: shear viscosity

if $\nabla \cdot \underline{u} = 0$ $\quad \underline{\underline{T}} = -p \underline{\underline{I}} + 2\mu \underline{\underline{E}}$

$$\left.\begin{aligned} \rho\left(\frac{\partial \underline{u}}{\partial t} + \underline{u} \cdot \nabla \underline{u}\right) &= \rho \underline{g} - \nabla p + \mu \nabla^2 \underline{u} \\ \nabla \cdot \underline{u} &= 0 \end{aligned}\right\} \text{Navier Stokes equations}$$

4 unknowns $(p, \underline{u})$

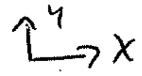
4 equations

assumptions $\rho =$ const (λ does not matter)

isotropic

$\mu =$ constant

Newtonian ($\underline{\underline{\tau}} \propto \nabla \underline{u}$)

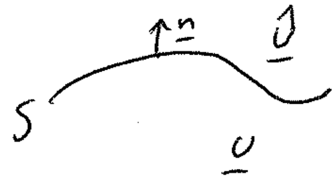


Boundary conditions

1) $\underline{u} = \underline{0}$ on a solid surface (noslip)

2) Free-slip $u_y = 0$
$\frac{\partial u_x}{\partial y} = 0$ (no shear stress)

3)



$\hat{\underline{u}} \cdot \underline{n} = \underline{u} \cdot \underline{n}$

kinematic condition

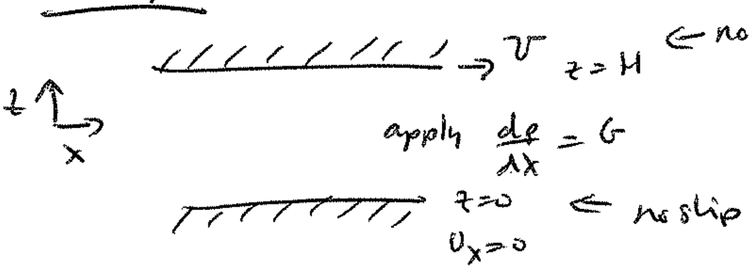
4) dynamic condition

tangential $\underline{u}$ is continuous

$3+4 \Rightarrow \quad \underline{u} = \hat{\underline{u}}$ on S

not true at contact lines

Example



What is $u(x,z)$?

$$\nabla p = \mu \nabla^2 \underline{v}$$

$$\frac{\partial p}{\partial z} = \mu\left(\cancel{\frac{\partial^2 v_z}{\partial x^2}}^{\,0} + \cancel{\frac{\partial^2 v_z}{\partial z^2}}^{\,0}\right)$$

$$\Rightarrow \frac{dp}{dz} = 0 \quad \Rightarrow p \text{ constant with respect to } z$$

$$\frac{\partial p}{\partial x} = \mu\left(\cancel{\frac{\partial^2 v_x}{\partial x^2}}^{\,0} + \frac{\partial^2 v_x}{\partial z^2}\right)$$

$$\frac{dp}{dx} = \mu \frac{d^2 v_x}{dz^2}$$

integrate wrt to z

$$\frac{dp}{dx} z = \mu \frac{dv_x}{dz} + c_1$$

integrate again

$$v_x = \frac{1}{2\mu}\frac{dp}{dx} z^2 + c_1 x + c_2$$

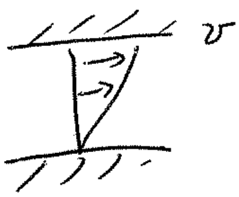
at $z = 0$, $v_x = 0 \Rightarrow c_2 = 0$

$z = H$, $v_x = \frac{H^2}{2\mu}\frac{dp}{dx} H^2 + c_1 H = 0$

$$\therefore v_x = \frac{1}{2\mu}(z^2 - Hz)\frac{dp}{dx} + v\, z/H$$

a) if $dp/dx = 0$

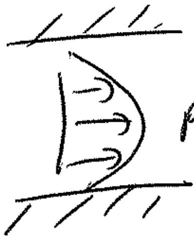
$v_x = v\, z/H$



linear

b) if $v = 0$

$$v_x = \frac{1}{2\mu}(z^2 - Hz)\frac{dp}{dx}$$

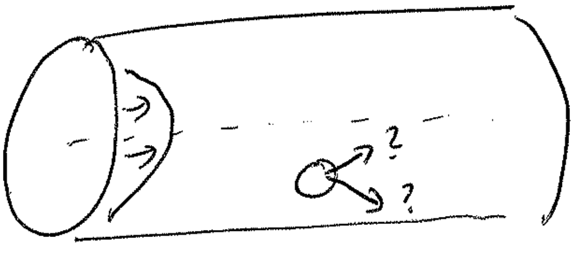


parabolic

superimpose owing to linearity

solve equivalent problem for a cylinder

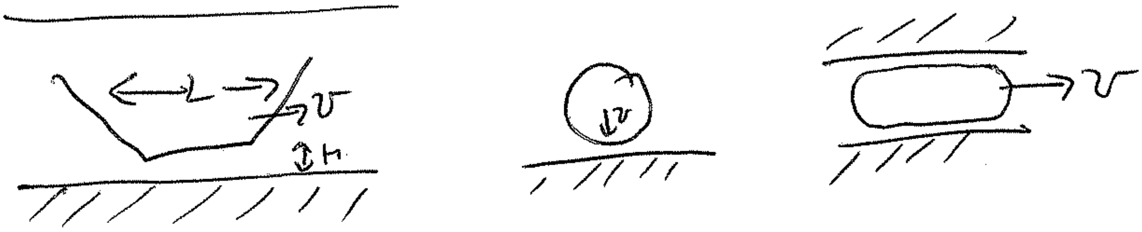
Linearity + reversibility of Stokes flows

