optical manipulation

University of St Andrews

Towards

Spin and orbital angular momentum

An optical eigenmode approach

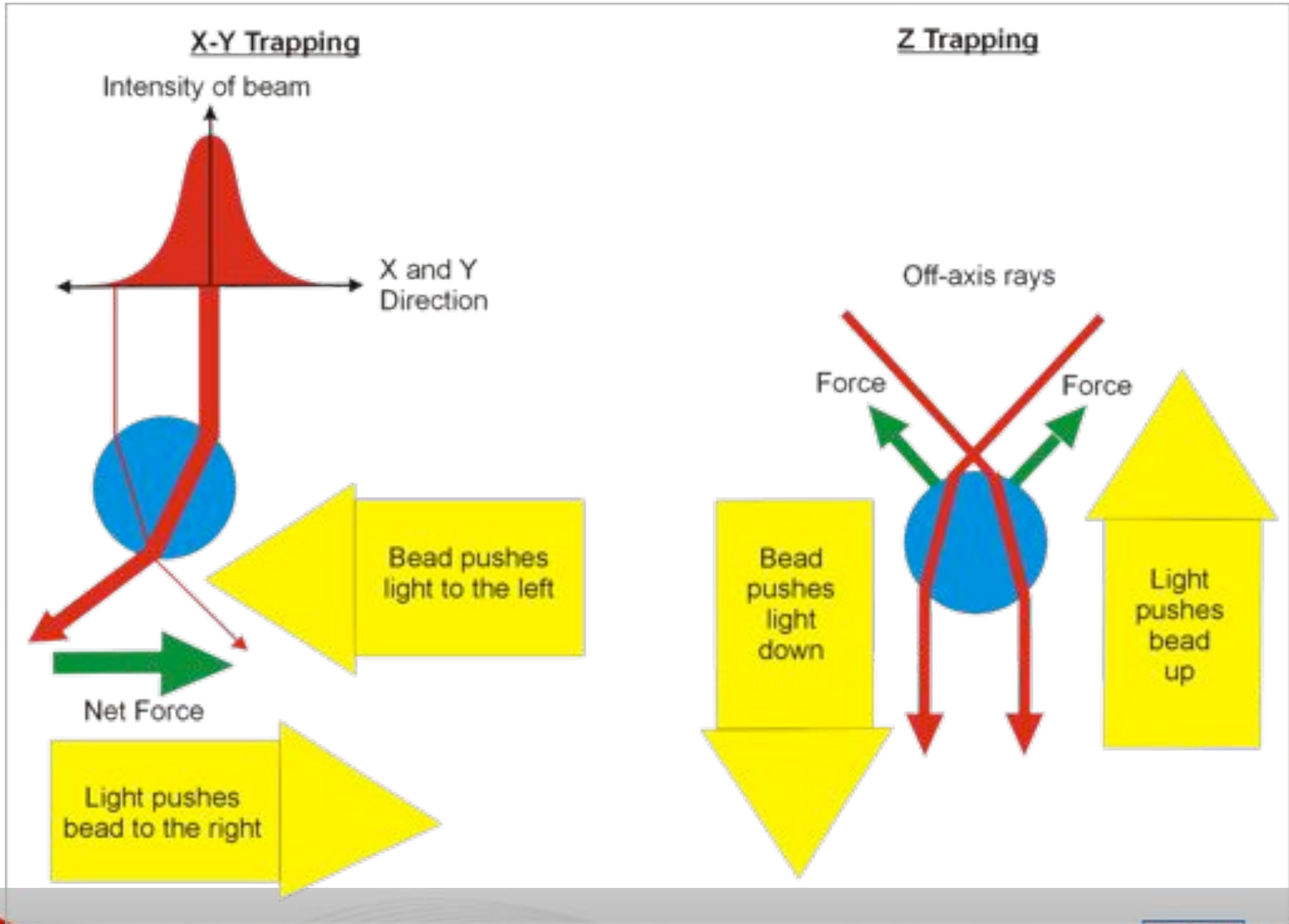
Michael Mazilu
SUPA, University of St. Andrews

