The Abdus Salam
International Centre for Theoretical Physics

United Nations
Educational, Scientific and
Cultural Organization

IAEA
International Atomic Energy Agency

2240-12

Advanced School on Scaling Laws in Geophysics: Mechanical and Thermal Processes in Geodynamics

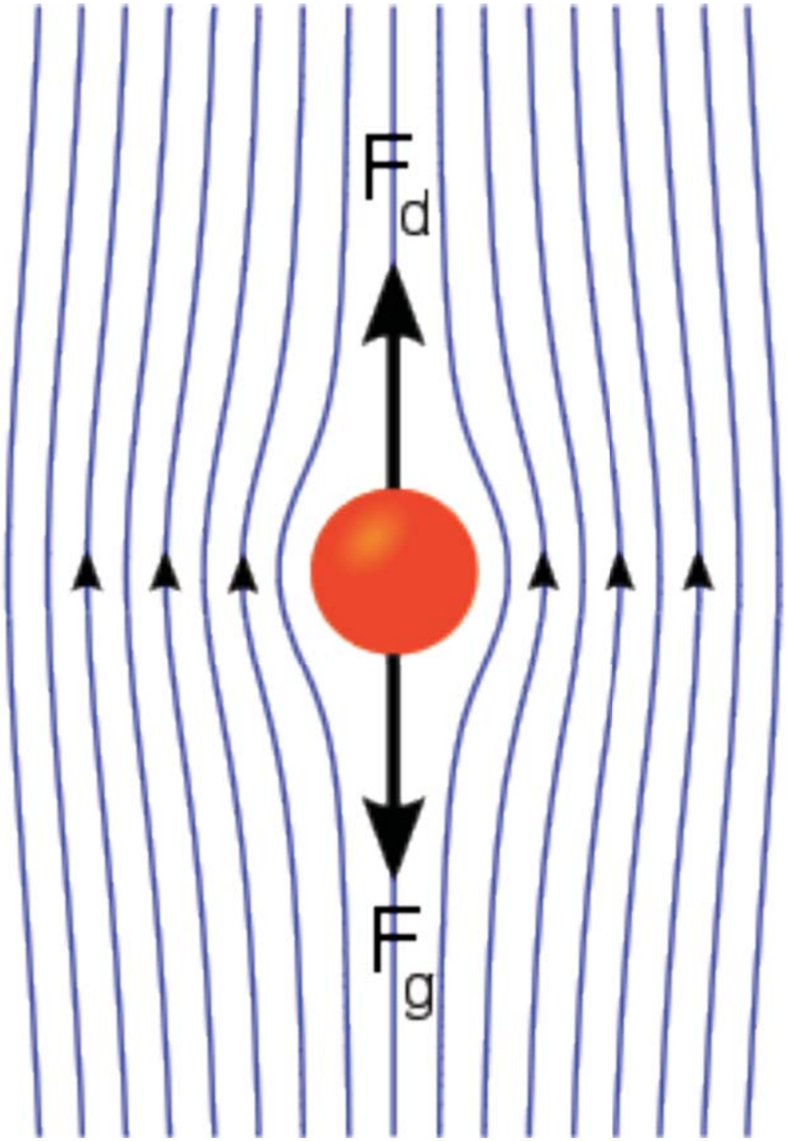
23 May - 3 June, 2011

Deflection of the surface above a sinking sphere and associated gravity anomalies and geoid

