The Abdus Salam
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Advanced School on Scaling Laws in Geophysics: Mechanical and Thermal Processes in Geodynamics

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Convection - Part II

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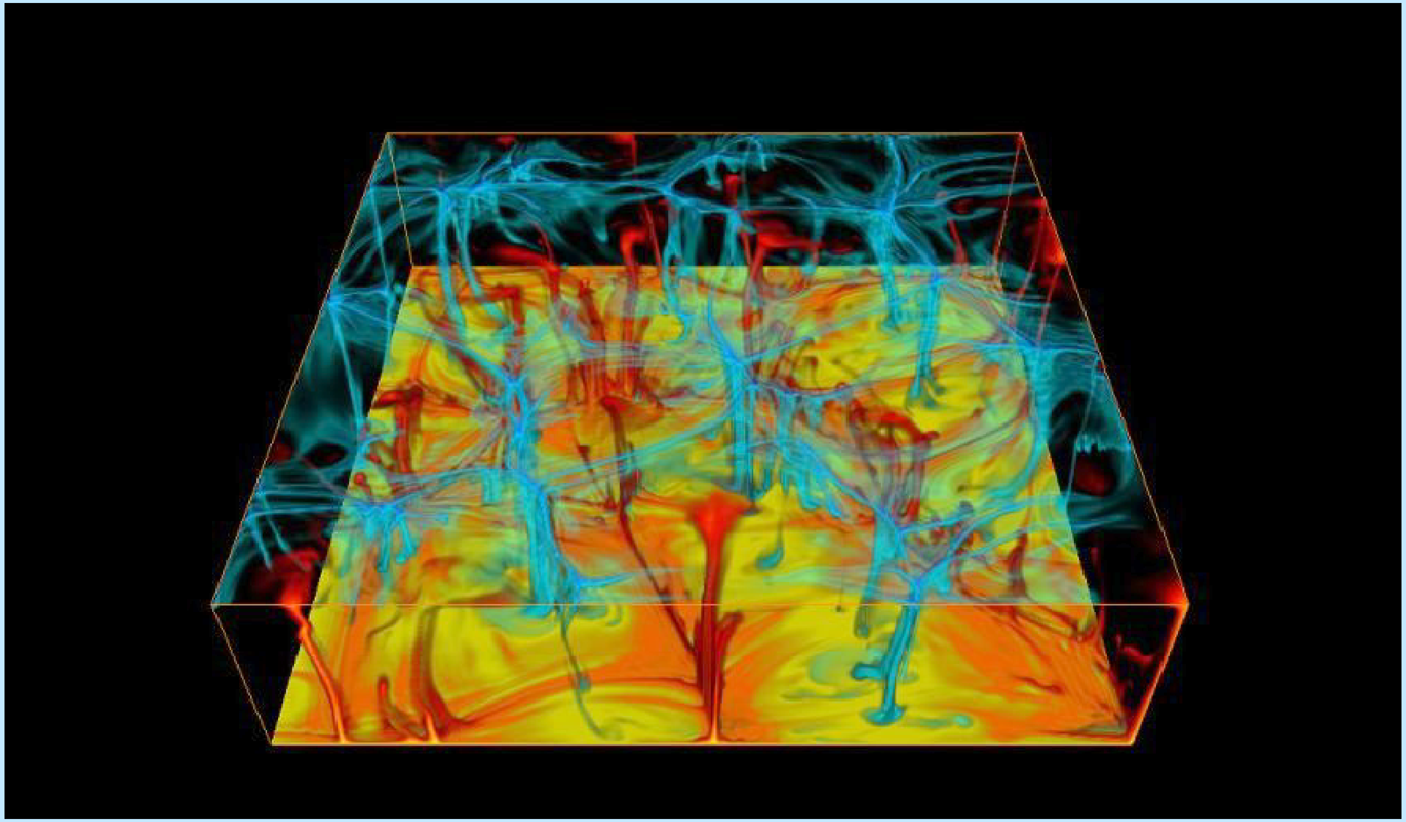


$Ra = 10^8$

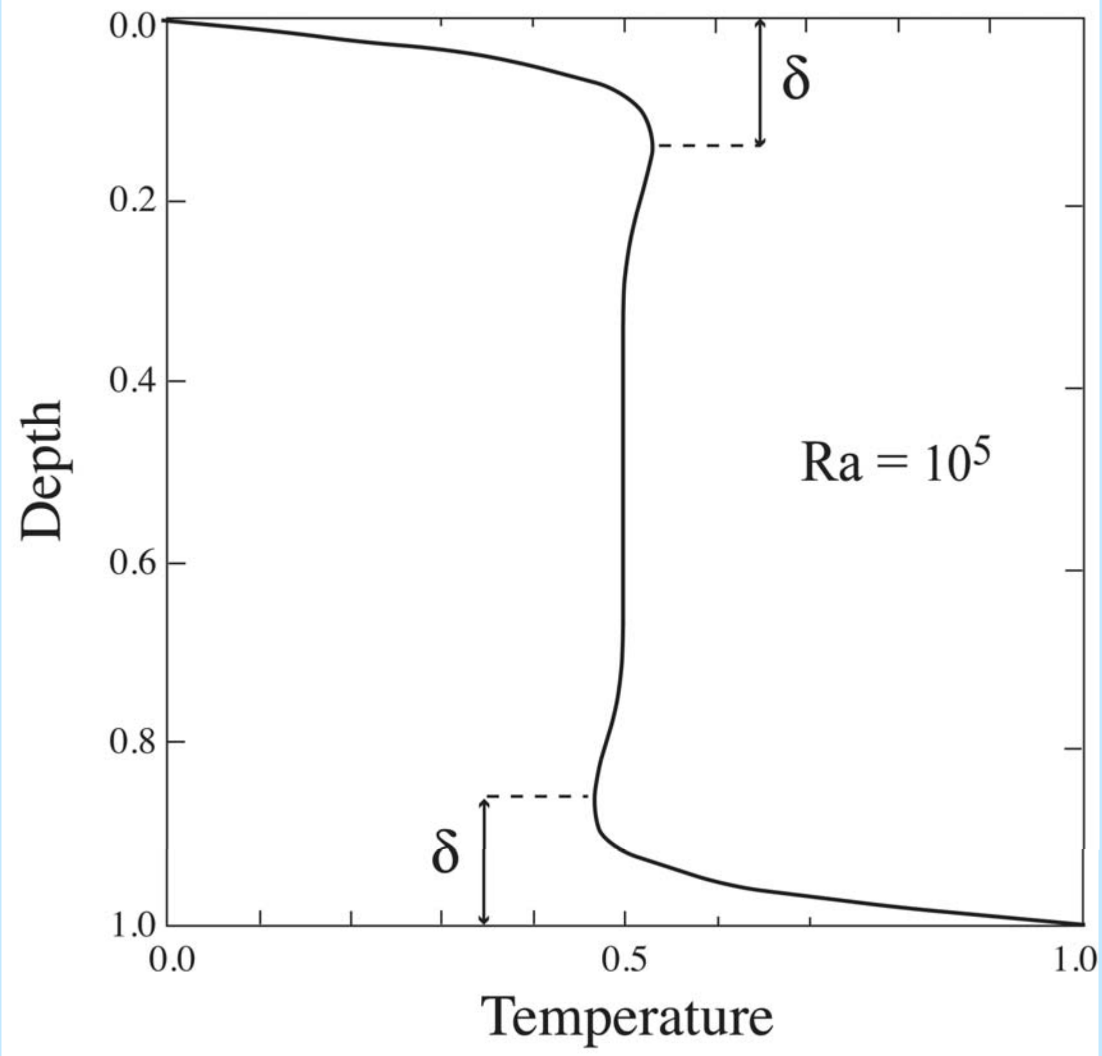
$$T = \overline{T}(z,t) + \theta(x,y,z,t).$$

$$\rho C_p \left[\frac{\partial \overline{T}}{\partial t} + \frac{\partial \overline{w\theta}}{\partial z} \right] = \lambda \frac{\partial^2 \overline{T}}{\partial z^2} \cdot$$