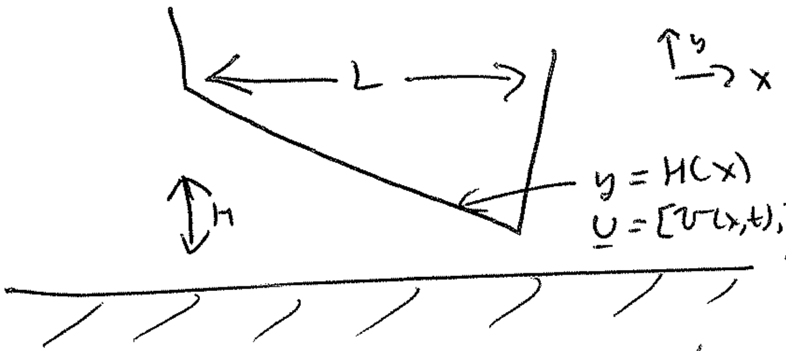


Re << 1 ; no buoyancy

Does sphere migrate towards or away from center line?

"Lubrication" model





$\epsilon = \frac{H}{L} << 1$

$y = H(x)$

$\underline{U} = [v(x,t), V(x,t), 0]$ steady flow

$v = 0$

let $U = U_c U'$; $V = V_c V'$

$$\frac{\partial}{\partial x} = \frac{1}{L}\frac{\partial}{\partial x'} \; ; \; \frac{\partial}{\partial y} = \frac{1}{H}\frac{\partial}{\partial y'}$$

continuity

$$\frac{\partial U}{\partial x} + \frac{\partial V}{\partial y} = 0$$

$$\frac{U_c}{L}\frac{\partial U'}{\partial x'} + \frac{V_c}{H}\frac{\partial V'}{\partial y'} = 0$$

$$V_c \sim \frac{H}{L} U_c = \epsilon U_c$$

X - component N.S.

$$\rho u \frac{\partial u}{\partial x} + \rho v \frac{\partial u}{\partial y} = -\frac{\partial p}{\partial x} + \mu\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right)$$

$$\rho \frac{v_c^2}{L} u' \frac{\partial u'}{\partial x'} + \rho \frac{v_c}{L^2} v' \frac{\partial u'}{\partial y'} = -\frac{p_c}{L}\frac{\partial p'}{\partial x'} + \mu \frac{v_c}{L^2}\frac{\partial^2 u'}{\partial x'^2} + \mu \frac{v_c}{H^2}\frac{\partial^2 u'}{\partial y'^2}$$

multiply by $H^2/\mu v_c$

$$\underbrace{\frac{\rho H^2 v_c}{L\mu}}\left(u' \frac{\partial u'}{\partial x'} + v' \frac{\partial u'}{\partial y'}\right) = -\frac{p_c H^2}{L\mu v_c}\frac{\partial p'}{\partial x'} + \cancel{\epsilon^2 \frac{\partial^2 u'}{\partial x'^2}}^{\,0} + \frac{\partial^2 u'}{\partial y'^2}$$

$$\frac{H^2}{L^2} Re = \epsilon^2 Re$$

$$p_c \sim \mu v_c \frac{L}{H^2} = \frac{1}{\epsilon^2}\left(\frac{\mu v_c}{L}\right)$$ ← normal characteristic pressure

y - component

$$\rho\left(u \frac{\partial v}{\partial x} + v \frac{\partial v}{\partial y}\right) = -\frac{\partial p}{\partial y} + \mu\left(\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2}\right)$$

$$\rho v_c^2 \frac{H}{L^2}\left(u' \frac{\partial v'}{\partial x'} + v' \frac{\partial v'}{\partial y'}\right) = -\mu \frac{v_c L}{H^3}\frac{\partial p'}{\partial y'} + \mu \frac{v_c H}{L^3}\left(\frac{\partial^2 v'}{\partial x'^2}\right) + \frac{\mu v_c}{L H^2}\frac{\partial^2 v'}{\partial y'^2}$$

multiply by $H^3/\mu v_c L$

$$\underbrace{\frac{\rho v_c H^4}{\mu L^3}}_{\epsilon^4 Re}\left(\qquad\right) = -\frac{\partial p'}{\partial y'} + \overbrace{\left(\frac{H^4}{L^4}\right)}^{\epsilon^4}\frac{\partial^2 v'}{\partial x'^2} + \overbrace{\left(\frac{H^2}{L^2}\right)}^{\epsilon^2}\frac{\partial^2 v'}{\partial y'^2}$$

$$\Rightarrow \frac{\partial p'}{\partial y'} = 0$$

Governing equations

$$(1) \quad \frac{dp}{dy} = 0$$

$$(2) \quad \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0$$

$$(3) \quad \frac{\partial p}{\partial x} = \mu \frac{\partial^2 u}{\partial y^2}$$

$\left.\right\}$ $Re\,\epsilon^2$ is small, ϵ is small

Q: Given V, what is Force?
-or- given force, what is V?

integrate (3) — can do because $\frac{\partial p}{\partial x}$ is indep of y (equation 1)

$$u(y) = \frac{1}{2\mu}\frac{dp}{dx} y^2 + c_1(x)\, y + c_2(x)$$

as before

$$u(y) = \frac{1}{2\mu}\frac{dp}{dx}\left(y^2 - H(x,t)\, y\right) + \frac{V(x,t)}{H}\, y$$

but what is dp/dx?