optical manipulation

University of St Andrews

Optical micromanipulation

- Trapping
- Guiding
- Tweezing



Optical momentum flow

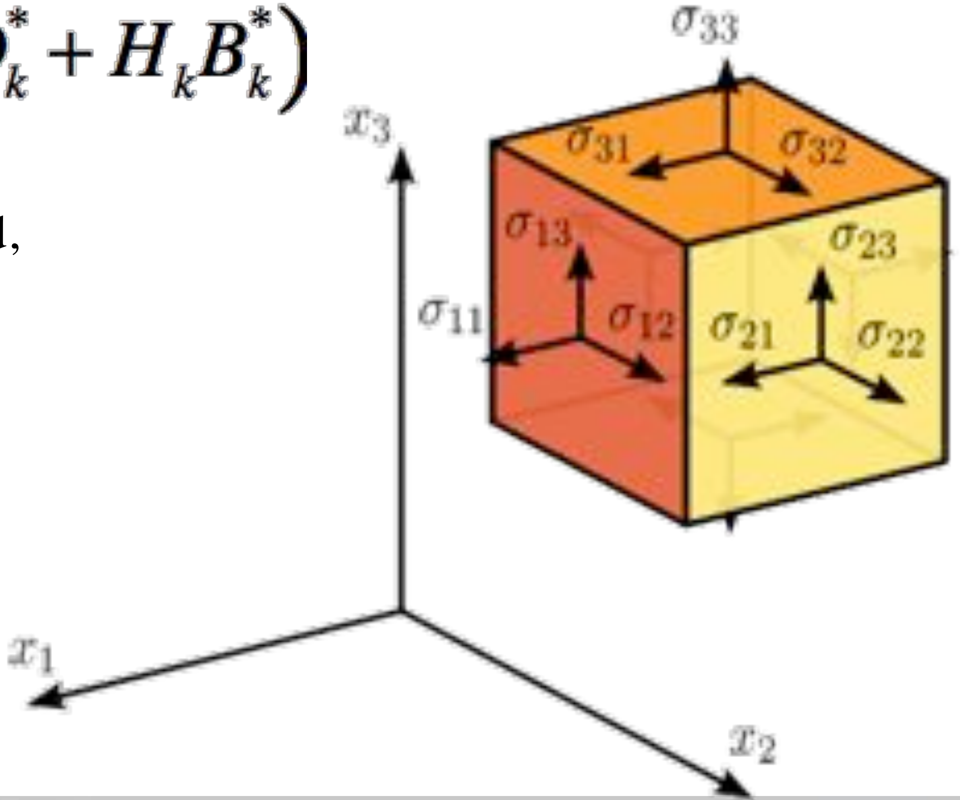
- Maxwell's stress tensor

$$T_{ij} = E_i D_j^* + H_i B_j^* - \frac{1}{2}\left(E_k D_k^* + H_k B_k^*\right)$$

E_i, D_i, H_i and B_i denote the electric field, the electric displacement, the magnetic field and magnetic flux respectively.

$$T_{ij} = \begin{bmatrix} \sigma_{11} & \sigma_{12} & \sigma_{13} \\ \sigma_{21} & \sigma_{22} & \sigma_{23} \\ \sigma_{31} & \sigma_{32} & \sigma_{33} \end{bmatrix}$$

σ_{11}, σ_{22}, σ_{33}: normal stresses (pressure like)
σ_{12}, σ_{13}, σ_{23}, σ_{21}, σ_{23}, σ_{32}: shear stresses



Picture from wikipedia

Optical forces

- Optical force density (momentum transfer)

$$f_i = -\partial_t g_i - \partial_j T_{ij}$$

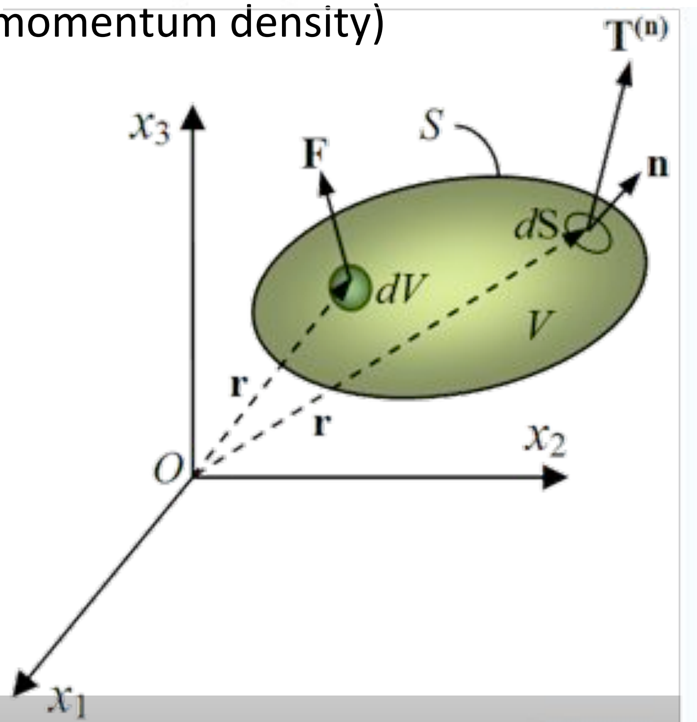
$$g_i = \varepsilon_{ijk} D_j B_k^*$$

(momentum density)

Total optical force, time averaged over the pulse repetition period Δt

$$< F_i > = \frac{1}{\Delta t} \iint f_i dV \, dt = \frac{1}{\Delta t} \iint T_{ij} ds_j dt$$

The momentum density, gi, cancels out in the averaging process.