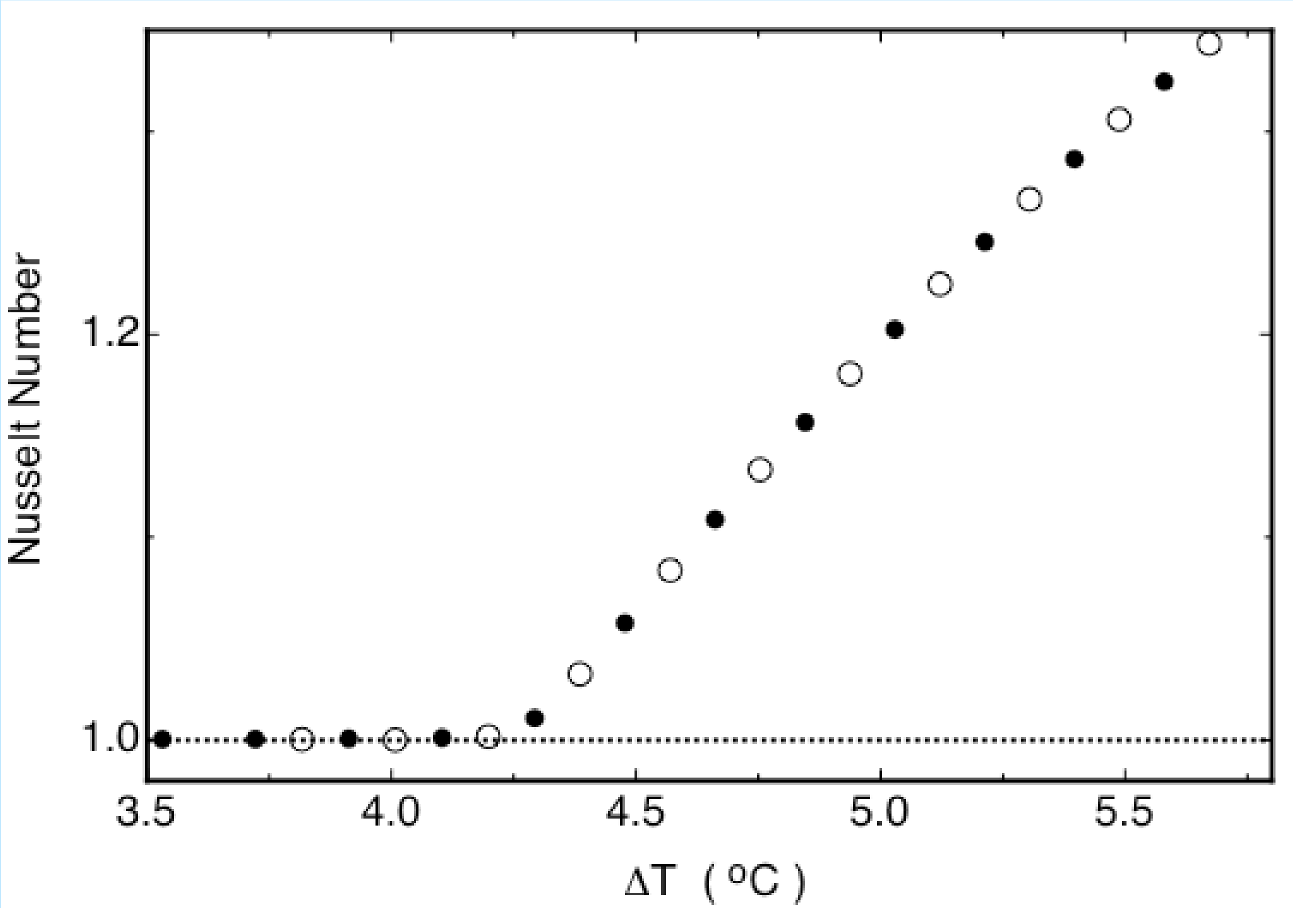
$$\rho C_p \frac{\partial \overline{T}}{\partial t} = -\frac{\partial}{\partial z} \left[-\lambda \frac{\partial \overline{T}}{\partial z} + \rho C_p \overline{w\theta} \right]$$

$$\rho C_p \frac{\partial \overline{T}}{\partial t} = -\frac{\partial \overline{q}}{\partial z}$$

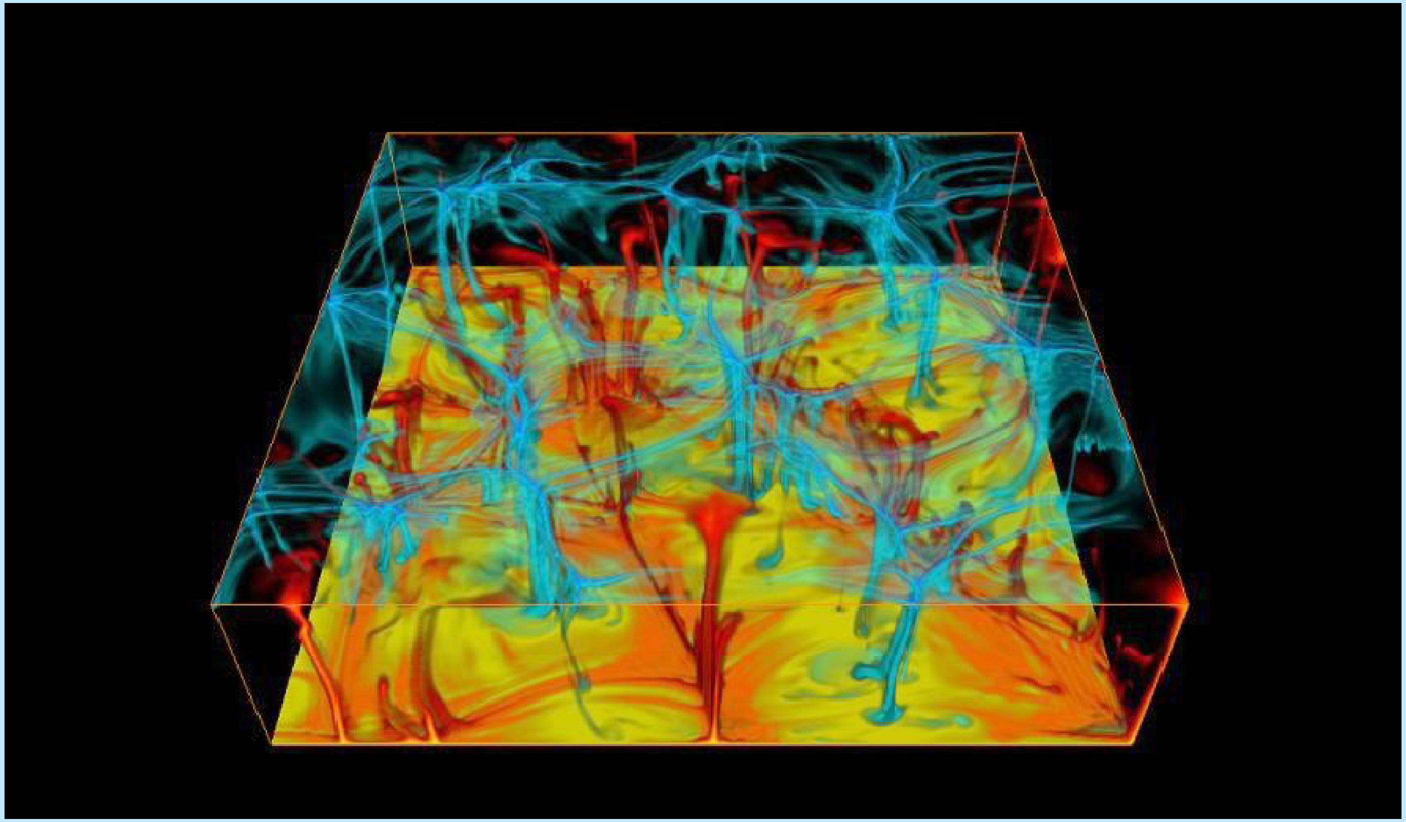
$$\overline{q} = -\lambda \frac{\partial \overline{T}}{\partial z} + \rho C_p \overline{w\theta}.$$



$$\overline{q} = -\lambda \frac{\partial \overline{T}}{\partial z} + \rho C_p \overline{w\theta} = \text{constant} = Q,$$



Nusselt Number
1.2
1.0
3.5
4.0
4.5
5.0
5.5
ΔT (°C)



$Ra = 10^8$

. *Physical characteristics of geological convective systems*

System	h	ΔT, K	μ, Pa s	Pr	Ra
Upper mantle	660 km	1300 †	5×10^{20}	10^{23}	10^{6}
Whole mantle	3000 km	3300 †	5×10^{21}	10^{24}	10^{7}
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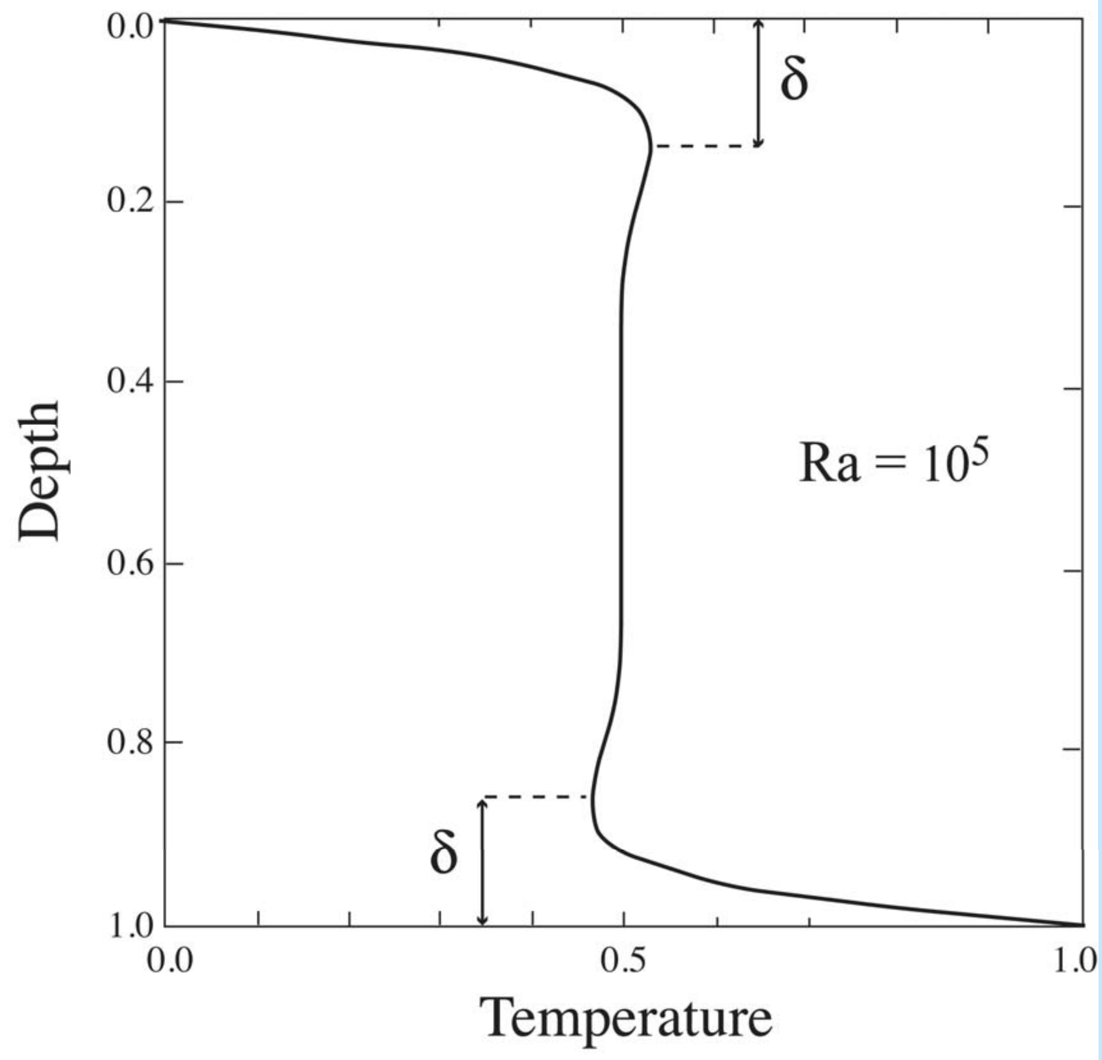
Values have been rounded off for clarity.
† True temperature difference deduced from the mantle geotherm of Figure 2.4.
‡ temperature difference across the actively convecting part of the system.

