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Advanced School on Scaling Laws in Geophysics: Mechanical and Thermal Processes in Geodynamics

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Laboratory Notes by Shijie Zhong

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Dynamic topography and geoid kernels

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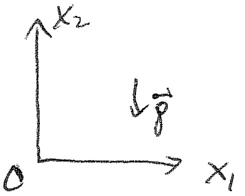
i. Stream function for 2-D flow



$$\begin{cases} -\nabla p + \eta \nabla^2 \vec{v} + \delta\rho \vec{g} = 0 & ① \\ \nabla \cdot \vec{v} = 0 & ② \end{cases}$$

where $\delta\rho$ is density anomalies and p is dynamic pressure.

Define stream function ψ such that $\vec{v} = \left(-\frac{\partial \psi}{\partial x_2}, \frac{\partial \psi}{\partial x_1}\right)$.

Such defined ψ automatically satisfies $\nabla \cdot \vec{v} = 0$.

Eliminate p by applying $\nabla \times$ to the equation of motion, ①, and assuming that η is a constant, and $\vec{\omega} = \nabla \times \vec{v}$

$$\eta \nabla^2 \vec{\omega} + \nabla \times (\delta\rho \vec{g}) = 0. \qquad ③$$

For 2-D flow, $\vec{\omega} = \nabla \times \vec{v} = \left(\frac{\partial v_2}{\partial x_1} - \frac{\partial v_1}{\partial x_2}\right) \hat{k}$, ④

where $\hat{k}$ is the unit vector $\perp$ X_1OX_2 and points to x_3 direction.

$$\nabla \times (\delta\rho \vec{g}) = -\frac{\partial \delta\rho}{\partial x_1} g \hat{k}, \qquad ⑤$$

where $\vec{g}$ is in the opposite direction to x_2.

③ becomes: $\eta \nabla^2 \left(\frac{\partial v_2}{\partial x_1} - \frac{\partial v_1}{\partial x_2}\right) - \frac{\partial \delta\rho}{\partial x_1} g = 0$

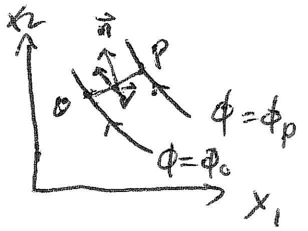
Substitute v_1 and v_2 with ψ:

$$\eta \nabla^2 \nabla^2 \psi - \frac{\partial \delta\rho}{\partial x_1} g = 0$$

$$\eta \nabla^4 \psi = \frac{\partial \delta\rho}{\partial x_1} g \qquad ⑥$$

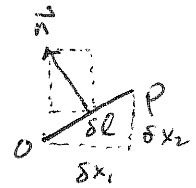
Remarks: Ⓐ $\frac{\partial \delta\rho}{\partial x_1}$ drives the flow as the source term.

B



$\phi_p - \phi_o$ is the mass flux across the line segment op. Easy to prove.

Let $\delta\ell$ be the length of $\overline{op}$, and dQ be the mass flux across $\overline{op}$, and $\vec n$ be the normal vector,



$$dQ = \vec v \cdot \vec n \, \delta\ell ,$$

$$\vec n = -\frac{\delta x_2}{\delta\ell}\vec e_1 + \frac{\delta x_1}{\delta\ell}\vec e_2$$

$$\vec v = v_1 \vec e_1 + v_2 \vec e_2$$

$$dQ = -v_1 \delta x_2 + v_2 \delta x_1 = \frac{\partial\psi}{\partial x_2}\delta x_2 + \frac{\partial\psi}{\partial x_1}\delta x_1 = d\phi$$

$$Q_{op} = \int_o^p d\phi = \int_o^p d\psi = \cancel{\Phi}\,\psi_p - \psi_o$$

2. General solution for $\eta \nabla^4 \psi = \frac{\partial \delta\ell}{\partial x_1} g$.

For homogeneous equation $\eta\nabla^4\psi = 0$ or $\nabla^4\psi = 0$

or $$\frac{\partial^4\psi}{\partial x_1^4} + 2\frac{\partial^4\psi}{\partial x_1^2 \partial x_2^2} + \frac{\partial^4\psi}{\partial x_2^4} = 0, \qquad (7)$$

Let $\psi = \sin k x_1 \, Y(x_2)$, where $k = \frac{2\pi}{\lambda}$, wavenumber,

then (7) $\rightarrow$ $$\frac{d^4 Y}{d x_2^4} - 2k^2 \frac{d^2 Y}{d x_2^2} + k^4 Y = 0 \qquad (8)$$

Let $Y = \exp(m x_2)$,

$m^4 - 2k^2 m^2 + k^4 = 0$, or $m = \pm k$, both are repeated roots.