Have not used boundary conditions on v (just on u)

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$$\frac{\partial v}{\partial y} = -\frac{\partial u}{\partial x} = -\frac{1}{2\mu}\, y(y-H)\frac{d^2 p}{dx^2} + \frac{1}{2\mu}\frac{dp}{dx}\, y\, \frac{dH}{dx} - \frac{dU}{dx}\frac{y}{H} + \frac{Uy}{H^2}\frac{dH}{dx}$$

integrate wrt to y to get v

$$v = \underbrace{-\frac{1}{2\mu}\frac{d^2 p}{dx^2}\left(\frac{y^3}{3} - \frac{y^2 H}{2}\right) + \frac{1}{2\mu}\frac{dp}{dx}\frac{dH}{dx}\frac{y^2}{2}} - \left(\frac{dU}{dx}\frac{1}{H} - \frac{U}{H^2}\frac{dH}{dx}\right)\frac{y^2}{2} + c_3(x)$$

$v = 0$ at $y = 0 \Rightarrow c_3 = 0$

at $y = H$, $v = V$

$$V = \underbrace{\frac{1}{12\mu}\frac{d^2 p}{dx^2}H^3 + \frac{1}{4\mu}\frac{dp}{dx}H^2\frac{dH}{dx}}_{\frac{1}{12\mu}\frac{d}{dx}\left(H^3\frac{dp}{dx}\right)} - \frac{1}{2}\frac{dU}{dx}H - \frac{1}{2}U\frac{dH}{dx}$$

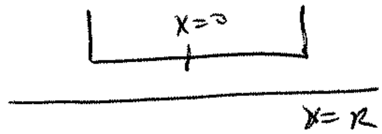
$$\boxed{\frac{d}{dx}\left(H^3_{(x)}\frac{dp}{dx}\right) = 6\mu\left[H(x)\frac{dU}{dx} - U\frac{dH(x)}{dx} + 2V\right]}$$

2^{nd} order equation for $p(x)$

Reynolds equation

solve for p in terms of $H(x), U, V$

need B.C. on p at edges of gap



if $H =$ constant

$$p = \frac{6\mu V}{H^3} x^2 + C_2$$

$$p_0 = \frac{6\mu V R^2}{H^3} + C_2 = 0$$

so $$p(x) = \frac{6\mu V}{H^3}(x^2 - R^2)$$

$$F = 2\int_0^R p(x)\,dx$$

$$= \frac{12\mu V}{H^3}\left(\frac{R^3}{3} - \frac{R^3}{2}\right)$$

$$= 2\mu V \frac{R^3}{H^3}$$

let $v =$ constand

$V = 0$

$$\frac{d}{dx}\left(H^3 \frac{dp}{dx}\right) = -6\mu v \frac{dH}{dx}$$

integrate once

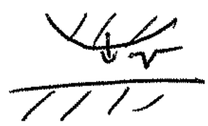
$$\frac{H^3}{6\mu}\frac{dp}{dx} = -vH + c_1$$

integrate again

$$p(x) = -6\mu v \int^x \frac{H(\xi)+a_1}{H^3(\xi)} + a_2$$

let $v = 0$

$$\frac{1}{12\mu}\frac{d}{dx}\left(H^3\frac{dp}{dx}\right) = V(t)$$

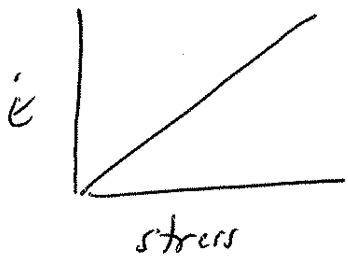


$$\frac{dp}{dx} = \frac{12\mu V(t)}{H^3}x + c \quad c \leftarrow 0$$ from symmetry

∇p scales with $\frac{1}{H^3}$!!

Beyond Newtonian Fluids

Newtonian fluid



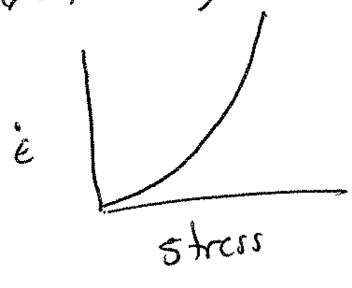
$\sigma = 2\mu\dot{\epsilon}$

Linear elastic material (Hooke's law)

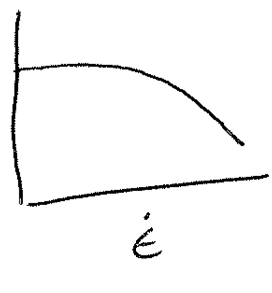
stess $\propto$ strain

$\sigma = E\epsilon$

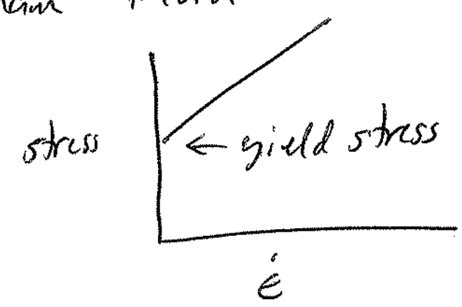
Shear-thinning fluid

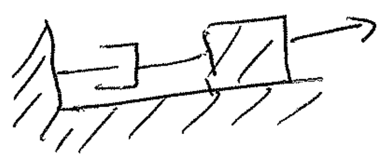


$\Rightarrow \quad \mu = \frac{\text{stress}}{\dot{\epsilon}}$



Bingham fluid





Maxwell viscoelastic

short time : elastic

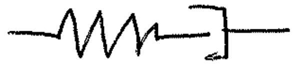
long time : fluid

superimpose strain rates

$$\epsilon_e = \sigma / E$$

$$\dot{\epsilon}_f = \frac{d\epsilon_f}{dt} = \frac{\sigma}{2\mu}$$

$$\dot{\epsilon} = \dot{\epsilon}_e + \dot{\epsilon}_f$$



$$\underbrace{\frac{d\epsilon}{dt} = \frac{\sigma}{2\mu} + \frac{1}{E}\frac{d\sigma}{dt}}_{\text{constitutive law}}$$