The "4/3" law for the convective heat flux at high Rayleigh number

$$\mathrm{Nu} = C_N(\mathrm{Pr}) f_N(\mathrm{Ra}),$$



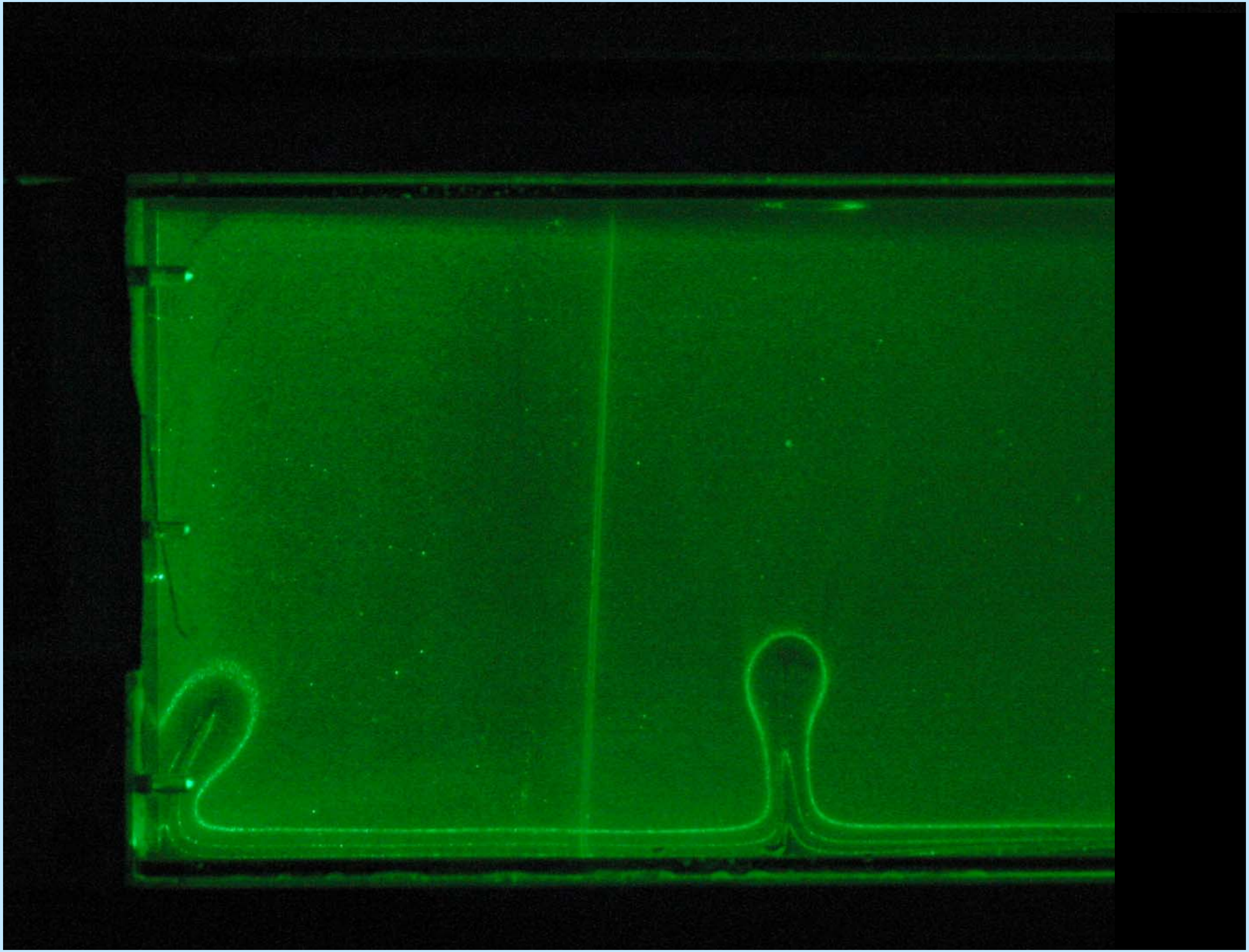
$$Q = k\frac{\Delta T}{2\delta},$$

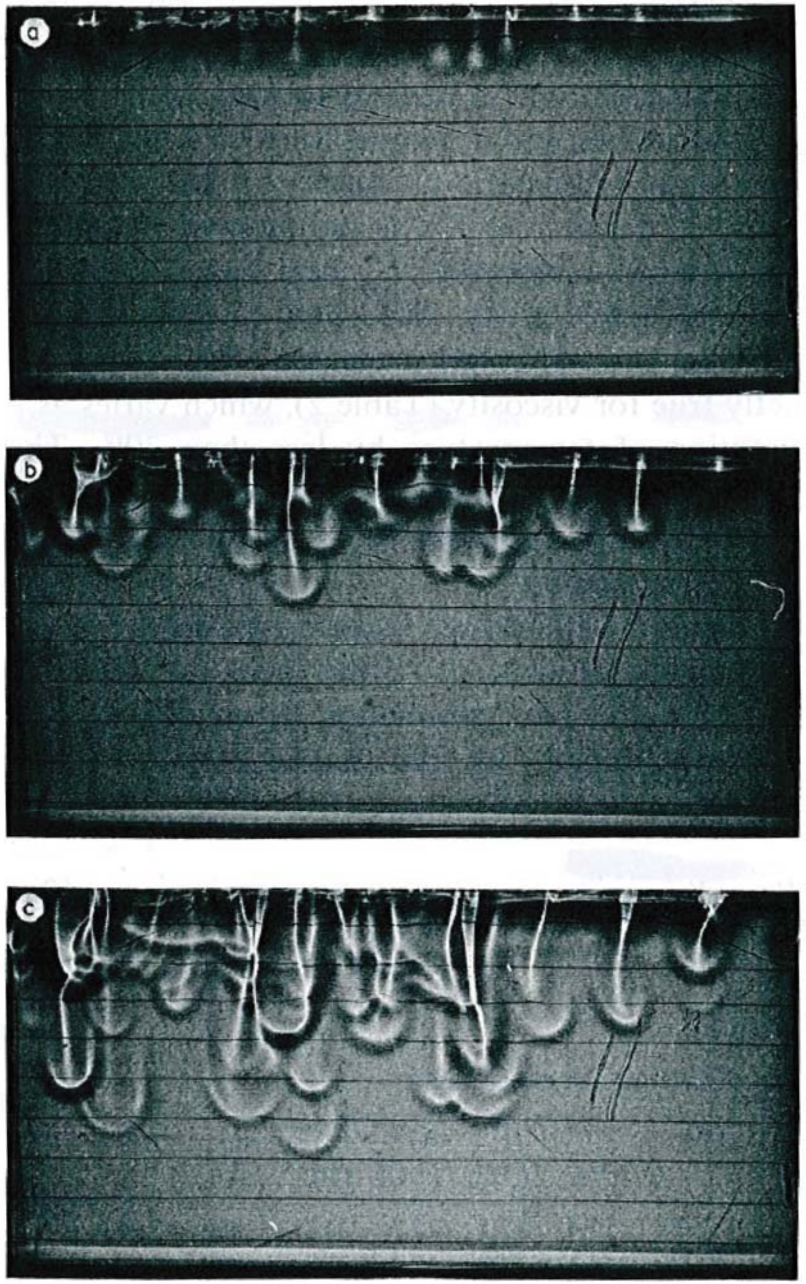
The "4/3" law for the convective heat flux at high Rayleigh number

$$\mathrm{Nu} = C_N(\mathrm{Pr}) f_N(\mathrm{Ra}),$$

$$Q : k\,\lambda \frac{\Delta T}{2\delta},$$

$$\mathrm{Nu} = \frac{h}{2\delta}.$$




a
b
c

$$Q = \mathrm{Nu}\, k \frac{\Delta T}{h} = C_N(\mathrm{Pr})\, k \frac{\Delta T}{h} f_N\left(\frac{\rho_o g \alpha \Delta T h^3}{\kappa \mu}\right),$$

