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Advanced School on Scaling Laws in Geophysics: Mechanical and Thermal Processes in Geodynamics

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What are scaling laws, why are they important in geodynamics and what problems in geophysics make them attractive

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Scaling laws in solid-earth geodynamics

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Quick Introduction

1. We will try to find a time for everyone to introduce herself or himself, maybe later today.
2. **We are informal**. Feel free to ask questions when we are not clear.
3. Tell us if we speak too rapidly!
4. This week, we are **Peter**, **Michael**, and **Greg**. You may call us "Professor" only if what we say is nonsense; otherwise, we are informal.

Why teach a course about scaling laws?

1. With the inexorable growth in computational power, the capability for building increasingly complex, and complicated, models also grows.
2. Such models are good for engineering, but the goal of science is to **understand**.
3. The ***common denominator*** of **understanding** is the **ability** to **predict** outcomes using basic principles and appropriate parameters.
4. **Scaling laws**, which commonly are algebraic expressions, offer a **short-cut** to making such **predictions**, and therefore to **understanding**.

“What it means to understand an equation ... was described by ***Dirac***. He said: ‘**I understand what an equation means if I have a way of figuring out the characteristics of its solution without actually solving it.**’ So if we have a way of knowing what should happen in given circumstances without actually solving the equations, then we ‘understand’ the equations, as applied to these circumstances.”

Richard Feynman, Chapter 2, page 2-1, *The Feynman Lectures on Physics, Volume I, The Electromagnetic Field*.

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Diffusion (of heat for example)

Rate of change in internal energy (in the absence of heat sources) equals the rate that heat is conducted away.

$$\rho C_P \frac{\partial T}{\partial t} = k\nabla^2 T\left(= -\nabla\cdot\left(-k\nabla T\right)\right)$$

Non-dimensionalize this using:

$$\kappa = \frac{k}{\rho C_P},\ t = \tau t',\ \left(x,y,z\right) = d\left(x',y',z'\right),\ T = \Delta T T'$$

Diffusion (of heat for example)

$$\frac{\Delta T}{\tau}\frac{\partial T'}{\partial t'} = \frac{\kappa \Delta T}{d^2}\nabla'^2 T', \text{ or } \frac{\partial T'}{\partial t'} = \frac{\kappa\tau}{d^2}\nabla'^2 T'$$

Hence, all solutions scale with $\kappa\tau / d^2$

κ

If there is a characteristic distance, *d*, then time scales with

$$\text{time} \sim d^2/\kappa$$

If there is a characteristic time scale, τ, then distances scale with

$$\text{distance} \sim \sqrt{\kappa\tau}$$

Dimensions: paraphrased from Gary Parker, ca. 2006

The British Empire exploited (and the United States, still, exploits) the English system of units: *feet, pounds, °F, etc.*

The French (like most of the world) use the Système International (SI): meters, kilograms, °C, etc.

God uses dimensionless quantities.

Consider viscous flow: Stokes's Problem of sinking solid sphere in a viscous fluid

$$F_{driving} = mg = \frac{4}{3}\pi R^3 \Delta\rho g$$

Resistance is due to viscous drag over the area of the sphere: stress x area.

Stress = pressure + 2 viscosity x strain rate.

Scale the strain rate to the ratio of the sinking speed to the radius of the sphere (we need a characteristic dimension).

$$\tau(=\text{stress}) = 2\eta\dot{\varepsilon} \sim \eta\frac{V}{R} \sim pressure$$

Sinking solid sphere in a viscous fluid

$$F_{driving} = mg = \frac{4}{3}\pi R^3 \Delta\rho g$$

With resistance proportional to stress x area.