$$t_{char} = t_{Maxwell} = 2\mu / E$$

Kelvin model

$$\sigma = \sigma_f + \sigma_e$$

$$= 2\mu \frac{d\epsilon}{dt} + \epsilon E$$

Scaling in Geophysics
Michael Manga
May 2011

Fluid flow

1. Show that $\mathbf{A} \cdot \mathbf{B} \wedge \mathbf{C} = \mathbf{A} \wedge \mathbf{B} \cdot \mathbf{C} = -\mathbf{C} \cdot \mathbf{B} \wedge \mathbf{A}$.

2. Use the very useful identity (you are also encouraged to try deriving this identity) called the $\epsilon\delta$ identity

$$\epsilon_{ijk}\epsilon_{imn} = \delta_{jm}\delta_{kn} - \delta_{jn}\delta_{km}$$

to show that

$$(\mathbf{a} \wedge \mathbf{b}) \wedge \mathbf{c} = (\mathbf{a} \cdot \mathbf{c})\mathbf{b} - (\mathbf{b} \cdot \mathbf{c})\mathbf{a}.$$

Here we use ϵ to denote the permutation symbol and δ is the Kronecker delta.

3. When we build models of river systems and ships, care is usually taken to scale the Froude number

$$Fr = U/\sqrt{gl},$$

where l is the length scale and g is gravity (this is the ratio of velocity to the speed of shallow water waves.

Suppose we would like to study the motion of a boat (length 100 m, speed 10 m/s) in the lab. Due to budget and space constraints, we can only make a model boat that is 1 m long.

a) How fast does the model boat have to move for Froude number scaling to hold?

b) What can we do if we want the Reynolds number to also be the same for the real boat and the model?

4. Show that the continuity equation can be written as

$$\frac{1}{V}\frac{DV}{Dt} = \nabla \cdot \mathbf{u}$$

where V is volume. Recall the chain rule. Explain why $\nabla \cdot \mathbf{u} = 0$ if the fluid is incompressible.

5. Consider two equal-size spherical particles in a very viscous fluid (e.g. a magma) sinking because they are more dense than the surrounding fluid. Assuming the Reynolds number is $\ll 1$, what can you say about the change in their relative orientation and separation distance? Why?

6. If the eruption rate of the Columbia River flood basalts was 1 km^3/day, what is the radius of the conduit (here assumed to be cylindrical and smooth) that transported the magma? Assume a viscosity of 100 Pa s, a density difference between the magma and surrounding rocks of 300 kg/m^3, and a magma density of 2600 kg/m^3; also assume that only the density difference between the magma and surrounding rocks drives the flow (i.e., the pressure gradient is $\Delta\rho g$).

Given the assumptions made above, would the flow of the magma through the conduit be laminar or turbulent?

7. Derive an expression for the spreading rate of a viscous fluid (Reynolds numbers much less than 1) over a flat surface by scaling analysis. You should find that for a constant volume of fluid, the radius R increases as time$^{1/8}$.

Perform an experiment to verify the $t^{1/8}$ spreading rate. To do this, you will need to measure the radius of a spreading blob of very viscous fluid (e.g., honey, syrup) as a function of time. By plotting your data on a log-log scale you should be able to determine the power law relationship (if there is one). You should try to let your experiment run for at least one day.

If your experimental data do not agree with the theory, describe several possible reasons for the disagreement.

Derive a similar expression for a two-dimensional flow in the same limit (low Re, flow due only to buoyancy forces, constant volume of fluid).

8. Two parallel plane, circular disks of radii R lie one above the other. They are separated by a distance H. The space between them is filled with an incompressible Newtonian fluid. One disk approaches the other at constant velocity V, displacing the fluid. The pressure at the edge of the upper disk is atmospheric.

a) Under what conditions are the lubrication equations (Stokes equations) valid? What is the appropriate choice for the characteristic pressure in the lubrication approximation?

b) What is the velocity as a function of radius r?

c) What is the dynamic pressure distribution?

d) Show that the hydrodynamic force resisting motion is

$$F = \frac{3\pi\mu R^4}{2H^3}\frac{dH}{dt}$$

Hopefully this solution helps you understand why separating microscope slides by pulling them apart (when there is water in between the slides) is not easy.

Many adhesion processes rely on lubrication theory. If gaps are thin (and especially if the fluid is very viscous) then large forces required to separate the surfaces at reasonable rates. Aparently, some insects use lubrication theory to help their feet stick to smooth surfaces and as result they can even walk upside down.

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Heat transfer by conduction

1. Consider two horizontal layers, layer A lying on top of layer B. The thermal conductivity of layers A and B are 2 and 5 W $K^{-1}m^{-1}$, respectively. Layer A has a thickness of 30 m and layer B a thickness of 70 m. The temperature at the bottom of layer B is 50 degrees C and the surface temperature is 0 degrees C.

What is the surface heat flow and the temperature between layers A and B?

2. Using the relation $\tau = l^2/\kappa$ and assuming $\kappa = 1 \times 10^{-6}$ m^2/s, determine the characteristic conduction time scales for conductive cooling of the Earth, Mars, Enceladus (the moon that is erupting water ice), and Mars' moon Phobos.

What are the implications of these time scales for the interiors of the planets?