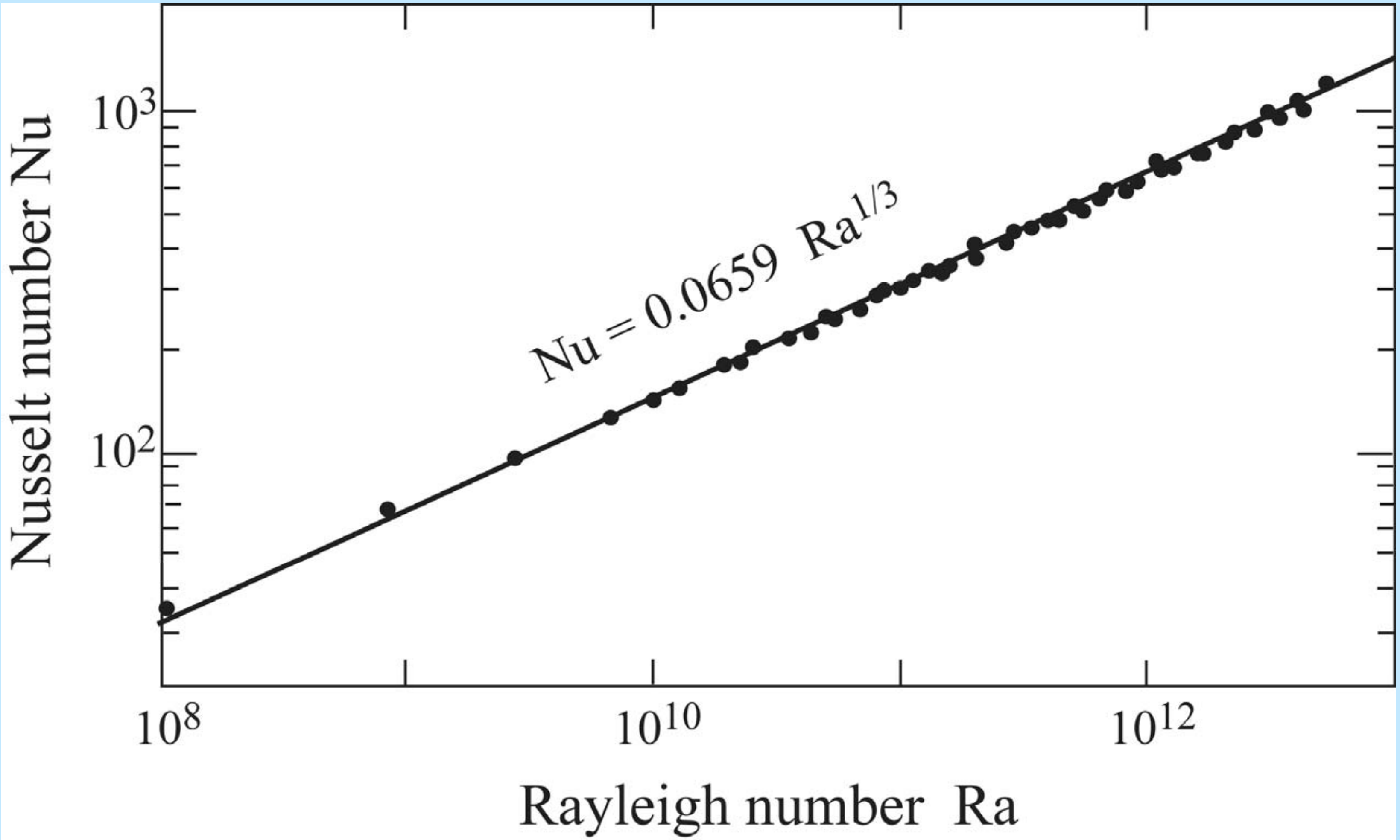
$$f_N(\mathrm{Ra}) \propto \mathrm{Ra}^{1/3}$$

$$\mathrm{Nu} = C_N \mathrm{Ra}^{1/3}.$$

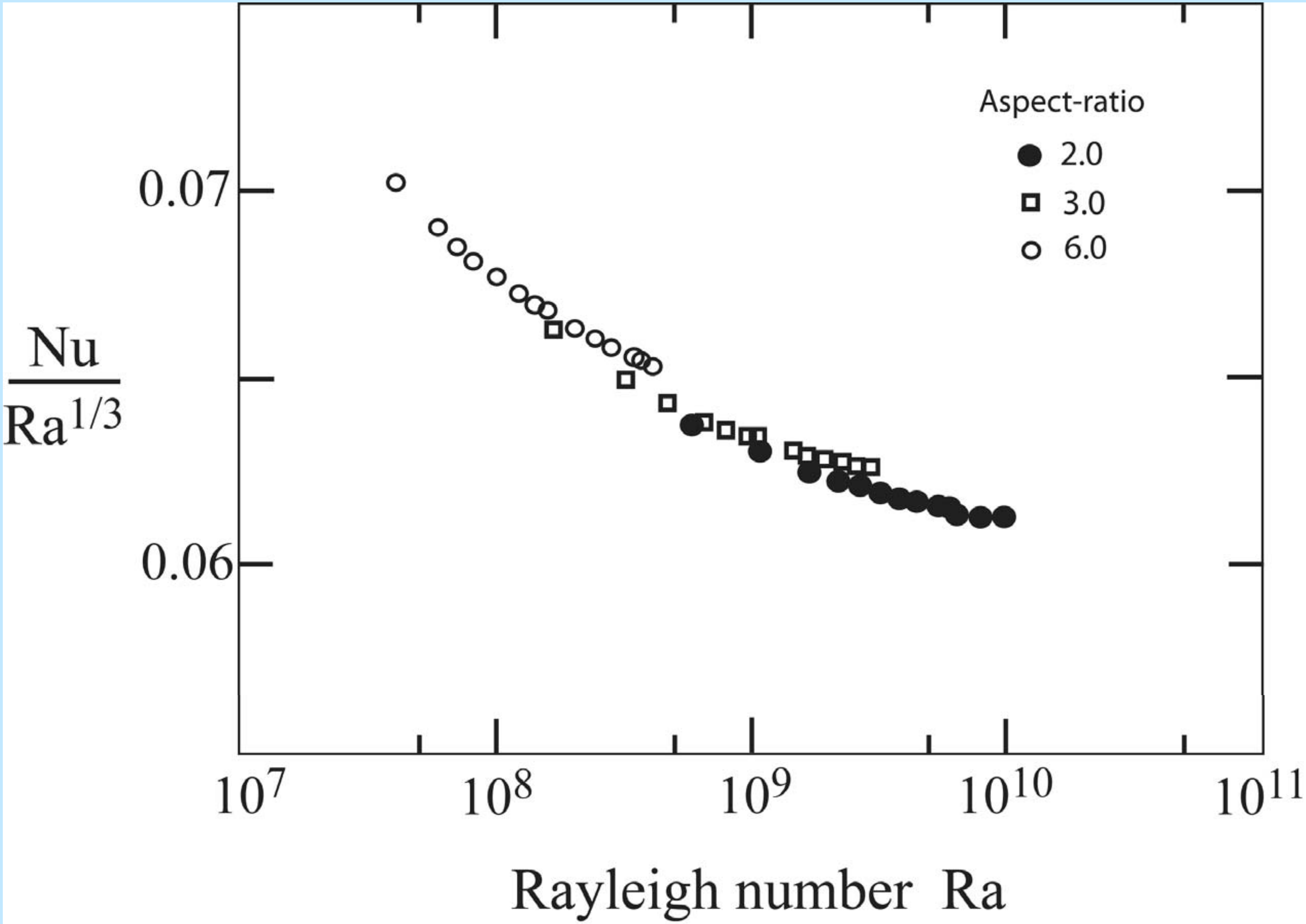
$$Q = C_Q k \left(\frac{g\alpha}{\kappa \nu}\right)^{1/3} \Delta T_\delta^{4/3},$$

$$C_Q = 2^{4/3} C_N$$

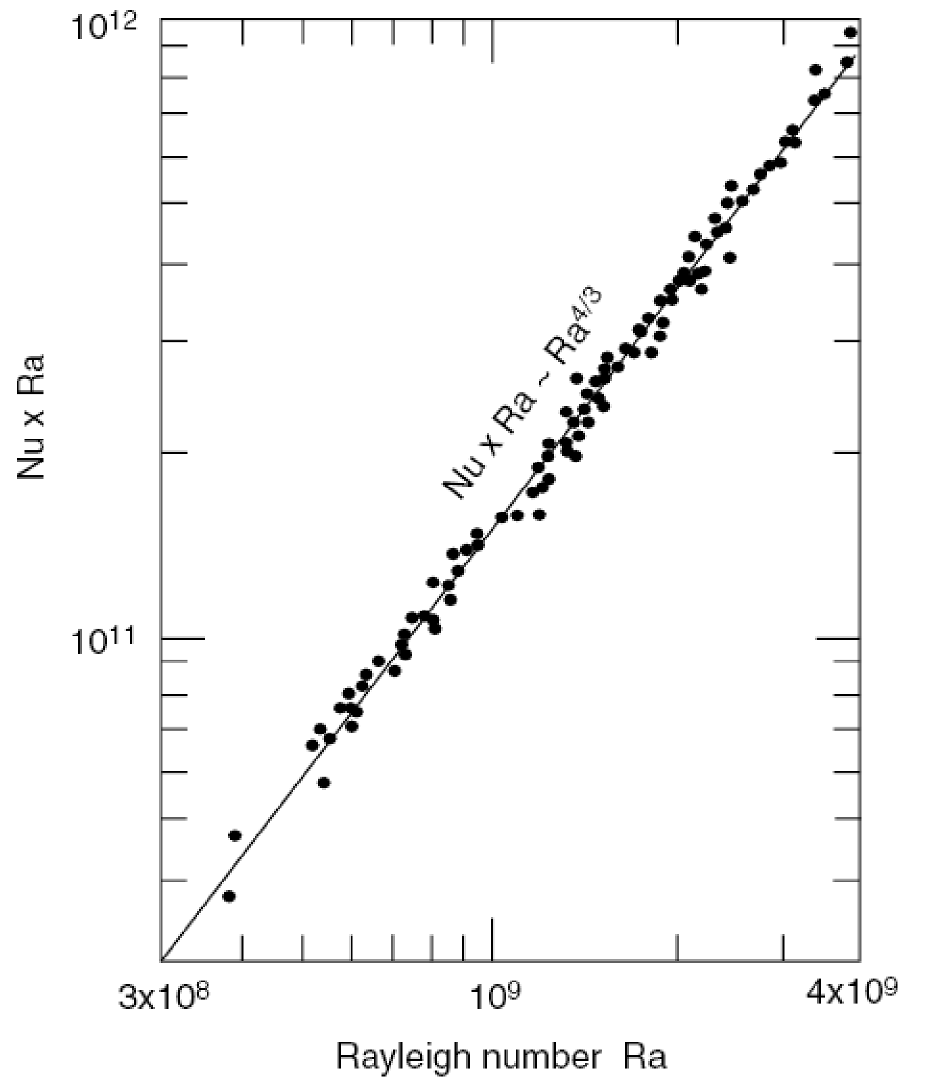
Rigid boundaries, Pr = 2750, L/h >1 (Goldstein et al., 1990)