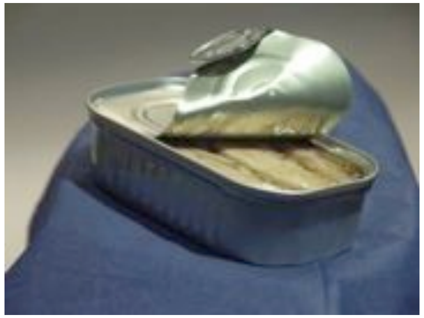
Picture from wikipedia

Outline

- Interference conservation
- Lorentz lemma ⇔ Noether's theorem
- Superposition principle & optical eigenmodes
- SAM and OAM for generalised Bessel vector beams
- Optical spin vector

Take away message

Conservation



con•ser•va•tion |ˌkänsərˈvā sн ən|
noun
the action of conserving something, in particular
- preservation, protection, or restoration of the natural environment, natural ecosystems, vegetation, and wildlife.
- preservation, repair, and prevention of deterioration of archaeological, historical, and cultural sites and artifacts.
- prevention of excessive or wasteful use of a resource.
- Physics the principle by which the total value of a physical quantity (such as energy, mass, or linear or angular momentum) remains constant in a system.

Simplest equation in the world

Convergence

EM

QM

optical manipulation

Picture from the web

University of St Andrews

Maxwell's equations

$$\nabla \cdot \epsilon_0 \mathbf{E} = 0,$$
$$\nabla \cdot \mu_0 \mathbf{H} = 0,$$
$$\nabla \times \mathbf{E} = -\mu_0 \partial_t \mathbf{H},$$
$$\nabla \times \mathbf{H} = \epsilon_0 \partial_t \mathbf{E},$$

E (energy density), **S** (Poynting vector) and $\tilde{\sigma}$ (Maxwell's stress tensor)

$$E(\mathcal{F}) = \frac{1}{2}\left(\epsilon_0 \mathbf{E}^* \cdot \mathbf{E} + \mu_0 \mathbf{H}^* \cdot \mathbf{H}\right),$$
$$\mathbf{S}(\mathcal{F}) = \frac{1}{2}\left(\mathbf{E}^* \times \mathbf{H} + \mathbf{E} \times \mathbf{H}^*\right),$$
$$\tilde{\sigma}(\mathcal{F}) = \frac{c^2}{2}\Big(\left(\epsilon_0 \mathbf{E}^* \cdot \mathbf{E} + \mu_0 \mathbf{H}^* \cdot \mathbf{H}\right)\tilde{I} - \epsilon_0 \mathbf{E}^* \otimes \mathbf{E} - \epsilon_0 \mathbf{E} \otimes \mathbf{E}^* - \mu_0 \mathbf{H}^* \otimes \mathbf{H} - \mu_0 \mathbf{H} \otimes \mathbf{H}^*\Big),$$

Conventions:
E and **H** are complex fields having only positive frequencies components

$\mathcal{F} = (\mathbf{E}, \mathbf{H})$.

Conservation:

$$\nabla \cdot \mathbf{S}(\mathcal{F}) + \partial_t E(\mathcal{F}) = 0,$$
$$\nabla \cdot \tilde{\sigma}(\mathcal{F}) + \partial_t \mathbf{S}(\mathcal{F}) = 0.$$

Interference conservation

$$\mathcal{F} = \mathcal{F}_1 + \mathcal{F}_2 \qquad E(\mathcal{F}) = E(\mathcal{F}_1) + E_{12}(\mathcal{F}_1, \mathcal{F}_2) + E_{21}(\mathcal{F}_1, \mathcal{F}_2) + E(\mathcal{F}_2),$$

$$\mathcal{F} = \mathcal{F}_1 + \mathrm{i}\mathcal{F}_2 \qquad E(\mathcal{F}) = E(\mathcal{F}_1) + iE_{12}(\mathcal{F}_1, \mathcal{F}_2) - i\,E_{21}(\mathcal{F}_1, \mathcal{F}_2) + E(\mathcal{F}_2),$$

E_{12} (interference energy)
$\mathbf{S}_{12}$ (Poynting vector)
$\tilde{\sigma}_{12}$ (Maxwell's stress tensor):

$$E_{12}(\mathcal{F}_1, \mathcal{F}_2) = \tfrac{1}{2}(\epsilon_0 \mathbf{E}_1^* \cdot \mathbf{E}_2 + \mu_0 \mathbf{H}_1^* \cdot \mathbf{H}_2),$$

$$\mathbf{S}_{12}(\mathcal{F}_1, \mathcal{F}_2) = \tfrac{1}{2}(\mathbf{E}_1^* \times \mathbf{H}_2 + \mathbf{E}_2 \times \mathbf{H}_1^*),$$

$$\tilde{\sigma}_{12}(\mathcal{F}_1, \mathcal{F}_2) = \frac{c^2}{2}((\epsilon_0 \mathbf{E}_1^* \cdot \mathbf{E}_2 + \mu_0 \mathbf{H}_1^* \cdot \mathbf{H}_2)\tilde{I} - \epsilon_0 \mathbf{E}_1^* \otimes \mathbf{E}_2 - \epsilon_0 \mathbf{E}_2 \otimes \mathbf{E}_1^* - \mu_0 \mathbf{H}_1^* \otimes \mathbf{H}_2 - \mu_0 \mathbf{H}_2 \otimes \mathbf{H}_1^*).$$

Lorentz lemma,
Feld-Tai reciprocity

$$\nabla \cdot \mathbf{S}_{12}(\mathcal{F}_1, \mathcal{F}_2) + \partial_t E_{12}(\mathcal{F}_1, \mathcal{F}_2) = 0,$$