Therefore, the four independent solutions for (8)

$\exp(k x_2)$, $x_2 \exp(k x_2)$, $\exp(-k x_2)$ and $x_2 \exp(-k x_2)$

The general solution for ψ is:

$$\psi = \sin kx_1 \left[A' \exp(-kx_2) + B' x_2 \exp(-kx_2) + C' \exp(kx_2) + D' x_2 \exp(kx_2) \right] \quad (9)$$

Introduce hyperbolic functions: $\mathrm{sh}\, x = \frac{e^x - e^{-x}}{2}$, $\mathrm{ch}\, x = \frac{e^x + e^{-x}}{2}$,

(Notice: $\frac{d\, \mathrm{sh} x}{dx} = \mathrm{ch}\, x$ and $\frac{d\, \mathrm{ch} x}{dx} = \mathrm{sh}\, x$)

It's easy to show that

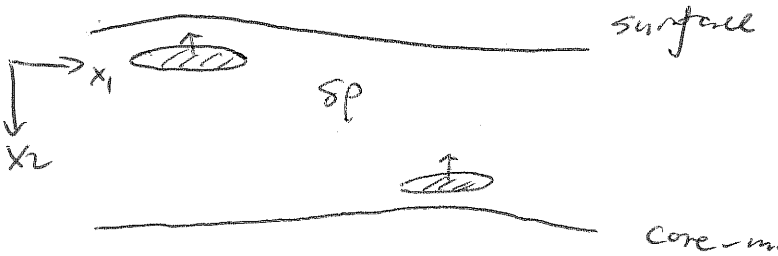
$$\psi = \sin kx_1 \left[A\, \mathrm{ch}\, kx_2 + B\, \mathrm{sh}\, kx_2 + C x_2\, \mathrm{ch}\, kx_2 + D x_2\, \mathrm{sh}\, kx_2 \right] \quad (10)$$

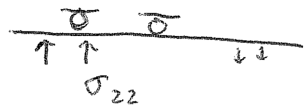
is also the general solution to (7),

where A, B, C & D are unknown constants.

For an application, constraints such that boundary conditions can be used to determine these constants. Once they are determined, velocity field $\vec{v}$ can be recovered and hence the full solution of p and stress fields.

3. Dynamic topography.





Often it is convenient to use zero-normal velocity boundary condition $v_2 = 0$, which linearizes the problem. However, $v_2 = 0$ at the boundary may introduce $\sigma_{22} \neq 0$ at the boundary. If the boundary were let go free, this σ_{22}

would cause the surface to deform in the vertical direction.
For long-wavelength approximation or the induced surface deformation $h \ll \lambda$, a good approximation for h is $h = \frac{\sigma_{22}}{\Delta\rho g}$, where $\Delta\rho$ is the density difference across the boundary.

In the new coordinate system where $x_2 \parallel \vec{g}$,

$$\eta \nabla^4 \psi = -\frac{\partial \delta\rho}{\partial x_1} g, \qquad (11)$$

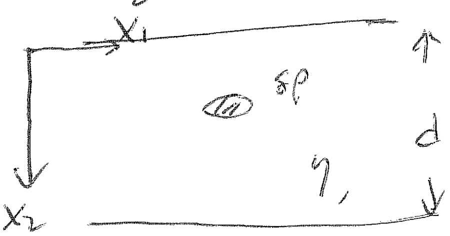
Normalizing $\delta\rho$ by ρ_0, x_i by d, and ψ by $\frac{\rho_0 g d^3}{\eta}$, the non-dimensional form

$$\nabla^4 \psi' = -\frac{\partial \delta\rho'}{\partial x_1'}, \qquad (12)$$

Drop out the primes, the nondimensional equation.