Rigid boundaries, Pr ≈ 5 (water)   (Funfschilling et al., 2005)



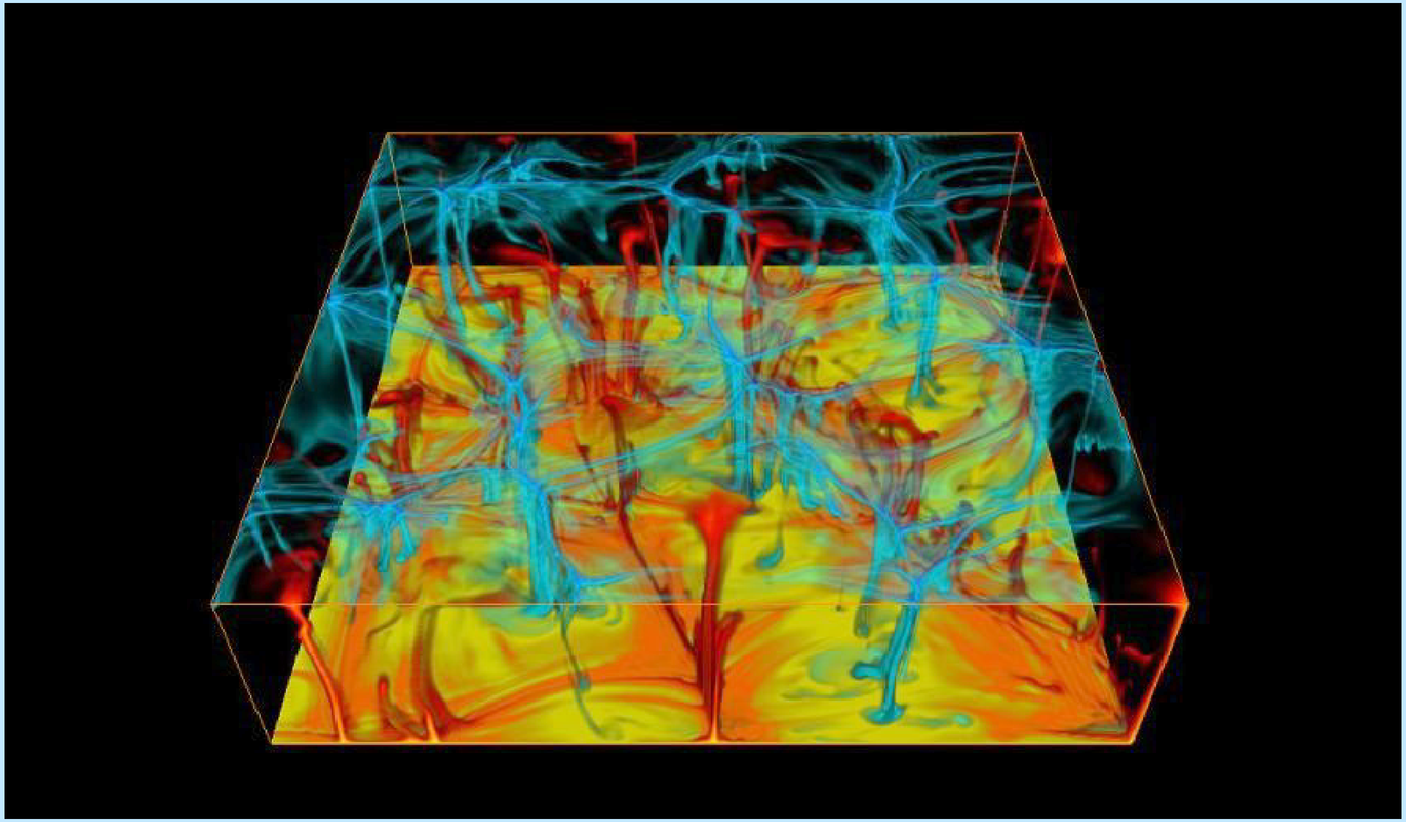
Free upper boundary (cooling from top only), Pr ≈ 5 (water) L/h = 1 {Katsaros et al., 1977)



Data on the convective heat flux in Rayleigh–Benard convection

Pr	Ra	Bound. cond.	Aspect ratio	C_N +	C_Q ++	Reference
4–6	$5 \times 10^{10} - 10^{11}$	Rigid	0.98	0.060	0.15	(Funfschilling *et al.*, 2005)
4–6	10^{10}	Rigid	2	0.062	0.16	(Funfschilling *et al.*, 2005)
4–6	$3 \times 10^8 - 4 \times 10^9$	Free	1	§	0.16†	(Katsaros *et al.*, 1977)
2750	$10^8 - 10^{13}$	Rigid	> 1	0.0659	0.17	(Goldstein *et al.*, 1990)
∞	$10^6 - 10^9$ ‡	Free	1.5	0.150¶	0.378¶	(Hansen *et al.*, 1992)

+ Constant in the Nu versus Ra relationship (5.81).
++ Constant in the local heat flux scaling law (5.82).
§ Only the boundary layer scaling can be determined in this transient cooling experiment.
† Value re-calculated for a 1/3 scaling exponent (instead of 0.33).
¶ Value re-calculated for a 1/3 scaling exponent at $Ra = 10^9$.
‡ Numerical calculations in 2D.



Ra = 10^8

$$0 = -\nabla P_h + \rho_o \left[1 - \alpha \left(\overline{T} - T_o\right)\right] \mathbf{g},$$

Irreversible kinetic dissipation

$$\psi = \mathbf{v} \cdot (\nabla \cdot \sigma \boldsymbol{\tau} - \nabla \cdot (\sigma \boldsymbol{\tau} \, \mathbf{v}).$$

$$\int_V \rho_o \frac{\partial e_c}{\partial t} dV + \int_S \rho_o e_c \mathbf{v} \cdot dS = -\int_S p\mathbf{v} \cdot dS + \int_V p \nabla \cdot \mathbf{v} dV$$
$$- \int_V \psi \, dV - \int_S (\boldsymbol{\tau} \cdot \mathbf{v}) \cdot dS + \int_V \alpha \rho_o w \theta g dV .$$

$$+\int_V \rho\alpha g w\theta \, dV - \int_V \psi \, dV = 0.$$

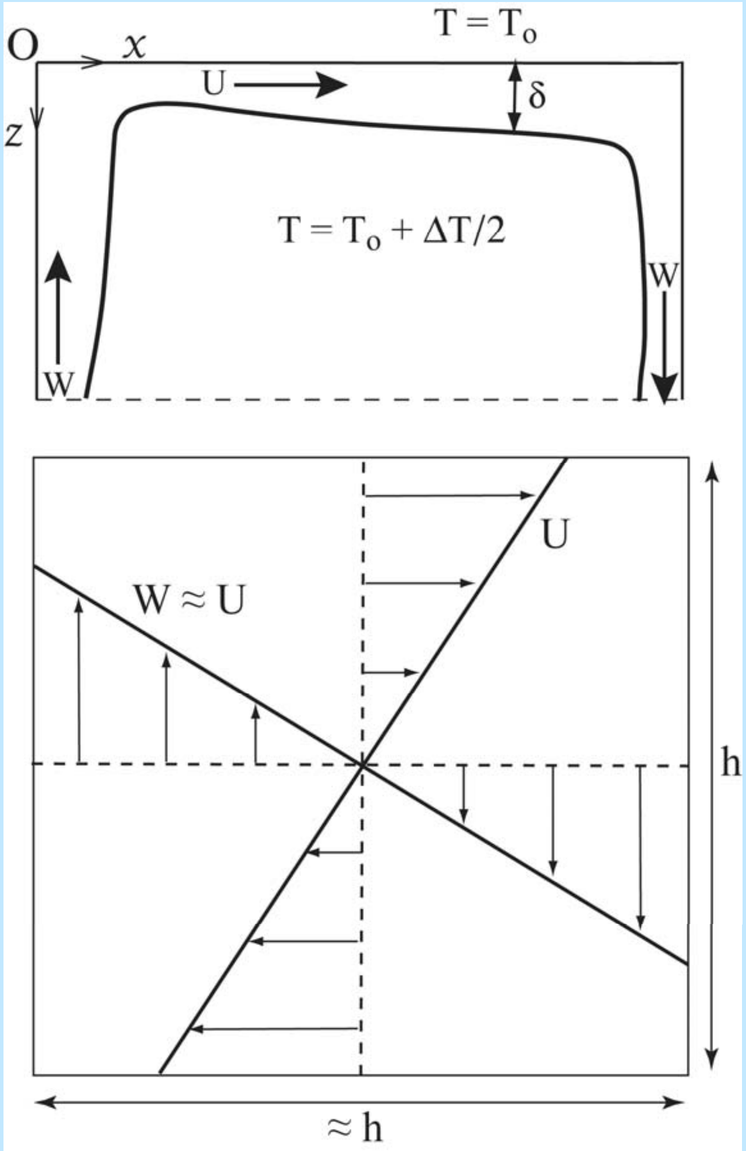
$$+\int_0^h \rho\alpha g \overline{w\theta} \, dz - \int_0^h \overline{\psi} \, dz = 0.$$

$$\int_0^h \rho C_p \overline{w\theta} \, dz = \int_0^h \left(Q + k \frac{d\overline{T}}{dz} \right) dz$$

$$= Qh + k[T(h) - T(0)] = Qh - k \Delta T.$$

$$\epsilon = \int_0^h \overline{\psi} \, dz = \mu \frac{\nu^2}{h^3} (\mathrm{Nu} - 1) \, \mathrm{Ra} \, \mathrm{Pr}^{-2}.$$

Free boundaries, large Prandtl number



$$\int_0^h \psi \, dz \sim \mu \frac{U^2}{h^2} h \sim \mu \frac{\nu^2}{h^3} \mathrm{Re}^2.$$

$$\mu \frac{\nu^2}{h^3} (\mathrm{Nu} - 1) \, \mathrm{Ra} \mathrm{Pr}^{-2} \sim \mu \frac{\nu^2}{h^3} \mathrm{Re}^2,$$

Remember that $Nu \sim h/\delta$.