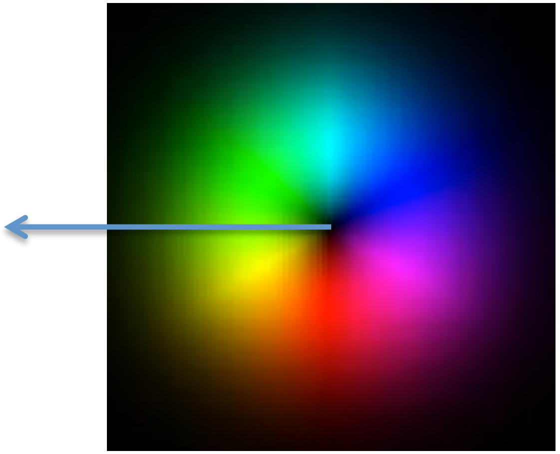
$$\nabla \cdot \tilde{\sigma}_{12}(\mathcal{F}_1, \mathcal{F}_2) + \partial_t \mathbf{S}_{12}(\mathcal{F}_1, \mathcal{F}_2) = 0.$$

Noether's theorem

The interference of any two fields is conserved: $\mathcal{F}_1$ and $\mathcal{F}_2$

In particular:

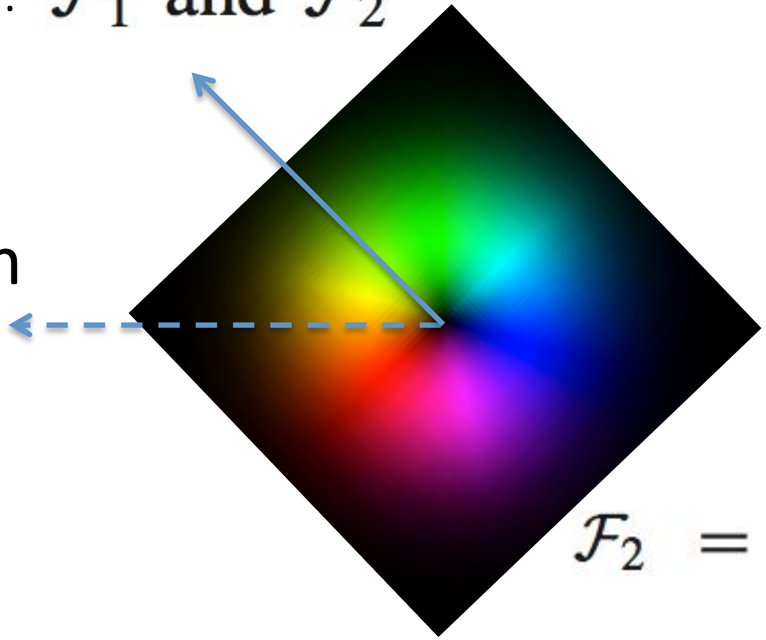
$$\mathcal{F}_1 = \mathcal{F}$$



$\mathbb{T}$

rotation

angle ϕ



$$\mathcal{F}_2 = \mathbb{T}\mathcal{F}$$

If $\mathcal{F}$ is a solution of Maxwell's equations, $\mathbb{T}$ is a transformation that leaves Maxwell's equations invariant and $\mathbb{T}\mathcal{F}$ is the transformed field, still a solution of Maxwell's equation, then $E_{12}(\mathcal{F}, \mathbb{T}\mathcal{F})$, $\mathbf{S}_{12}(\mathcal{F}, \mathbb{T}\mathcal{F})$ and $\tilde{\sigma}_{12}(\mathcal{F}, \mathbb{T}\mathcal{F})$ define the canonical energy density, current and stress tensor associated with this transformation or symmetry.

Conserving quantities: Identity

$$\mathbb{A} = \mathbb{I},$$

$$E_{12}(\mathcal{F}, \mathbb{A}\mathcal{F}) = \tfrac{1}{2}(\epsilon_0 \mathbf{E}^* \cdot \mathbf{E} + \mu_0 \mathbf{H}^* \cdot \mathbf{H}),$$

$$\mathbf{S}_{12}(\mathcal{F}, \mathbb{A}\mathcal{F}) = \tfrac{1}{2}(\mathbf{E}^* \times \mathbf{H} + \mathbf{E} \times \mathbf{H}^*),$$

$$\tilde{\boldsymbol{\sigma}}_{12}(\mathcal{F}, \mathbb{A}\mathcal{F}) = \frac{c^2}{2}((\epsilon_0 \mathbf{E}^* \cdot \mathbf{E} + \mu_0 \mathbf{H}^* \cdot \mathbf{H})\tilde{I} - \epsilon_0 \mathbf{E}^* \otimes \mathbf{E} - \epsilon_0 \mathbf{E} \otimes \mathbf{E}^* - \mu_0 \mathbf{H}^* \otimes \mathbf{H} - \mu_0 \mathbf{H} \otimes \mathbf{H}^*).$$

$\mathbb{A}$: Symmetry transformation

Conserving quantities: energy and momentum?