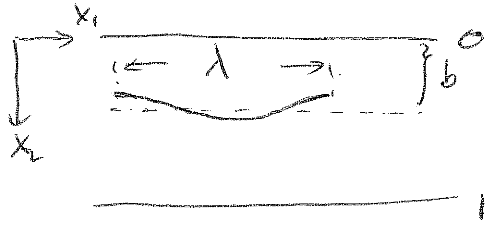
$$\nabla^4 \psi = -\frac{\partial \delta\rho}{\partial x_1} \qquad (13)$$

with $x_2 \in [0, 1]$.



4. Dynamic topography kernels.

For the source term $\delta\rho(x_1, x_2)$, we may consider a special form $\delta\rho = \cos kx_1 \, \delta(x_2 - b)$,



$\delta(x_2 - b)$ is the delta function localizing at $x_2 = b$,

$$\begin{cases} \int_0^1 f(x_2)\, \delta(x_2 - b)\, dx_2 = f(b) \\ \delta(x_2 - b) = 0 \quad \text{for } x_2 \neq b. \end{cases} \qquad (14)$$

This is the Green's function approach.

Suppose that the surface dynamic topography in response to the δ-function source is $H(b,k)$ then for buoyancy force $\delta\rho(x_2,k)$, then the total topography may be written as

$$\mathcal{H}(k) = \int_0^1 H(x_2', k)\, \delta\rho(x_2', k)\, dx_2'. \qquad (15)$$

Therefore $H(b,k)$ is called topography kernels or response functions. Notice that the horizontal dependence is always considered to take a form of $\cos kx_1$.

OK, now let's consider this problem:

$$\nabla^4 \psi = k \sin kx_1\, \delta(x_2 - b) \quad (16), \quad \Leftarrow \left[-\frac{\partial\, \delta\rho}{\partial x_1}\right].$$

For the layer above or below $x_2 = b$, the equation is still a homogeneous equation as $\delta(x_2-b) = 0$ for $x_2 \neq b$.

$$\psi_b = \sin kx_1 [A_b \operatorname{ch} k(x_2-1) + B_b \operatorname{sh} k(x_2-1) + C_b (x_2-1)\operatorname{ch} k(x_2-1) + D_b (x_2-1) \operatorname{sh} k(x_2-1)] \quad \text{for } x_2 > b \qquad (17)$$

$$\psi_t = \sin kx_1 [A_t \operatorname{ch} kx_2 + B_t \operatorname{sh} kx_2 + C_t x_2 \operatorname{ch} kx_2 + D_t x_2 \operatorname{sh} kx_2] \quad \text{for } 0 \le x_2 < b \qquad (18)$$

8 unknown parameters: A_t, B_t, C_t, D_t, A_b, B_b, C_b, & D_b.

[Note that (17) with $x_2 - 1$ remains to be the general solution. With $x_2 - 1$, the solutions are simpler.]

We need 8 constraints to determine these 8 constants.

$$\left.\begin{aligned} V_{2t} &= \tau_{12}^{t} = 0 \quad \text{at } x_2 = 0 \\ V_{2b} &= \tau_{12}^{b} = 0 \quad \text{at } x_2 = 1 \end{aligned}\right\} \text{free-slip.} \quad \Leftarrow 4 \text{ constraints.}$$

The other 4 constraints are from conditions at $x_2 = b$.

$$\left.\begin{aligned} V_{1t} &= V_{1b} \\ V_{2t} &= V_{2b} \\ \tau_{12}^{t} &= \tau_{12}^{b} \end{aligned}\right\} \text{ 3 continuity conditions.}$$

And the last one is by integrating (16) vertically from $x_2 = b^-$ to $x_2 = b^+$.