3. If the mean surface heat flow on the Earth (80 mW/m^2) is attributed entirely to the cooling of the Earth, what is the **mean** rate of cooling (degrees/million years)? Assume the mean specific heat is 1.2 kJ kg^{-1} K^{-1}. This problem involves only performing an energy balance.

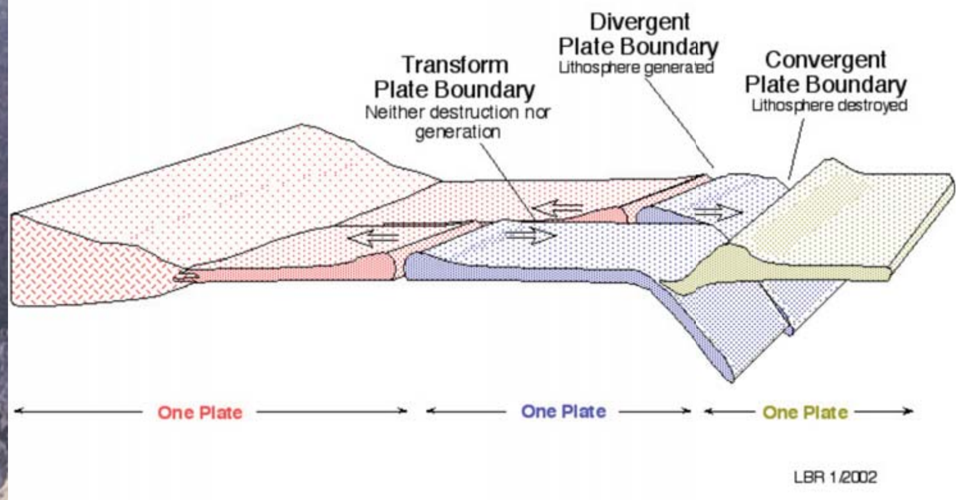
4. Calculate the maximum depth to which frost can penetrate at a latitude where the annual surface temperature varies sinusoidally between -10 degrees C and 20 degrees C (water pipes should be buried below this depth). Assume that the water content of the ground is sufficiently small that the latent heat associated with freezing and thawing can be ignored. Assume the thermal diffusivity of the soil is 1×10^{-6} m^2/s.

5. A body of water at 0 degrees C is subjected to a constant surface temperature of -10 degrees C for 10 days. How thick is the layer of ice that develops? Assume the latent heat is 300 kJ/kg, the thermal conductivity is 2 W K^{-1} m^{-1}, the specific heat is 4 kJ kg^{-1} K^{-1}, and the density is 1000 kg/m^3.

6. Pseudotachylites are rocks found in fault zones that appear to have been melted (see Kanamori, Anderson, and Heaton (1998) Frictional melting during the rupture of the 1994 Bolivian earthquake, *Science, 279*, 839-842 for a short discussion).

Assume a constant sliding speed u on a fault during an earthquake that results in friction heat production $u\sigma$ where σ is the stress on the fault (what are the units of heat production here?). If $u = 20$ m/s, the total displacement is 5 m, $\sigma = 20$ MPa, the thermal conductivity is 2 W K^{-1} m^{-1} and the thermal diffusivity is 1×10^{-6} m^2/s, what is the temperature increase on the fault (assuming the rocks do not melt)? Will the temperature high enough to melt rocks?

Scaling



Relationship between processes and properties

Premise: Physical laws do not depends on arbitrarily chosen units of measurement

Barenblatt, G.I. (1996) "Scaling, self-similarity, and intermediate asymptotics", Cambridge Univ Press.

Bridgman, P. (1931) "Dimensional analysis", Yale Univ Press.

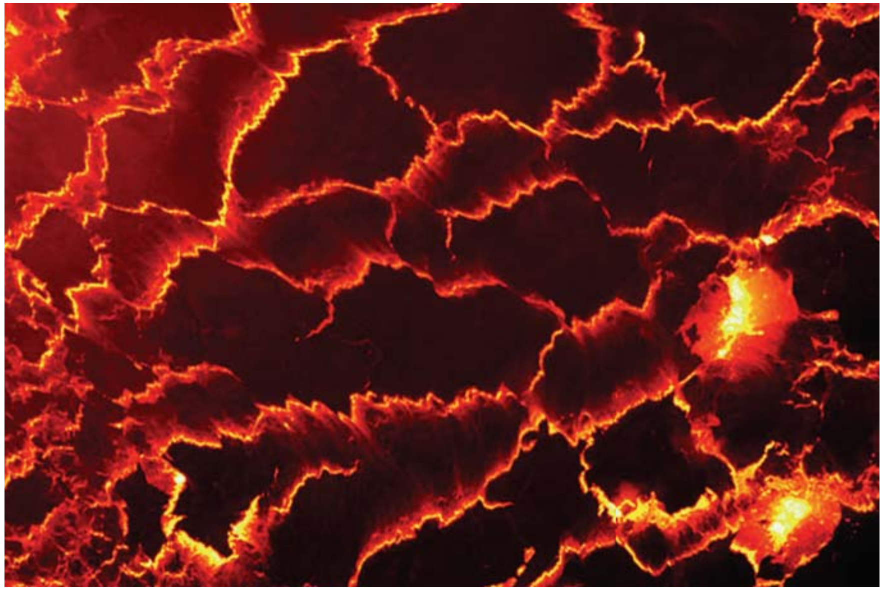
Why?