P. Molnar

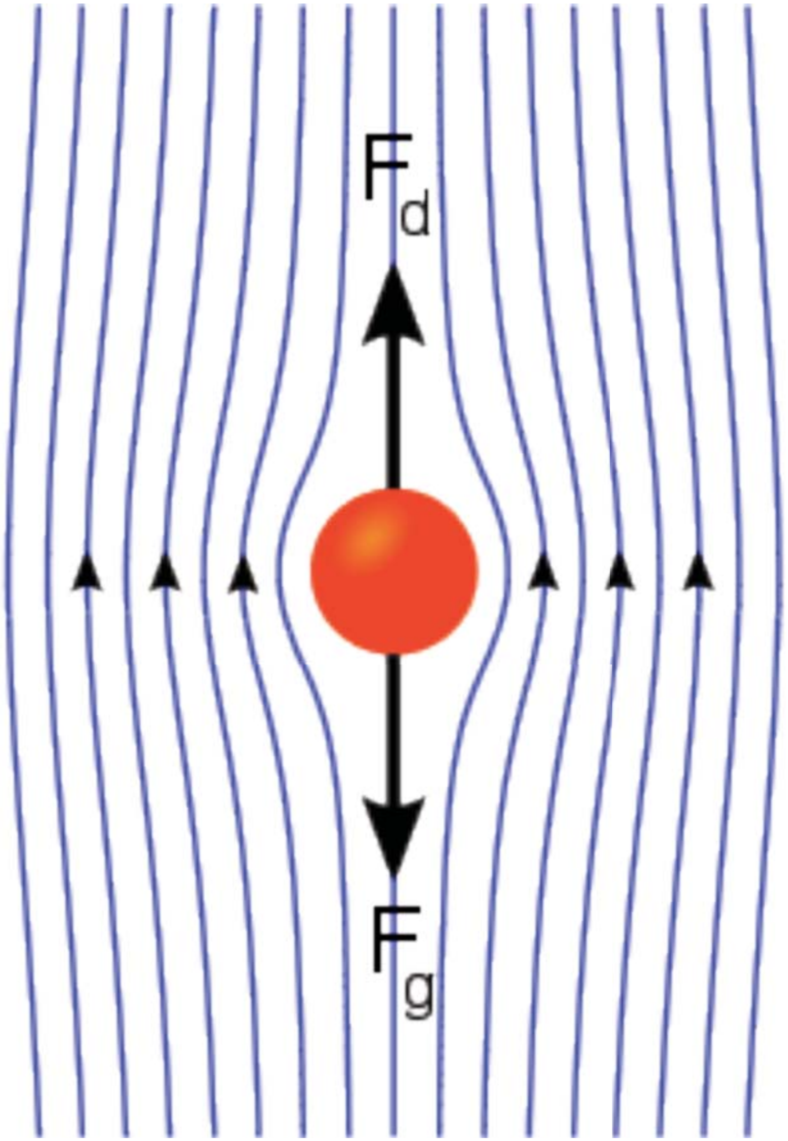
Univ. of Colorado at Boulder
USA

Strada Costiera 11, 34151 Trieste, Italy - Tel.+39 040 2240 111; Fax +39 040 224 163 - sci_info@ictp.it



Stokes's Theory: sphere sinking through a viscous fluid

Gravity pulls it down, but viscous drag prevents it from accelerating.



Coordinate system

Spherical coordinates, centered at the sphere, so that the surrounding fluid, at large distances, moves at:

$$\vec{u}(r \rightarrow \infty) = \vec{V} = V\cos\vartheta\hat{r} - V\sin\vartheta\hat{\vartheta}$$

Basic equations

$$\nabla p = \eta \nabla^2 \vec{u} = -\eta \nabla \times \vec{\omega}, \quad \nabla \cdot \vec{u} = 0$$

Here, p is the perturbation to pressure due to flow; η is viscosity, $\vec{u}$ is velocity; and $\vec{\omega}$ is vorticity.

$$\vec{\omega} = \nabla \times \vec{u}$$

The first equation above expresses a balance of forces (Newton's second law), for the case of very high viscosity, so that accelerations are negligible. The second equation states that mass is conserved, and the fluid is incompressible.

Simplified equations

$$\nabla p = \eta \nabla^2 \vec{u} = -\eta \nabla \times \vec{\omega}, \quad \nabla \cdot \vec{u} = 0$$

Taking the divergence of the first of these, and using the second gives.

$$\nabla^2 p = 0$$

Taking the curl of the first gives:

$$\nabla^2 \vec{\omega} = 0$$

Moreover, for this case, we may write components of velocity in terms of a stream function.

Stream function in spherical coordinates

$$u_r = \frac{1}{r^2 \sin\vartheta} \frac{\partial\psi}{\partial\vartheta}, \quad u_\vartheta = -\frac{1}{r \sin\vartheta} \frac{\partial\psi}{\partial r}$$

Moreover, for a fluid with constant viscosity and density, the stream function obeys

$$\nabla^4 \psi = 0$$

The equation for pressure

$$\nabla^2 p = 0$$

$$\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial p}{\partial r}\right) + \frac{1}{r^2\sin\vartheta}\frac{\partial}{\partial\vartheta}\left(\sin\vartheta\frac{\partial p}{\partial\vartheta}\right) + \frac{1}{r^2\sin^2\vartheta}\frac{\partial^2 p}{\partial\varphi^2} = 0$$

Symmetry requires that p depend on only r and ϑ

$$\frac{\partial}{\partial r}\left(r^2\frac{\partial p}{\partial r}\right) + \frac{1}{\sin\vartheta}\frac{\partial}{\partial\vartheta}\left(\sin\vartheta\frac{\partial p}{\partial\vartheta}\right) = 0$$

The solution for pressure

$$\frac{\partial}{\partial r}\left(r^2\frac{\partial p}{\partial r}\right)+\frac{1}{\sin\vartheta}\frac{\partial}{\partial\vartheta}\left(\sin\vartheta\frac{\partial p}{\partial\vartheta}\right)=0$$

To match the condition that p vanish at large distances and have opposite signs in front of and behind the sphere, the solution takes a form like:

$$p = C\frac{\eta V\cos\vartheta}{r^2}$$

As we shall see, the constant C must have units of distance.