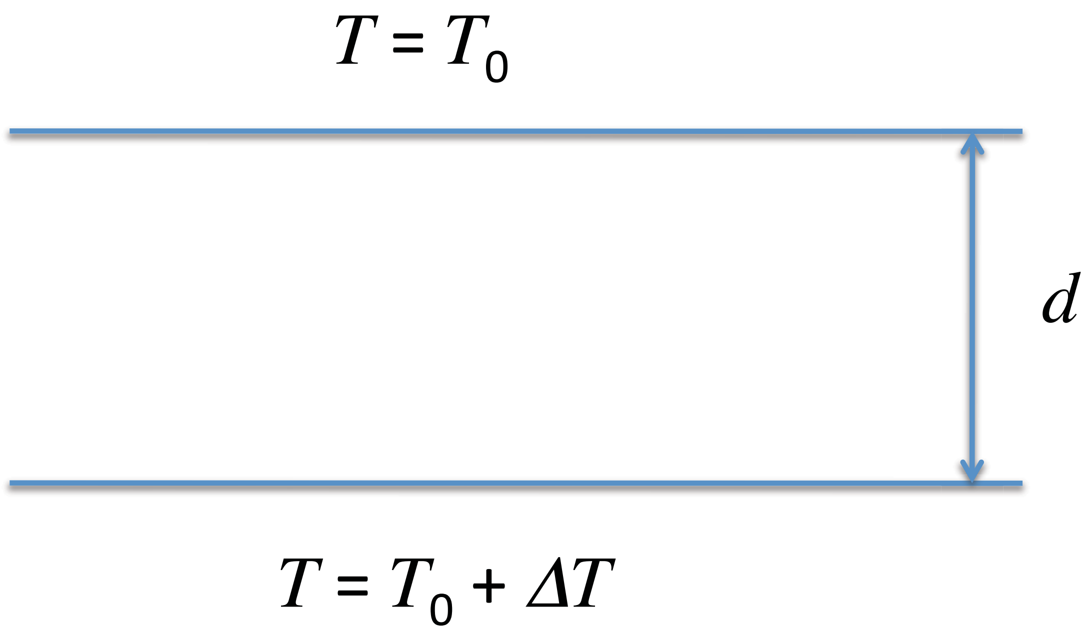
$$F_{drag} \sim \eta \frac{V}{R} 4\pi R^2 = 4\pi\eta V R$$

Balance (equilibrium) is reached when the two forces are equal:

$$\frac{4}{3}\pi R^3 \Delta\rho g \sim 4\pi\eta V R$$

$$V \sim \frac{\Delta\rho g R^2}{3\eta}; \text{ actually}: \; V = \frac{2\Delta\rho g R^2}{9\eta}$$

Thermal convection: layer of thickness, *d*, with temperature along the bottom boundary warmer by ΔT than along the top boundary.



Key parameters: density (ρ), thermal expansion (α: $\Delta\rho=\rho\alpha\Delta T$), gravity ($g$), thermal diffusivity ($\kappa$), viscosity ($\eta$), and d and ΔT.

Thermal convection: layer of thickness, *d*, with temperature along the bottom boundary warmer by ΔT than along the top boundary.

Consider two time scales to move heat across the layer:

Diffusive heat transfer: d^2/κ

Advective heat transfer: d/V

What is V? $$V\left(=\frac{2\Delta\rho g R^2}{9\eta}\right)\sim\frac{\rho\alpha\Delta T g d^2}{\eta}$$

Here, we use $$\Delta\rho=\rho\alpha\Delta T$$

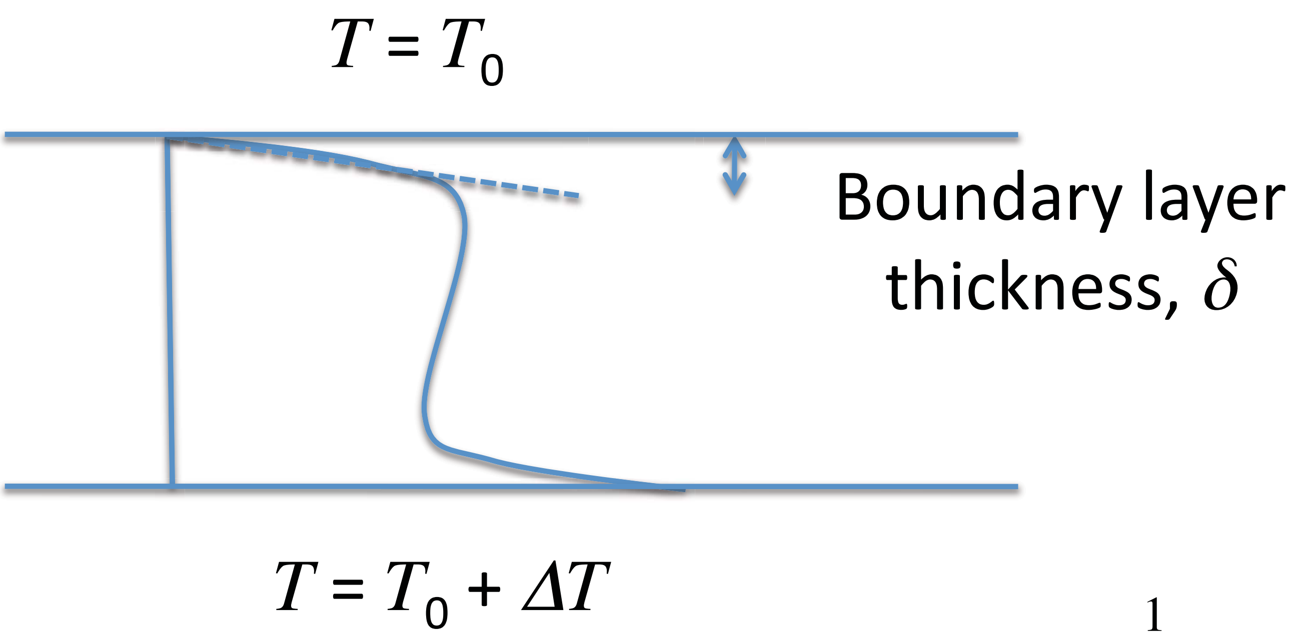
Thermal convection: consider ratio of time scales.