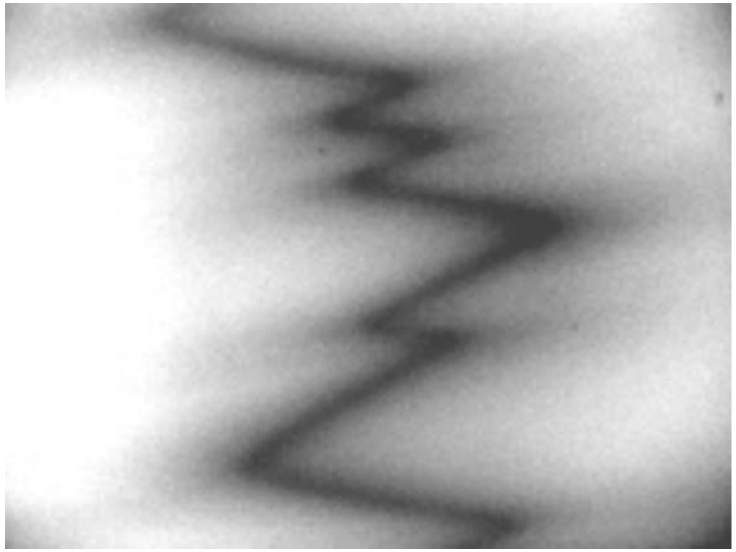
- Provides dimensionless parameters and scaling relationships
- Organize thinking
- Helps analysis of field data and lab experiments
- Reduce number of variables for analysis
- Generalize results
- Relative importance of different effects

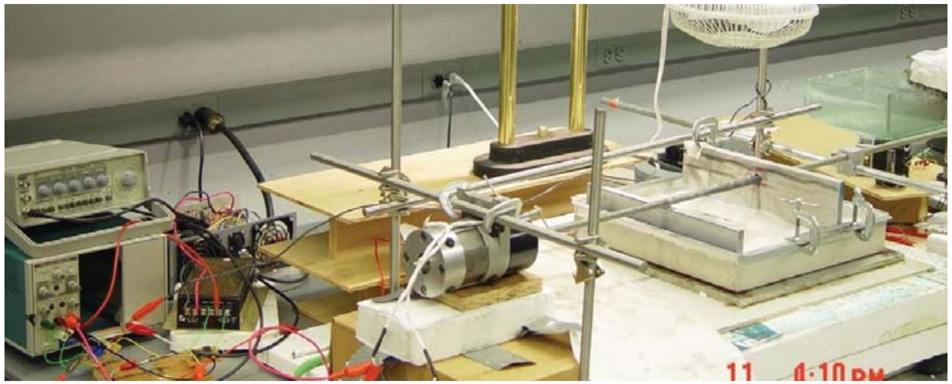
How do evolution and dynamics differ?

- Temperature? Surface dynamics? Mountain height?

Are lab models relevant?







Making equations dimensionless

Example on board

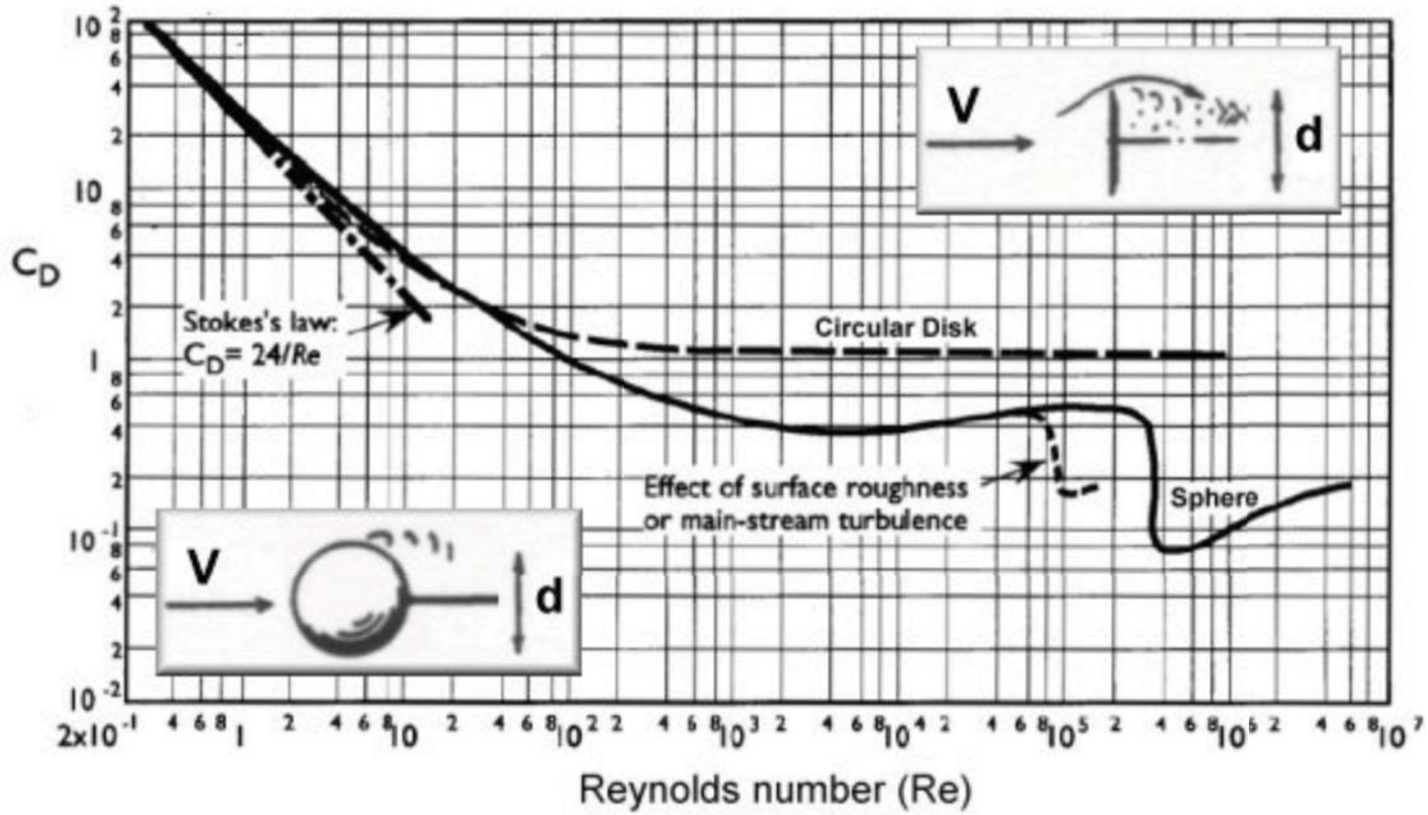
$$\overbrace{\rho\Big(\underbrace{\frac{\partial \mathbf{v}}{\partial t}}_{\substack{\text{Unsteady}\\\text{acceleration}}} + \underbrace{\mathbf{v}\cdot\nabla\mathbf{v}}_{\substack{\text{Convective}\\\text{acceleration}}}\Big)}^{\text{Inertia}} = \underbrace{-\nabla p}_{\substack{\text{Pressure}\\\text{gradient}}} + \underbrace{\mu\nabla^2\mathbf{v}}_{\text{Viscosity}} + \underbrace{\mathbf{f}}_{\substack{\text{Other}\\\text{forces}}}$$