The solution for vorticity

From $\nabla p = -\eta \nabla \times \vec{\omega}$, and $p = C \dfrac{\eta V \cos\vartheta}{r^2}$

Thus, $$\vec{\omega} = \frac{C\vec{V} \times \vec{r}}{r^3}$$

By symmetry, again, there can be only one component of vorticity:

$$\omega_\varphi = \frac{CV \sin\vartheta}{r^2}$$

Check

$$\nabla p = -\eta \nabla \times \vec{\omega}, \;\; p = C\frac{\eta V \cos\vartheta}{r^2}$$

$$\vec{\omega} = \frac{C\vec{V}\times\vec{r}}{r^3} = \hat{\varphi}\frac{CV\sin\vartheta}{r^2}$$

$$\nabla p = \hat{r}\frac{\partial p}{\partial r} + \hat{\vartheta}\frac{1}{r}\frac{\partial p}{\partial \vartheta} = -2C\frac{\eta V\cos\vartheta}{r^3}\hat{r} - C\frac{\eta V \sin\vartheta}{r^3}\hat{\vartheta}$$

$$\nabla\times\vec{\omega} = \frac{\hat{r}}{r\sin\vartheta}\frac{\partial\omega_\varphi \sin\vartheta}{\partial r} - \frac{\hat{\vartheta}}{r}\frac{\partial r\omega_\varphi}{\partial r}$$

$$= CV\frac{\hat{r}}{r^3\sin\vartheta}\frac{\partial \sin^2\vartheta}{\partial r} - CV\sin\vartheta\frac{\hat{\vartheta}}{r}\frac{\partial}{\partial r}\left(\frac{1}{r}\right)$$

$$= \hat{r}\frac{2CV\cos\vartheta}{r^3} + \hat{\vartheta}\frac{CV\sin\vartheta}{r^3}$$

An equation for the stream function

With

$$\omega_\varphi = \frac{1}{r}\frac{\partial r u_\vartheta}{\partial r} - \frac{1}{r}\frac{\partial u_r}{\partial \vartheta} = \frac{CV\sin\vartheta}{r^2}$$

and

$$u_r = \frac{1}{r^2\sin\vartheta}\frac{\partial\psi}{\partial\vartheta}, \quad u_\vartheta = -\frac{1}{r\sin\vartheta}\frac{\partial\psi}{\partial r}$$

we have

$$-\frac{1}{r\sin\vartheta}\frac{\partial^2\psi}{\partial r^2} - \frac{1}{r^3}\frac{\partial}{\partial\vartheta}\left(\frac{1}{\sin\vartheta}\frac{\partial\psi}{\partial\vartheta}\right) = \frac{CV\sin\vartheta}{r^2}$$

or

$$\frac{\partial^2\psi}{\partial r^2} + \frac{\sin\vartheta}{r^2}\frac{\partial}{\partial\vartheta}\left(\frac{1}{\sin\vartheta}\frac{\partial\psi}{\partial\vartheta}\right) = -\frac{CV\sin^2\vartheta}{r}$$

The solution for the stream function

With

$$\frac{\partial^2\psi}{\partial r^2}+\frac{\sin\vartheta}{r^2}\frac{\partial}{\partial\vartheta}\left(\frac{1}{\sin\vartheta}\frac{\partial\psi}{\partial\vartheta}\right)=-\frac{CV\sin^2\vartheta}{r}$$

the solution must be of the form:

$$\psi=V\sin^2\vartheta f(r)$$

Hence

$$\frac{\partial^2 f}{\partial r^2}-\frac{2f}{r^2}=-\frac{C}{r}$$

Thus,

$$f(r)=\frac{1}{2}Cr+\frac{A}{r}+Br^2$$

Boundary conditions at large r

With $$u_r = \frac{1}{r^2 \sin\vartheta}\frac{\partial\psi}{\partial\vartheta}, \quad u_\vartheta = -\frac{1}{r\sin\vartheta}\frac{\partial\psi}{\partial r}$$

$$\vec{u} = \left(\frac{1}{2}Cr + \frac{A}{r} + Br^2\right)\frac{V}{r^2}2\cos\vartheta\hat{r}$$

$$-\frac{V}{r}\left(\frac{1}{2}C - \frac{A}{r^2} + 2Br\right)\sin\vartheta\hat{\vartheta}$$

As $r \to \infty$,

$$\vec{u}\left(r \to \infty\right) = \vec{V} = V\cos\vartheta\hat{r} - V\sin\vartheta\hat{\vartheta}$$

Thus, B = 1/2.