$$V_1 = -\frac{\partial\psi}{\partial x_2}, \quad V_2 = \frac{\partial\psi}{\partial x_1}$$

$$V_{1t} = -k \sin kx_1 \left[\left(A_t + C_t x_2 + \frac{D_t}{k}\right) \operatorname{sh} kx_2 + \left(B_t + D_t x_2 + \frac{C_t}{k}\right) \operatorname{ch} kx_2 \right]$$

$$V_{2t} = k \cos kx_1 \left[A_t \operatorname{ch} kx_2 + B_t \operatorname{sh} kx_2 + C_t x_2 \operatorname{ch} kx_2 + D_t x_2 \operatorname{sh} kx_2 \right]$$

$$V_{1b} = -k \sin kx_1 \left\{ \left[A_b + C_b (x_2 - 1) + \frac{D_b}{k}\right] \operatorname{sh} k(x_2-1) + \left[B_b + D_b (x_2-1) + \frac{C_b}{k}\right] \cdot \operatorname{ch} k(x_2 - 1) \right\}$$

$$V_{2b} = k \cos kx_1 \left[A_b \operatorname{ch} k(x_2-1) + B_b \operatorname{sh} k(x_2-1) + C_b (x_2-1) \operatorname{ch} k(x_2-1) + D_b (x_2-1) \operatorname{sh} k(x_2-1) \right]$$

Stress normalized by $\rho_0 g d$,

$$\tau_{12}^{t} = \left(\frac{\partial V_{1t}}{\partial x_2} + \frac{\partial V_{2t}}{\partial x_1} \right) \quad \text{for the top layer above } x_2 = b$$

$$\tau_{12}^{b} = \left(\frac{\partial V_{1b}}{\partial x_2} + \frac{\partial V_{2b}}{\partial x_2} \right) \quad \text{for the bottom layer below} \ldots$$

With free-slip B.C. at $x_2 = 0$ and $x_2 = 1$, we have

$$A_b = A_t = D_t = D_b = 0 \qquad (\text{4 are gone!})$$

Here it is easy to show $A_b = A_t = 0$, by considering $V_{2t} = 0$ at $x_2 = 0$ and $V_{2b} = 0$ at $x_2 = 1$. $D_t = D_b = 0$ would require some derivations.

At $x_2 = b$, continuity of V_1, V_2 and τ_{12} requires continuity of ψ, $\frac{\partial \psi}{\partial x_2}$, and $\frac{\partial^2 \psi}{\partial x_2^2}$ at $x_2 = b$.

For example, $V_{1b} = -\frac{\partial \psi_b}{\partial x_2}$, $V_{1t} = -\frac{\partial \psi_t}{\partial x_2}$,

$V_{1b} = V_{1t} \rightarrow \frac{\partial \psi_b}{\partial x_2} = \frac{\partial \psi_t}{\partial x_2}$ at $x_2 = b$.

And $V_{2b} = \frac{\partial \psi_b}{\partial x_1}$ and $V_{2t} = \frac{\partial \psi_t}{\partial x_1}$, since x_1 dependence in ψ is $\sin kx_1$, therefore $V_{2b} = V_{2t}$ at $x_2 = b$ implies that $\psi_t = \psi_b$ at $x_2 = b$.

$\frac{\partial^2 \psi_t}{\partial x_2^2} = \frac{\partial^2 \psi_b}{\partial x_2^2}$ at $x_2 = b$ is derived from $\tau_{12}^t = \tau_{12}^b$.

Now let's look at $\int_{b^-}^{b^+} \nabla^4 \psi \, dx_2 = \int_{b^-}^{b^+} k \sin kx_1 \, \delta(x_2 - b) \, dx_2$

RHS: $= k \sin kx_1$

LHS: $= \int_{b^-}^{b^+} \left(\frac{\partial^4 \psi}{\partial x_1^4} + 2 \frac{\partial^4 \psi}{\partial x_1^2 \partial x_2^2} \right) dx_2 + \int_{b^-}^{b^+} \left(\frac{\partial^4 \psi}{\partial x_2^4} \right) dx_2$

$= 0 + \frac{\partial^3 \psi_b}{\partial x_2^3} - \frac{\partial^3 \psi_t}{\partial x_2^3}$ let $b^- \to b$ and $b^+ \to b$

due to $\psi_t = \psi_b$, $\frac{\partial \psi_t}{\partial x_2} = \frac{\partial \psi_b}{\partial x_2}$ at $x_2 = b$.