Add balance between horizontal advection
and vertical diffusion
(remember scalings for laminar plumes)

$$\delta \sim \sqrt{\kappa h / U}$$

$$\mathrm{Re}^{1/2} \mathrm{Pr}^{1/2} \sim \mathrm{Nu}.$$

$$\mathrm{Nu} \sim \mathrm{Ra}^{1/3}\ ,\ \mathrm{Re} \sim \mathrm{Ra}^{2/3}\mathrm{Pr}^{-1},$$

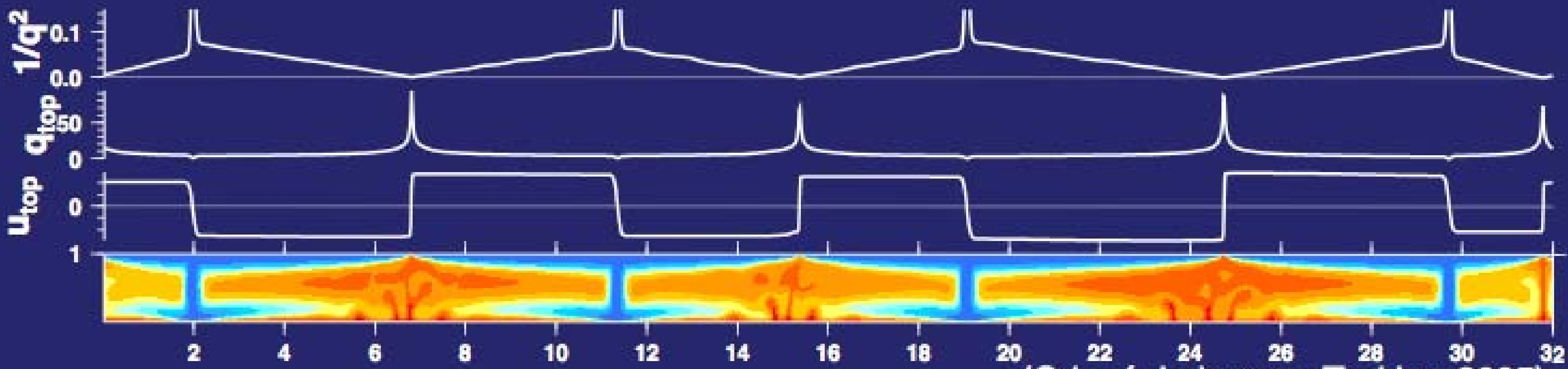
$$\delta \sim h\mathrm{Ra}^{-1/3}\ ,\ U \sim \frac{\kappa}{h}\mathrm{Ra}^{2/3}.$$

$$\delta \sim \sqrt{\kappa h / \dot{U}}$$

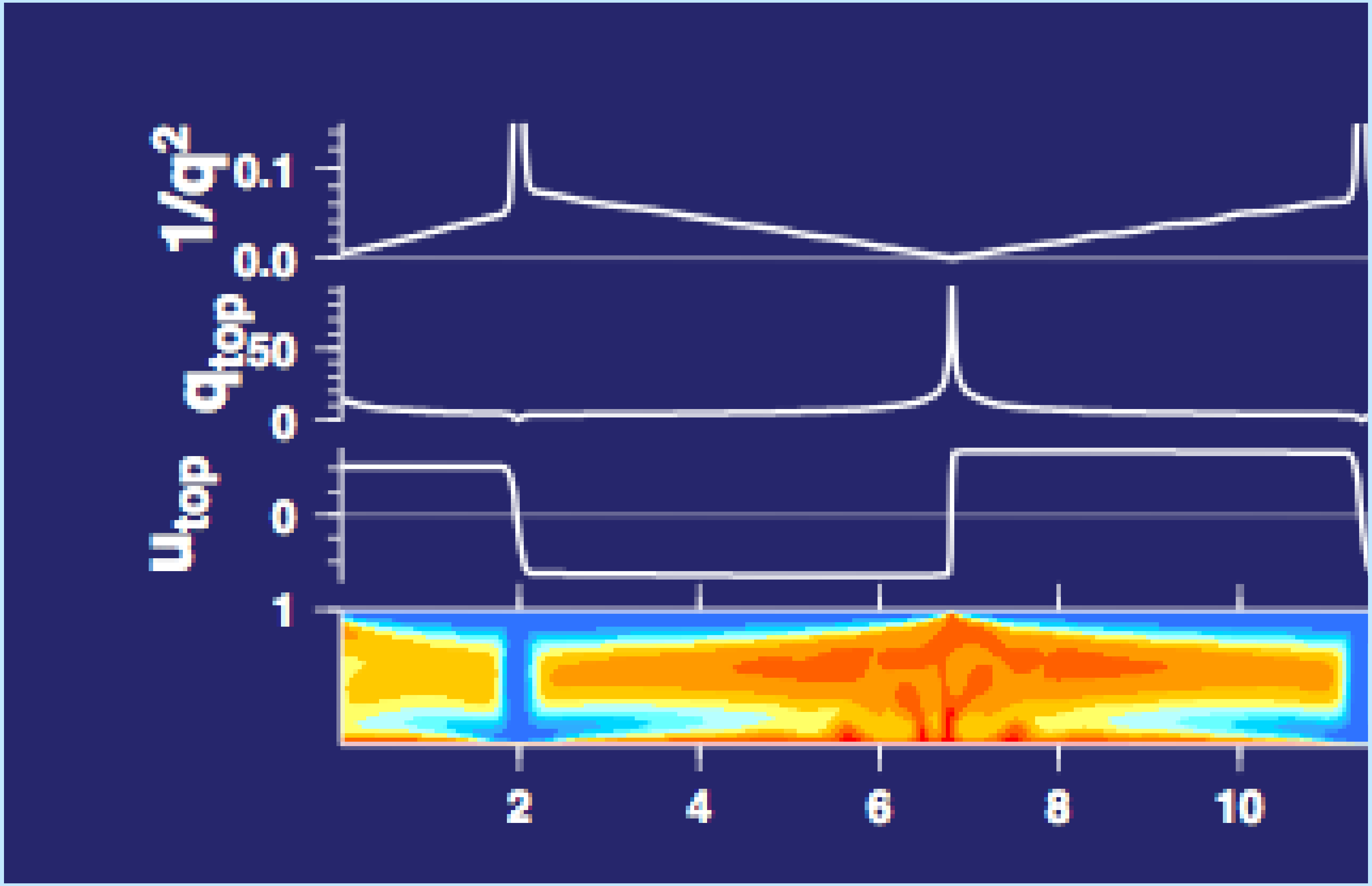
This is the average boundary layer thickness.
The local
thickness at distance ***x*** from upwelling is

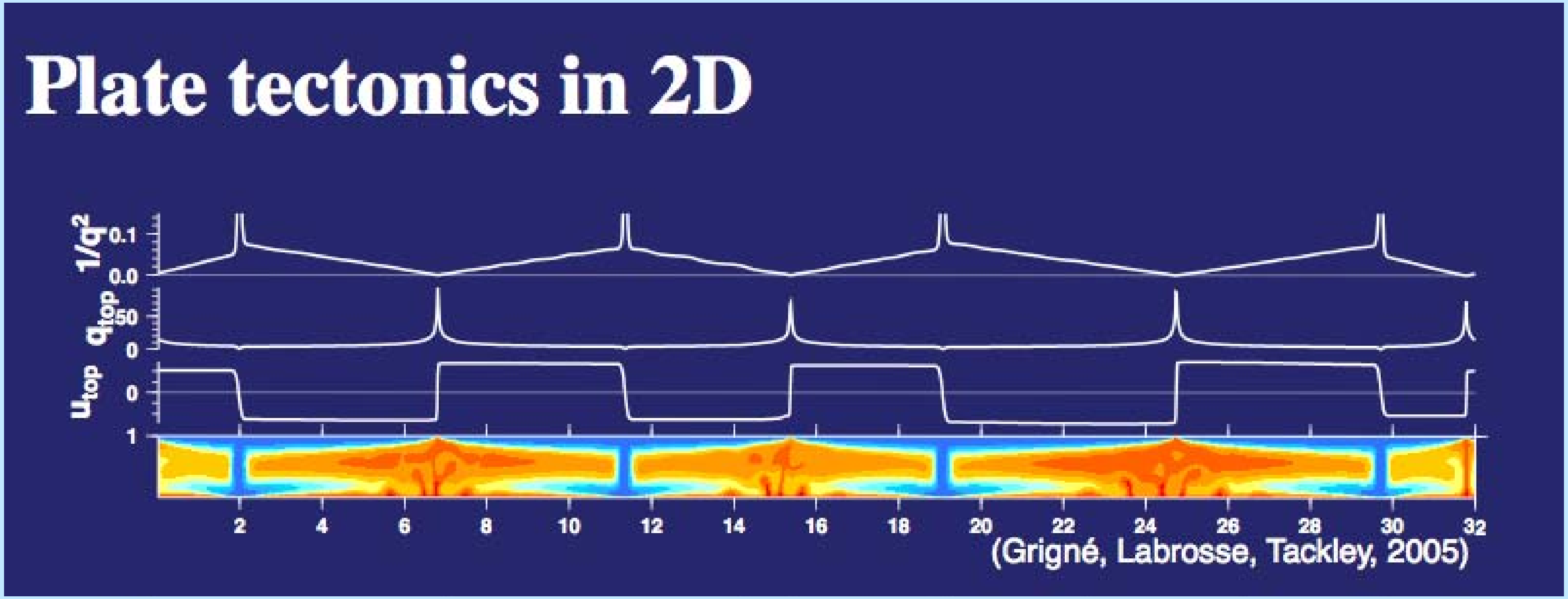
$$\delta \sim \sqrt{\kappa x / \dot{U}}$$