Buckingham Pi theorem

- If we have *n* physical variables and *k* independent physical dimensions (e.g., mass, length, time, temperature), there are *n-k* dimensionless parameters
- Do we know *n*?
- Not necessarily meaningful dimensionless groups

Example on the board

Drag on a sphere



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Example: As the number of rowers (N) increases, how does race time decrease?

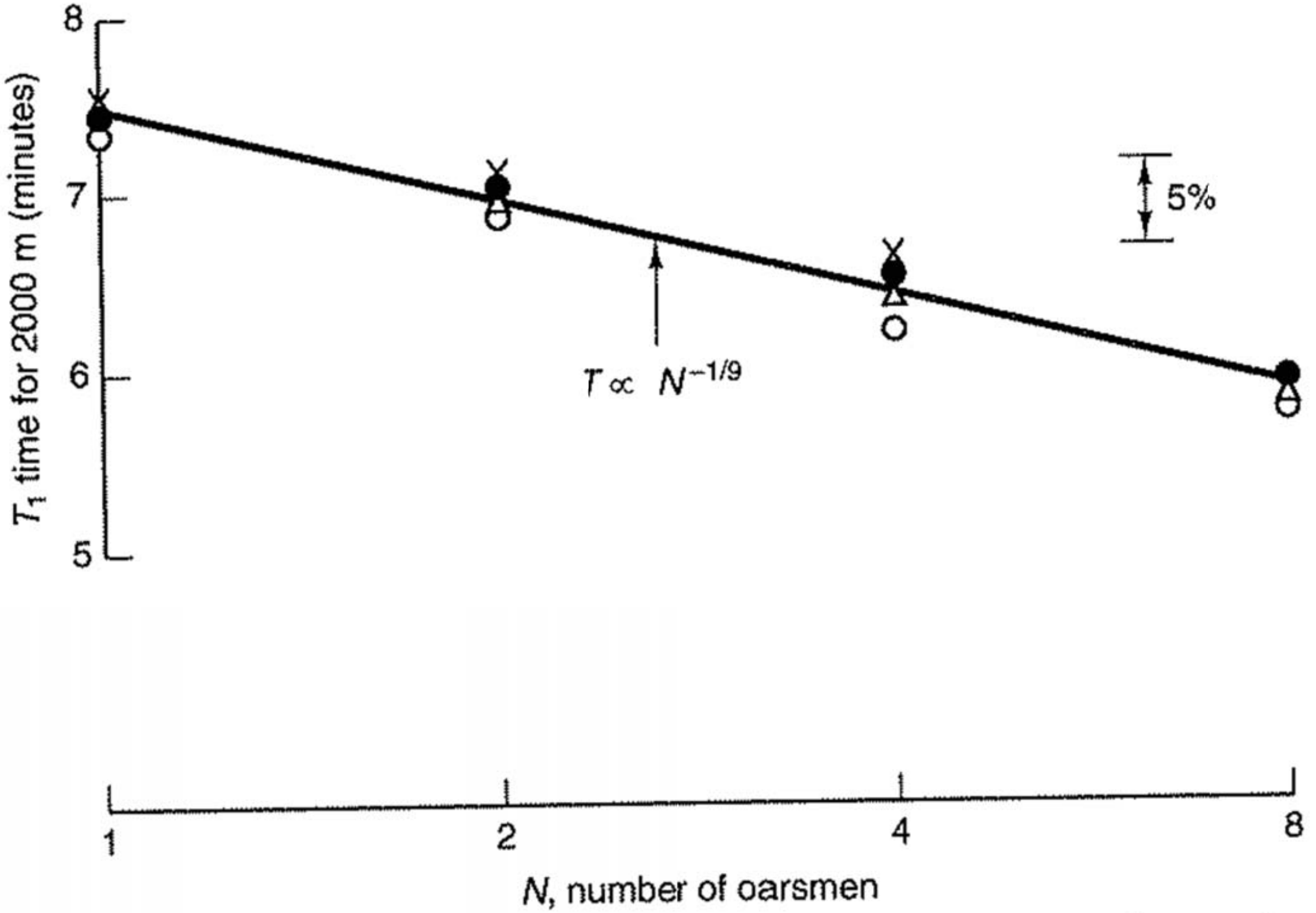
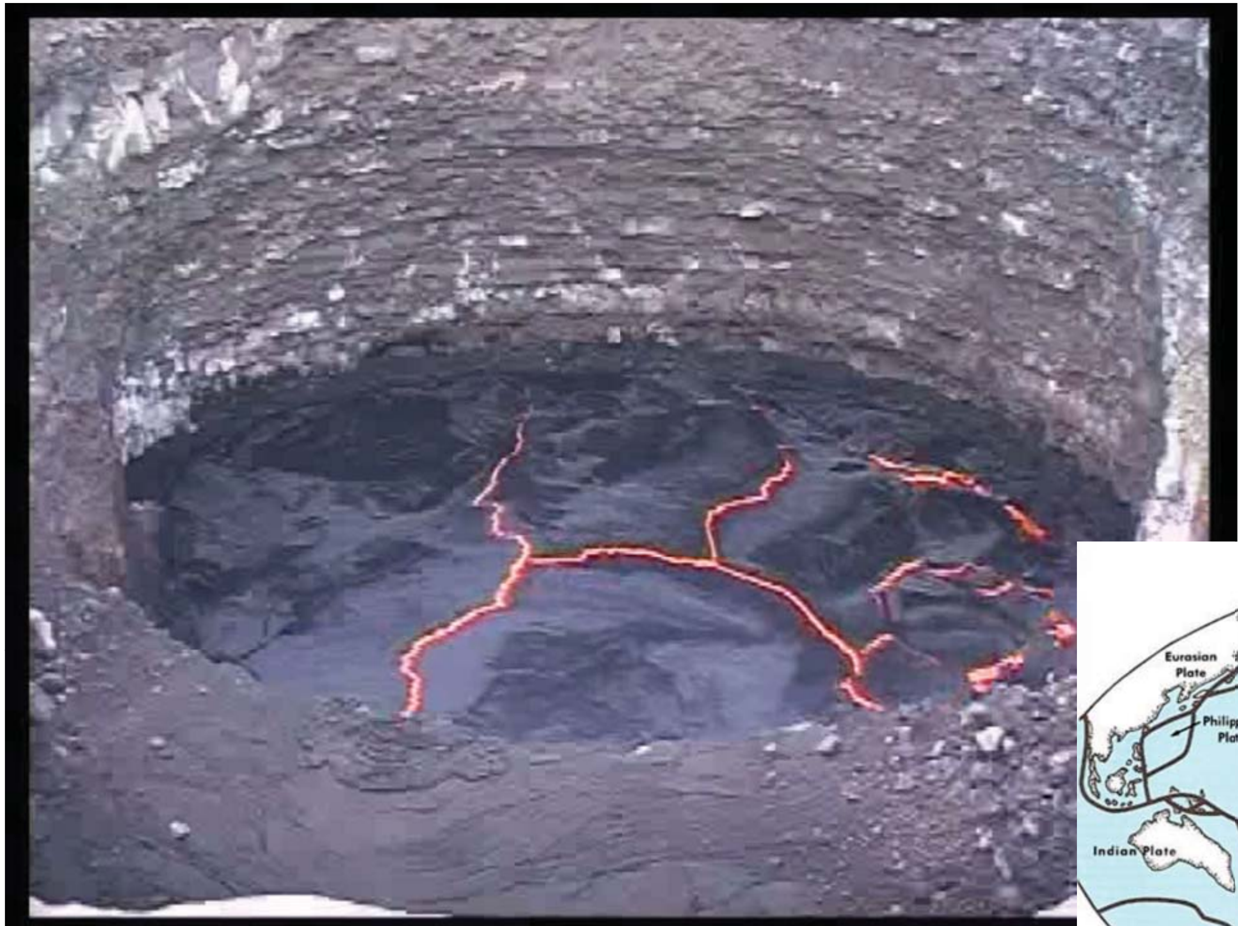