$$\frac{diffusive}{advective} = \frac{\frac{d^2}{\kappa}}{\frac{\eta d}{\rho\alpha\Delta T g d^2}} = \frac{\rho\alpha\Delta T g d^3}{\kappa\eta} = Ra$$

Ra is the Rayleigh number.

Everything depends on the Rayleigh number (and only on Ra, plus boundary conditions).

Thermal convection: Everything depends on the Rayleigh number.



Boundary layer thickness: $\delta \sim d\, Ra^{-\frac{1}{3}}$

Heat flux:

$$Q = k\frac{\Delta T}{2\delta} \sim k\frac{\Delta T}{2d} Ra^{\frac{1}{3}}$$

Thermal convection: typical speeds of flow.

Heat conducted through the boundary layer.

$$Q = k\frac{\Delta T}{2\delta}d$$

Heat advected in rising and sinking sheets:

$$Q_{adv} \sim V\rho C_p \Delta T \delta$$

Set them equal (approximately):

$$V\rho C_p \Delta T \delta \sim k\frac{\Delta T}{2\delta}d$$

Using: $\delta \sim d\, Ra^{-\frac{1}{3}}$

$$V \sim \frac{k}{\rho C_p}\frac{d}{4\delta^2} \sim \frac{\kappa}{d}Ra^{\frac{2}{3}}$$

Mountain building:
two sources of resistance

- To build a mountain belt, work must be done against gravity and against dissipative processes.

- Gravitational potential energy/unit area

- Work done against dissipative processes (friction, viscous dissipation, etc.)

Mountain building: resistance imposed by gravity

- Gravitational potential energy

$$PE = mgh$$

- Gravitational potential energy/unit area

$$GPE_{grav} = \frac{1}{2}\rho_c g h^2 \left(\text{plus PE stored below}\right)$$

$$\sim \rho_c g L^2\text{, or, more precisely, } \rho_c \frac{\rho_m - \rho_c}{\rho_m} g L^2$$

Mountain building: energy lost in dissipation

- Work/area done against dissipative stresses: stress x thickness of layer.
- Rock-forming minerals deform by non-Newtonian viscosity:

$$\tau = B\dot{\varepsilon}^{\frac{1}{n}},\ n \sim 3 - 10$$

- So,

$$W_{diss} \sim B\dot{\varepsilon}^{\frac{1}{n}} L \sim B\left(\frac{V}{L}\right)^{\frac{1}{n}} L$$

Mountain building: consider ratio of GPE and dissipative energy loss

$$\frac{\rho_c \dfrac{\rho_m - \rho_c}{\rho_m} g L^2}{B\left(\dfrac{V}{L}\right)^{\frac{1}{n}} L} = \frac{\rho_c \dfrac{\rho_m - \rho_c}{\rho_m} g L^{1+\frac{1}{n}}}{B V^{\frac{1}{n}}} = Ar$$

Ar is the Argand number: large-scale continental deformation varies with the Argand number.

Large Ar (weak rock): fill the bathtub (high plateau).

Small Ar (strong rock): localize deformation near boundaries (narrow mountain belt).

Messages

Many seemingly complex phenomena are dictated by a small number of dimensionless numbers: Ra, Ar, etc.

Dirac said: "'I understand what an equation means if I have a way of figuring out the characteristics of its solution without actually solving it." Scaling laws define such characteristics.

If you know how different phenomena and properties scale, you know how to plot experimental data.

Basic physical understanding can be simple, when the mathematics is not.

"The trouble with most people in this business is that they think that because they have a bigger, faster computer, they can make **more complicated** models that are more realistic.

"They should realize that because they cannot solve the equations and are stuck having to use the big machine, they must make the model **simpler**, if they want to **UNDERSTAND** the results."

Dan McKenzie, ca. 1975