Plate tectonics in 2D



(Grigné, Labrosse, Tackley, 2005)

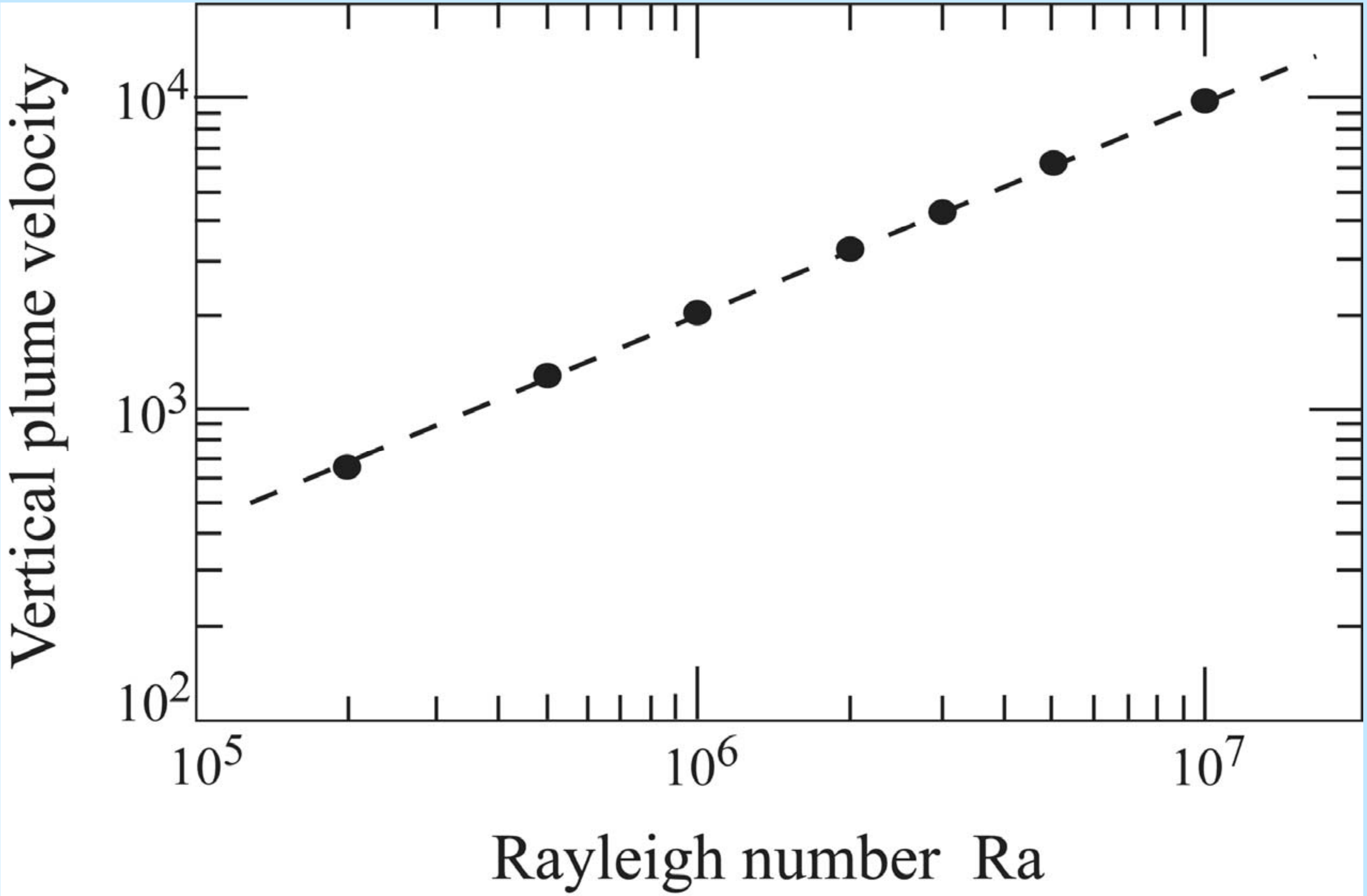

1/q^2
0.1
0.0
q_top
50
0
u_top
0
1
2
4
6
8
10



Heat flux ~ $(\text{distance})^{-1/2}$

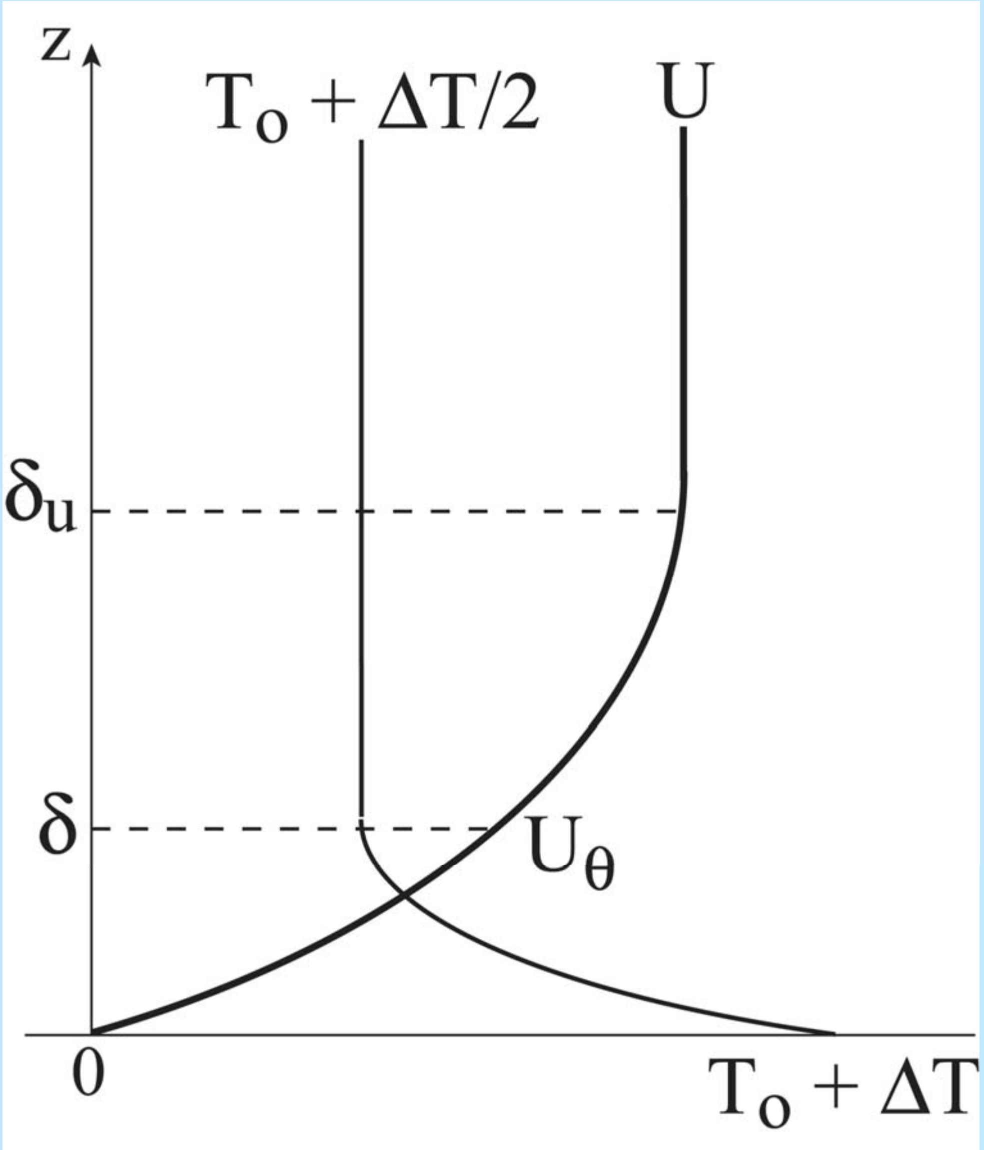
$$\mathrm{Nu} \sim \mathrm{Ra}^{1/3} \ , \ \mathrm{Re} \sim \mathrm{Ra}^{2/3}\mathrm{Pr}^{-1},$$

$$\delta \sim h\mathrm{Ra}^{-1/3} \ , \ U \sim \frac{\kappa}{h}\mathrm{Ra}^{2/3}.$$



(Galsa & Lenkey, Phys. Fluids 2007)