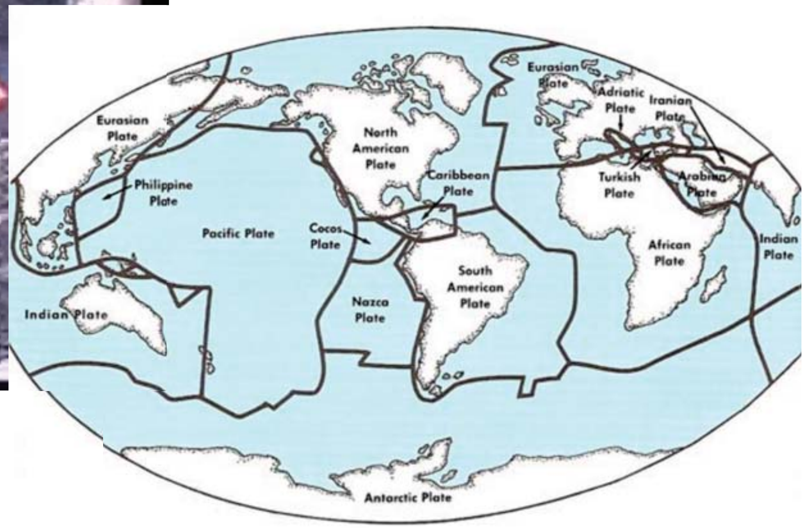


Figure 1.9. The -1/9 power-law dependence of rowing time on the number of oarsmen (solid line) compared with racing times for 2000 m, all at calm or near calm conditions. Δ, 1964 Olympics, Tokyo; $\bullet$, 1968 Olympics, Mexico City; $\times$, 1970 World Rowing Championships, Ontario; $\circ$, 1970 Lucerne International Championships. After McMahon (1971).

Is a planet with plates analogous to a lava lake?





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