Boundary conditions at $r = R$

With

$$\vec{u} = \left(Cr + \frac{2A}{r} + r^2\right)\frac{V\cos\vartheta}{r^2}\hat{r} - \left(\frac{1}{2}C - \frac{A}{r^2} + r\right)\frac{V\sin\vartheta}{r}\hat{\vartheta}$$

at $r = R$, $\vec{u} = 0$. Thus, $C = -\frac{3}{2}R,\ A = \frac{R^3}{4}$
which give

$$\vec{u} = \left(1 - \frac{3R}{2r} + \frac{R^3}{2r^3}\right)V\cos\vartheta\hat{r} - \left(1 - \frac{3R}{4r} - \frac{R^3}{4r^3}\right)V\sin\vartheta\hat{\vartheta}$$

and $$p = -\frac{3R\eta V\cos\vartheta}{2r^2}$$

Viscous drag on the sphere

To calculate the drag on the sphere, we must evaluate the various components of stress on the sphere, and this requires evaluating components of strain rate at $r = R$.

$$\dot{\varepsilon}_{rr} = \frac{\partial u_r}{\partial r},\ \dot{\varepsilon}_{\vartheta\vartheta} = \frac{1}{r}\frac{\partial u_\vartheta}{\partial \vartheta} + \frac{u_r}{r},\ \dot{\varepsilon}_{r\vartheta} = \frac{r}{2}\frac{\partial}{\partial r}\left(\frac{u_\vartheta}{r}\right) + \frac{1}{2r}\frac{\partial u_r}{\partial \vartheta}$$

We must then sum the components of these stresses parallel the direction that the sphere moves.

Evaluation of the strain rates

$$\dot{\varepsilon}_{rr} = \frac{\partial u_r}{\partial r},\ \dot{\varepsilon}_{\vartheta\vartheta} = \frac{1}{r}\frac{\partial u_\vartheta}{\partial \vartheta} + \frac{u_r}{r},\ \dot{\varepsilon}_{r\vartheta} = \frac{r}{2}\frac{\partial}{\partial r}\left(\frac{u_\vartheta}{r}\right) + \frac{1}{2r}\frac{\partial u_r}{\partial \vartheta}$$

$$\vec{u} = \left(1 - \frac{3R}{2r} + \frac{R^3}{2r^3}\right)V\cos\vartheta\hat{r} - \left(1 - \frac{3R}{4r} - \frac{R^3}{4r^3}\right)V\sin\vartheta\hat{\vartheta}$$

It turns out that two of these vanish on the sphere

$$\dot{\varepsilon}_{r\vartheta} = -\frac{RV}{2}\left(-\frac{1}{R^2} + \frac{3R}{2R^3} + \frac{4R^3}{4R^5}\right)\sin\vartheta = -\frac{3V}{4R}\sin\vartheta$$

Evaluation of the viscous drag

The component of $\tau_{r\theta}$ that is parallel to velocity of the sphere is

$$\tau_{r\vartheta}\sin\vartheta = 2\eta\dot{\varepsilon}_{r\vartheta}\sin\vartheta = -\frac{3V\eta}{2R}\sin^2\vartheta$$

We must also consider the pressure, and its component parallel to velocity of the sphere:

$$p\cos\vartheta = -\frac{3\eta V\cos\vartheta}{2R}\cos\vartheta = -\frac{3\eta V}{2R}\cos^2\vartheta$$

Thus, the drag (stress) is constant over the sphere (*a remarkable result*), and the drag force is

$$F_d = -\frac{3\eta V}{2R}4\pi R^2 = -6\pi R\eta V$$

Downward speed of sphere

If the sphere sinks because of gravity acting on it, then that force is

$$F_g = mg = \frac{4}{3}\pi R^3 \Delta\rho g$$

It must equal the drag force. Thus,

$$\frac{4}{3}\pi R^3 \Delta\rho g = F_d = 6\pi R \eta V$$

and

$$V = \frac{2R^2 \Delta\rho g}{9\eta}$$