$$\mathbb{A}_t = \mathrm{i}\partial_t,$$

$$E_{12}(\mathcal{F}, \mathbb{A}_t\mathcal{F}) = \frac{\mathrm{i}}{2}(\epsilon_0 \mathbf{E}^* \cdot \partial_t \mathbf{E} + \mu_0 \mathbf{H}^* \cdot \partial_t \mathbf{H}),$$

$$\mathbf{S}_{12}(\mathcal{F}, \mathbb{A}_t\mathcal{F}) = \frac{\mathrm{i}}{2}(\mathbf{E}^* \times \partial_t \mathbf{H} + \partial_t \mathbf{E} \times \mathbf{H}^*),$$

$$\tilde{\boldsymbol{\sigma}}_{12}(\mathcal{F}, \mathbb{A}_t\mathcal{F}) = \frac{\mathrm{i}c^2}{2}((\epsilon_0 \mathbf{E}^* \cdot \partial_t \mathbf{E} + \mu_0 \mathbf{H}^* \cdot \partial_t \mathbf{H})\tilde{I} - \epsilon_0 \mathbf{E}^* \otimes \partial_t \mathbf{E} - \epsilon_0 \partial_t \mathbf{E} \otimes \mathbf{E}^* - \mu_0 \mathbf{H}^* \otimes \partial_t \mathbf{H} - \mu_0 \partial_t \mathbf{H} \otimes \mathbf{H}^*).$$

Conserving quantities: Orbital momentum?

$$\mathbb{A}_\phi = \mathrm{i}\partial_\phi,$$

$$E_{12}(\mathcal{F}, \mathbb{A}_\phi\mathcal{F}) = \frac{\mathrm{i}}{2}(\epsilon_0 \mathbf{E}^* \cdot \partial_\phi \mathbf{E} + \mu_0 \mathbf{H}^* \cdot \partial_\phi \mathbf{H}),$$

$$\mathbf{S}_{12}(\mathcal{F}, \mathbb{A}_\phi\mathcal{F}) = \frac{\mathrm{i}}{2}(\mathbf{E}^* \times \partial_\phi \mathbf{H} + \partial_\phi \mathbf{E} \times \mathbf{H}^*),$$

$$\tilde{\boldsymbol{\sigma}}_{12}(\mathcal{F}, \mathbb{A}_\phi\mathcal{F}) = \frac{\mathrm{i}c^2}{2}((\epsilon_0 \mathbf{E}^* \cdot \partial_\phi \mathbf{E} + \mu_0 \mathbf{H}^* \cdot \partial_\phi \mathbf{H})\tilde{I} - \epsilon_0 \mathbf{E}^* \otimes \partial_\phi \mathbf{E} - \epsilon_0 \partial_\phi \mathbf{E} \otimes \mathbf{E}^* - \mu_0 \mathbf{H}^* \otimes \partial_\phi \mathbf{H} - \mu_0 \partial_\phi \mathbf{H} \otimes \mathbf{H}^*).$$

Conserving quantities: Spin?

$$\mathbb{A}_s = \begin{pmatrix} 0 & iZ_0 \\ -i/Z_0 & 0 \end{pmatrix}$$

Duality transformation applied to (**E**,**H**)

(**E**,**H**) ⟷ (**H**,-**E**)

$$E_{12}(\mathcal{F}, \mathbb{A}_s\mathcal{F}) = \frac{i}{2c}(\mathbf{E}^* \cdot \mathbf{H} - \mathbf{H}^* \cdot \mathbf{E}),$$

$$\mathbf{S}_{12}(\mathcal{F}, \mathbb{A}_s\mathcal{F}) = \frac{ic}{2}(\epsilon_0\mathbf{E}^* \times \mathbf{E} + \mu_0\mathbf{H} \times \mathbf{H}^*),$$

$$\tilde{\boldsymbol{\sigma}}_{12}(\mathcal{F}, \mathbb{A}_s\mathcal{F}) = \frac{ic}{2}((\mathbf{E}^* \cdot \mathbf{H} - \mathbf{H}^* \cdot \mathbf{E})\tilde{I} - \mathbf{E}^* \otimes \mathbf{H} - \mathbf{H} \otimes \mathbf{E}^* + \mathbf{H}^* \otimes \mathbf{E} + \mathbf{E} \otimes \mathbf{H}^*).$$

Question time

Definition: *The total EM field is composed from a superposition of partial fields*.