$$\left\{ \begin{array}{ll} \frac{\partial^3 \phi_b}{\partial x_2^3} - \frac{\partial^3 \psi_t}{\partial x_2^3} = k \sin kx_1 \quad \text{at } x_2 = b & (19) \\ \psi_t = \psi_b \quad \text{at } x_2 = b & (20) \\ \frac{\partial \psi_t}{\partial x_2} = \frac{\partial \psi_b}{\partial x_2} \quad \text{at } x_2 = b & (21) \\ \frac{\partial^2 \psi_t}{\partial x_2^2} = \frac{\partial^2 \psi_b}{\partial x_2^2} \quad \text{at } x_2 = b & (22) \end{array} \right.$$

Solving (19) — (22), for B_t, C_t, B_b and C_b.

$$B_t = \frac{1}{2k^3 \, sh^2 k}\left[kb \, shk \; ch\,k(b-1) - k \, sh\,kb - sh\,k \; sh\,k(b-1)\right]$$

$$C_t = \frac{sh\,k(b-1)}{2k^2 sh\,k}$$

$$B_b = \frac{1}{2k^3 sh^2 k}\left[k(b-1) sh\,k \; ch\,kb - k \, sh\,k(b-1) - sh\,k \; sh\,kb\right]$$

$$C_b = \frac{sh\,kb}{2k^2 \, sh\,k},$$

With B_t, C_t, B_b and C_b, we have ψ_b and ψ_t, i.e., the whole flow field generated by $\delta p = \cos kx_1 \, \delta(x_2 - b)$.

To determine dynamic topography, we need to compute σ_{22} at the surface, $\sigma_{22}^t(x_2=0)$

$$\sigma_{22}^t = -p^t + 2\eta \frac{\partial V_{2t}}{\partial x_2}. \quad (23)$$

where V_{2t} is easy to compute from ψ_t, and p^t can be obtained by integrating the x_1 component of the momentum equation.

$$-\frac{\partial p^t}{\partial x_1} + \eta\left(\frac{\partial^2 V_{1t}}{\partial x_1^2} + \frac{\partial^2 V_{1t}}{\partial x_2^2}\right) = 0 \quad (24)$$

The nondimensional form of (23) and (24):

$$\sigma_{22}^t = -p^t + 2\frac{\partial V_{2t}}{\partial x_2}, \quad (25)$$

$$-\frac{\partial p^t}{\partial x_1} + \frac{\partial^2 V_{1t}}{\partial x_1^2} + \frac{\partial^2 V_{1t}}{\partial x_2^2} = 0 \quad (26)$$

[v by $\frac{\rho_0 g d^2}{\eta}$
p & σ by $\rho_0 g d$]

From (26), $p^t = \cos kx_1 \left(-k^2 Y_t(x_2) + \frac{d^2 Y_t(x_2)}{dx_2^2}\right)$ (27)

where $Y_t(x_2) = C_t x_2 \,\mathrm{sh}\, kx_2 + (B_t + \frac{C_t}{k}) \,\mathrm{ch}\, kx_2$

[i.e., the x_2 part of $\psi_t(x_1, x_2)$]

Now for topography, the dimensional form is

$$H_s = \frac{\sigma_{22}^t(x_2=0)}{\Delta\rho_s g} \qquad (28)$$

where $\Delta\rho_s$ is the density contrast across the surface (e.g., mantle vs air, $\Delta\rho_s = \rho_m$).

Given that σ_{22} is scaled by $\rho_0 g d$, the topography is scaled by $\frac{\rho_0}{\Delta\rho_s} d$. And the nondimensional topography at the surface is

$$H_s = \sigma_{22}^t(x_2=0) = 2k^3 B_t \cos kx_1 \qquad (29)$$

Similarly, σ_{22} and hence H_{cmb} are expected at the bottom boundary or CMB. The dimensionless CMB topography (scaled by $\frac{\rho_0}{\Delta\rho_{CMB}} d$) is