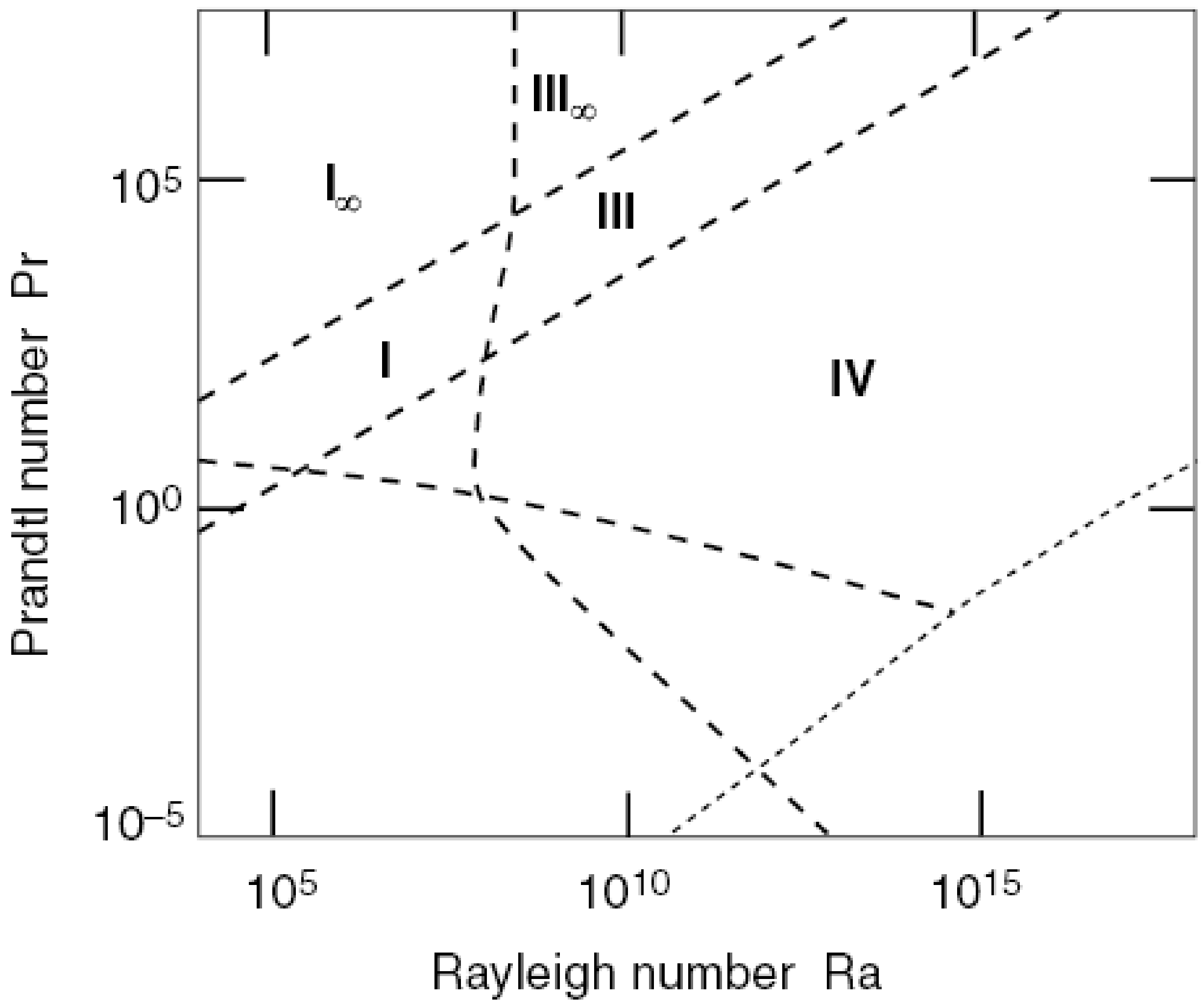
With rigid boundaries



The regimes of Rayleigh-Bénard convection in rigid enclosures (Grossmann & Lohse, J. Fluid Mech. 2000, Phys. Fluids. 2001)

Regime	Dominant dissipation §	Nu	Re
I	$(u,B)-(\theta,B)$	$0.31\ \mathrm{Ra}^{1/4}\mathrm{Pr}^{-1/12}$	$0.073\ \mathrm{Ra}^{1/3}\mathrm{Pr}^{-5/6}$
I_∞ ($\mathrm{Pr} \gg 1$)	$(u,B)-(\theta,B)$	$0.35\ \mathrm{Ra}^{1/5}$	$0.054\ \mathrm{Ra}^{3/5}\mathrm{Pr}^{-1}$
III	$(u,B)-(\theta,I)$	$0.018\ \mathrm{Ra}^{3/7}\mathrm{Pr}^{-1/7}$	$0.023\ \mathrm{Ra}^{4/7}\mathrm{Pr}^{-6/7}$
III_∞ ($\mathrm{Pr} \gg 1$)	$(u,B)-(\theta,I)$	$0.027\ \mathrm{Ra}^{1/3}$	$0.015\ \mathrm{Ra}^{2/3}\mathrm{Pr}^{-1}$
IV	$(u,I)-(\theta,I)$	$0.060\ \mathrm{Ra}^{1/3}$	$0.088\ \mathrm{Ra}^{4/9}\mathrm{Pr}^{-2/3}$

§Dominant contributions to kinetic and thermal dissipation (see text). u and θ stand for the kinetic and thermal dissipation, respectively, and symbols I and B indicate interior and boundary-layer contributions, respectively.



III∞
I∞
III
I
IV
10^5
10^0
10^{-5}
Prandtl number Pr
10^5
10^{10}
10^{15}
Rayleigh number Ra

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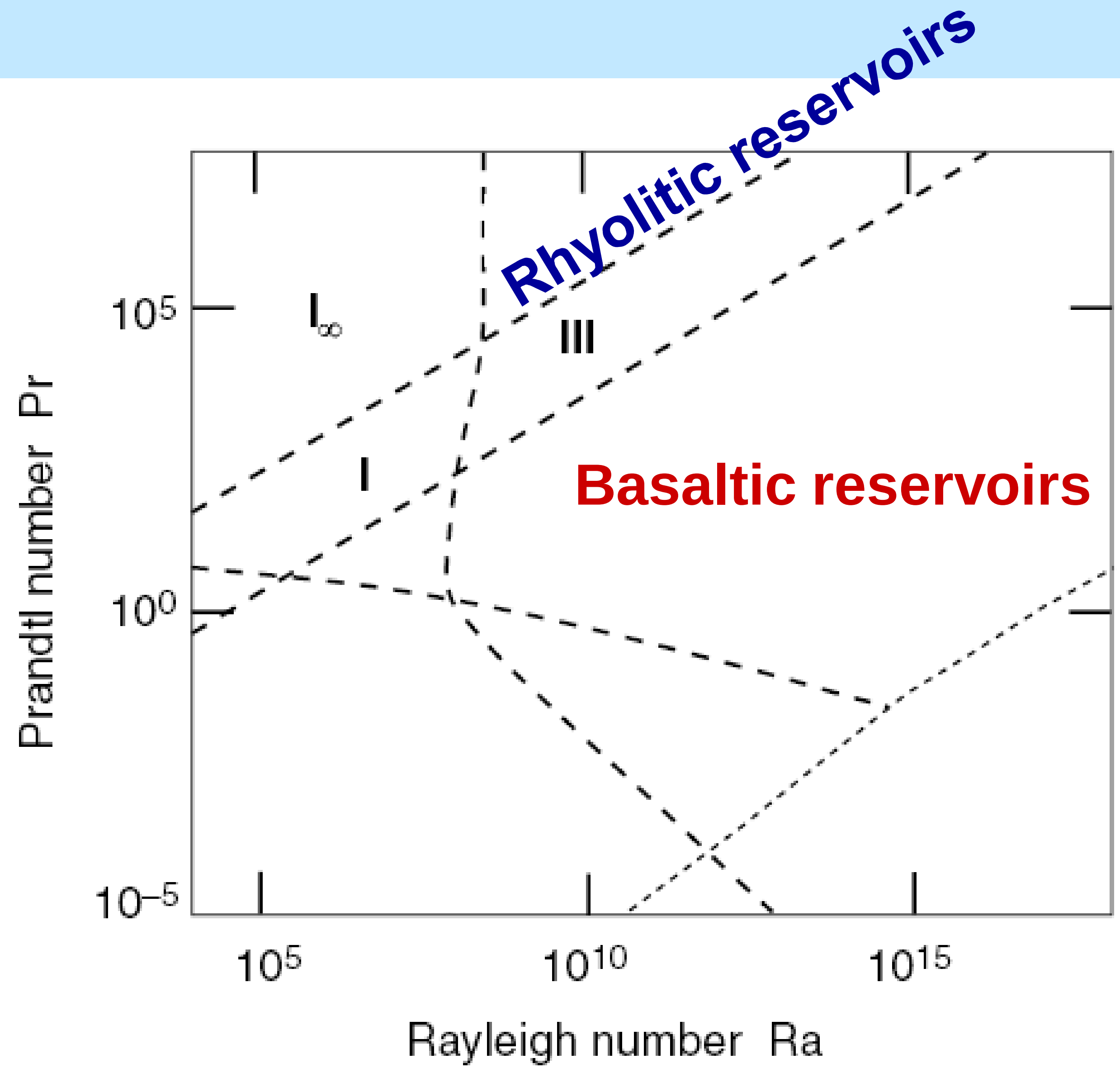

Rhyolitic reservoirs
Basaltic reservoirs
I_∞
I
III
10^5
10^0
10^{-5}
Prandtl number Pr
10^5
10^{10}
10^{15}
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