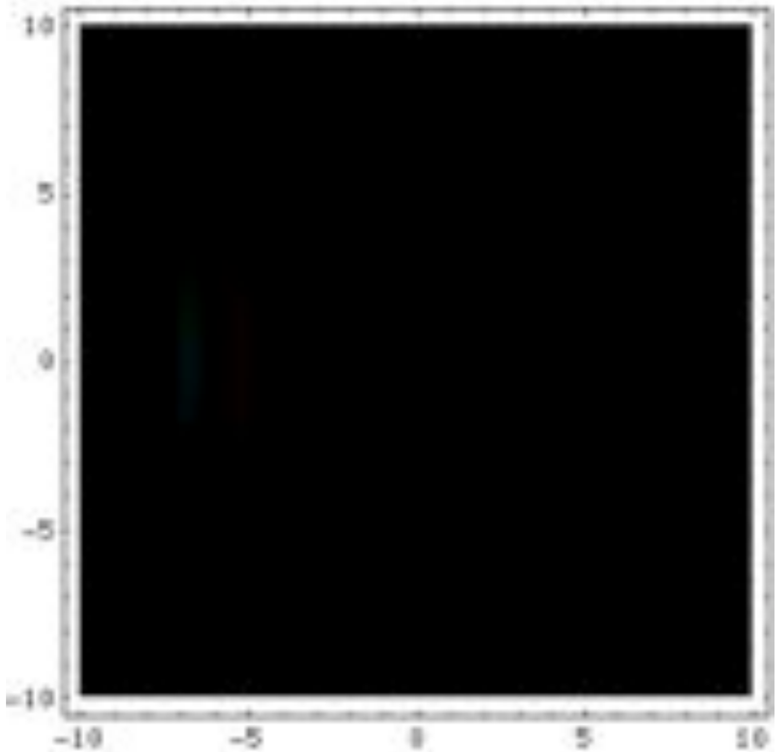
Can we associate a conserving quantity to a partial field?

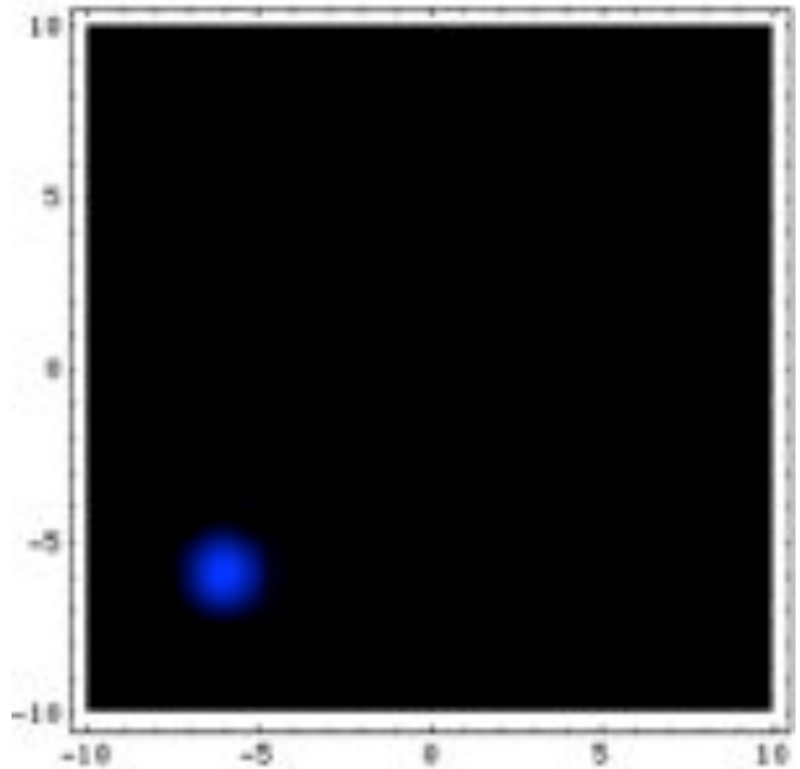
Why would we want this?

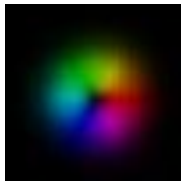
What is the challenge?

How many simultaneously conserving quantity can be associated with a partial field?