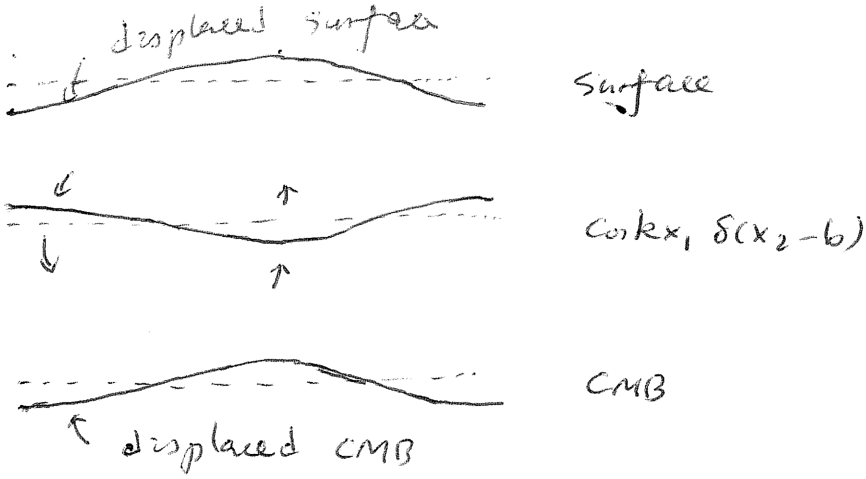
$$H_{cmb} = \sigma_{22}^b(x_2=1) = -2k^3 B_b \cos kx_1 \qquad (30)$$

In (29) and (30), if we drop out $\cos kx_1$, then the kernels are

$$H_s(k,b) = 2k^3 B_t \qquad (31)$$

$$H_{cmb}(k,b) = -2k^3 B_b \qquad (32)$$

Note: $B_t = 2k^3$ for $b=0$ (i.e., the source at the surface) $\rightarrow H_s(k,0)=1$
$B_t = 0$ for $b=1$ (i.e., the source at CMB), $\rightarrow H_s(k,1)=0$.
Likewise, $H_{CMB}(k,1) = 1$ and $H_{CMB}(k,0) = 0$.



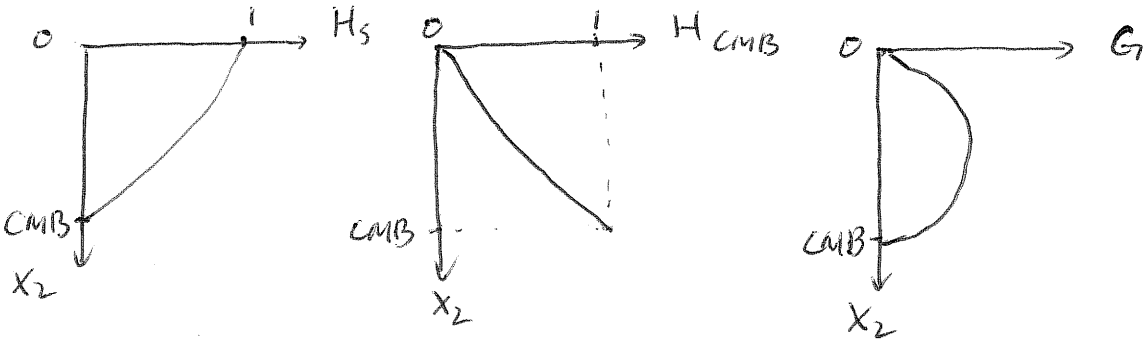
5. Geoid kernels (i.e., gravitational potential)

The geoid anomaly at the surface in response to interoir density anomaly $\delta\rho = \cos kx_1\, \delta(x_2-b)$ for a dynamic Earth must include contributions from the surface and CMB topography (i.e., H_s and H_{cmb}), in addition to $\delta\rho$. The dimensionless geoid kernel

$$G(k,b) = H_s(k,b) + e^{-k} H_{cmb}(k,b) - e^{-kb}, \qquad (33)$$

Clearly, $G(k,0) = G(k,1) = 0$,

i.e., geoid kernels are zero at the surface & CMB.



A) These curves vary for different wavenumbers k.

B) Near surface buoyancy force is perfectly compensated at the surface but produces zero ~~stuff~~ topography at the bottom boundary (i.e., CMB). The same can be said for the CMB. That is, the Airy/Pratt isostasy is predicted.

c). $G=0$ at the surface and CMB implies that the geoid is caused by density anomalies in the mantle interiors, and that near surface or CMB density anomalies do not produce the geoid. This explains why the geoid at long wavelength is uncorrelated with surface features such as ocean-continent distribution.