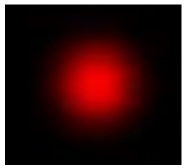
Decomposing an LG beam





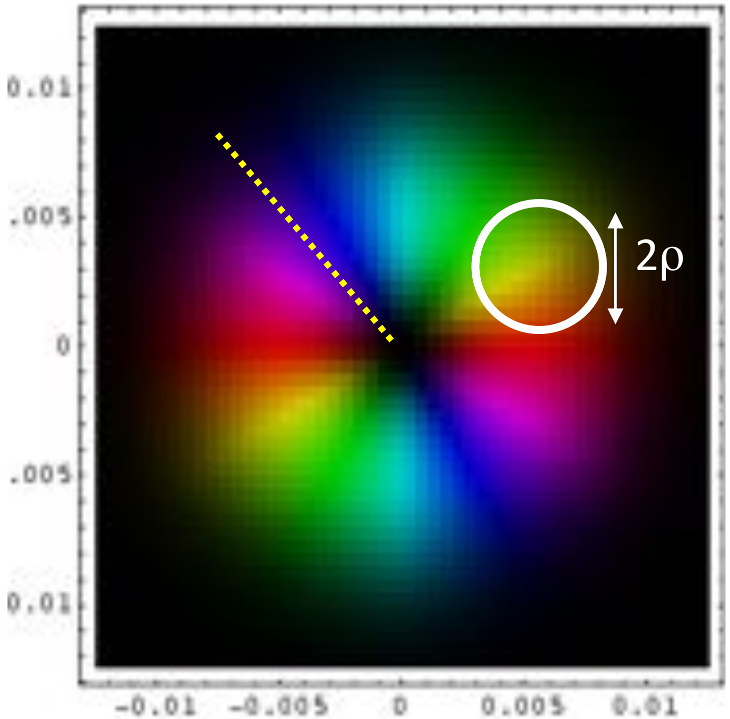


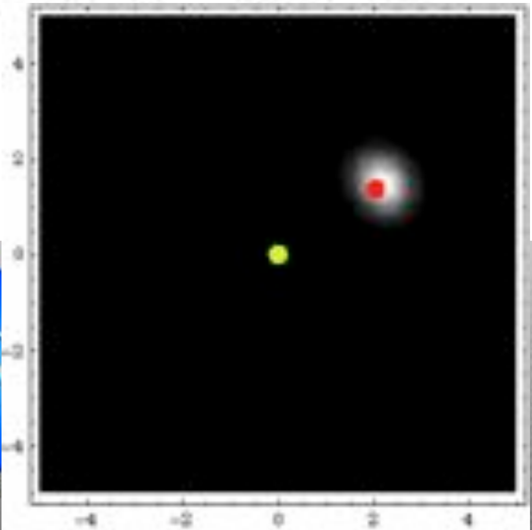
Vortlet decomposition



Gaussian beamlet decomposition

Pinhole & LG beam





- Energy conservation

$$\mathcal{E} = \text{const.} = \iint uu^* dxdy$$

- Momentum conservation

$$\mathbf{p} = \text{const.} = \iint \Im(u\nabla_2 u^*)dxdy$$

- Linear motion of the center of gravity

$$\begin{aligned} <\mathbf{r}> &= \frac{1}{\mathcal{E}}\iint \mathbf{r}uu^* dxdy \\ &= \mathbf{r} + \mathbf{v}(z - z_1) \end{aligned}$$

- Angular momentum

$$\bar{M}_z = <x> v_y - <y> v_x$$

$$\begin{aligned} M_z &= \frac{1}{\mathcal{E}}\iint \Im(xu\partial_y u^* - yu\partial_x u^*)dxdy \\ &= l \end{aligned}$$

Eigenmode superposition

Definition:

$$\langle E_{12}(\mathcal{F})\rangle_{\mathbb{A}} = \int E_{12}(\mathcal{F}, \mathbb{A}\,\mathcal{F})\,\mathrm{d}^3 r,$$

Decomposition in partial fields

$$\mathcal{F} = \sum_i \mathcal{F}_i$$

$$\langle E_{12}(\mathcal{F})\rangle_{\mathbb{A}} = \left\langle E_{12}\left(\sum_i \mathcal{F}_i\right)\right\rangle_{\mathbb{A}},$$

Orthogonally

$$\langle E_{12}(\mathcal{F})\rangle_{\mathbb{A}} = \sum_i \lambda_i \langle E_{12}(\mathcal{F}_i)\rangle_{\mathbb{A}}$$

IF: $\lambda_i \mathcal{F}_i = \mathbb{A}\mathcal{F}_i$

Simultaneous conservation

For a partial field to simultaneously conserve multiple quantities the eigenfunctions of the associated operators need to be shared.

This is only possible for an ensemble of Hermitian operators that commute pairwise.

Compatible operators

$$\mathbb{A}_t = i\partial_t$$

$$\mathbb{A}_x = i\partial_x$$

$$\mathbb{A}_y = i\partial_y$$

$$\mathbb{A}_z = i\partial_z$$

($\mathbb{A}_x$, $\mathbb{A}_y$, $\mathbb{A}_z$) → Plane waves

Incompatible operators

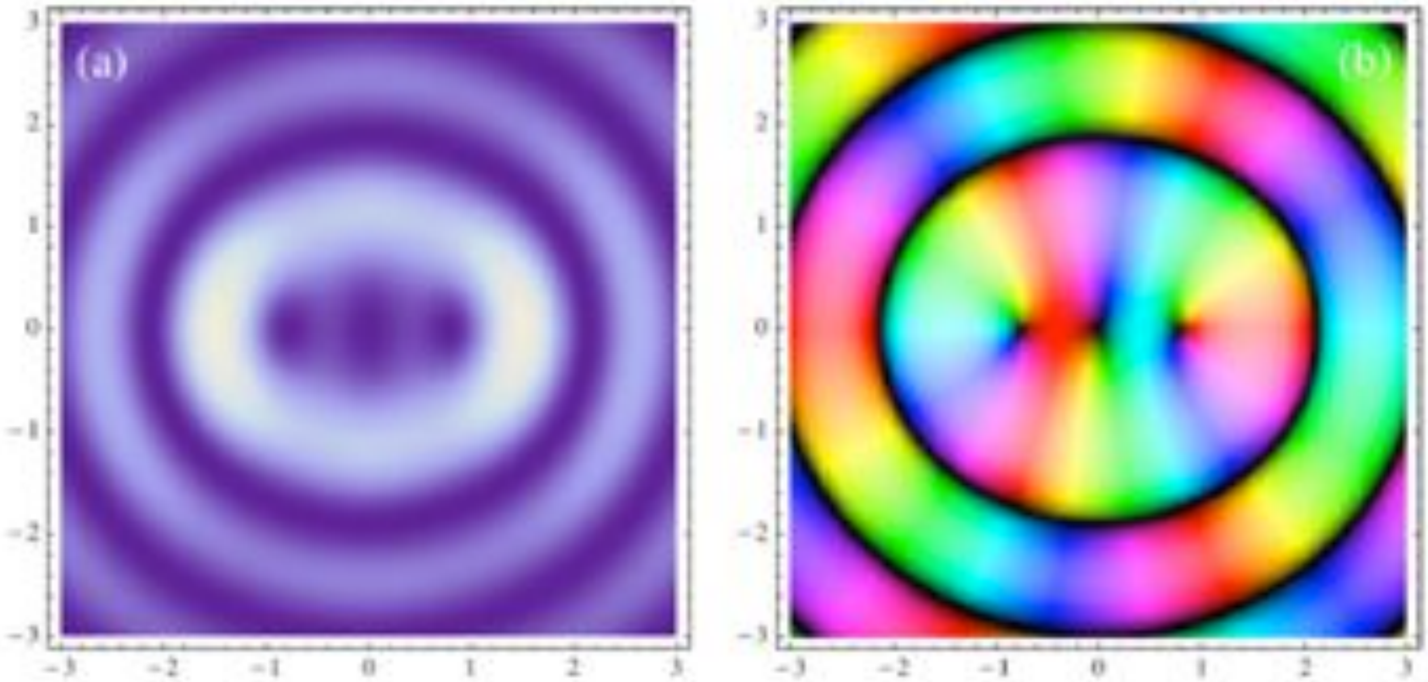
$$\mathbb{A}_\phi = i\partial_\phi$$

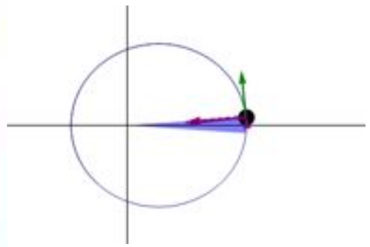
$$\mathbb{A}_x = i\partial_x$$

$$\mathbb{A}_y = i\partial_y$$

Eigenmodes: Example

Mathieu beam; L=3





Picture from wikipedia

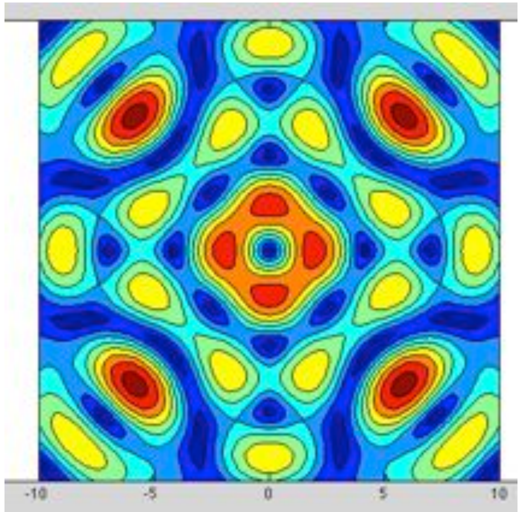
$$\mathbb{A} = \mathbb{A}_{\phi 1}\mathbb{A}_{\phi 2} + \mathbb{A}_{\phi 2}\mathbb{A}_{\phi 1}$$

Eigenmodes: multi-point OAM

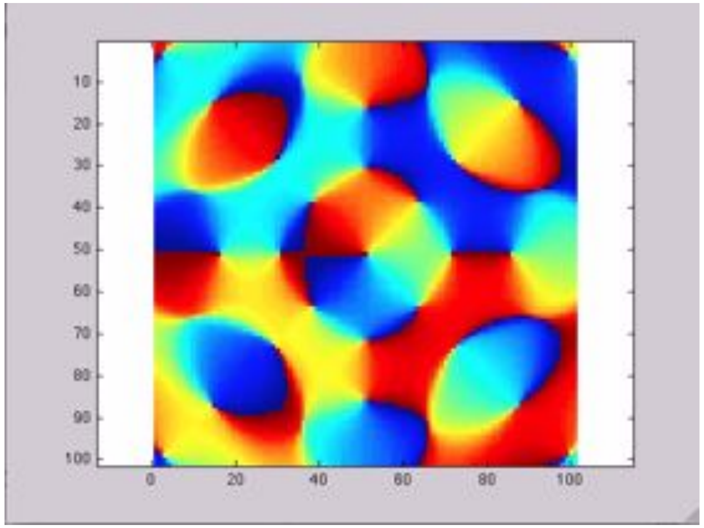
4 point ; L=3

Intensity four fold symmetry; phase three fold symmetry

Intensity



Phase



$$\mathbb{A} = \sum \mathbb{A}_{\phi 1}\mathbb{A}_{\phi 2}\mathbb{A}_{\phi 3}\mathbb{A}_{\phi 4}$$

Sum over all permutations

Summary eigenmodes

Optical eigenmodes are solutions of Maxwell's equations that are additive with respect to a conserving optical quantity.

Properties:

associated with a symmetry / operator

orthogonal i.e. not globally interfering

associated with a density, current and current flux

commuting operators lead to simultaneously additive properties

Next:

What is more important in the EM & QM convergence?

Operator or Symmetry

Bessel beams

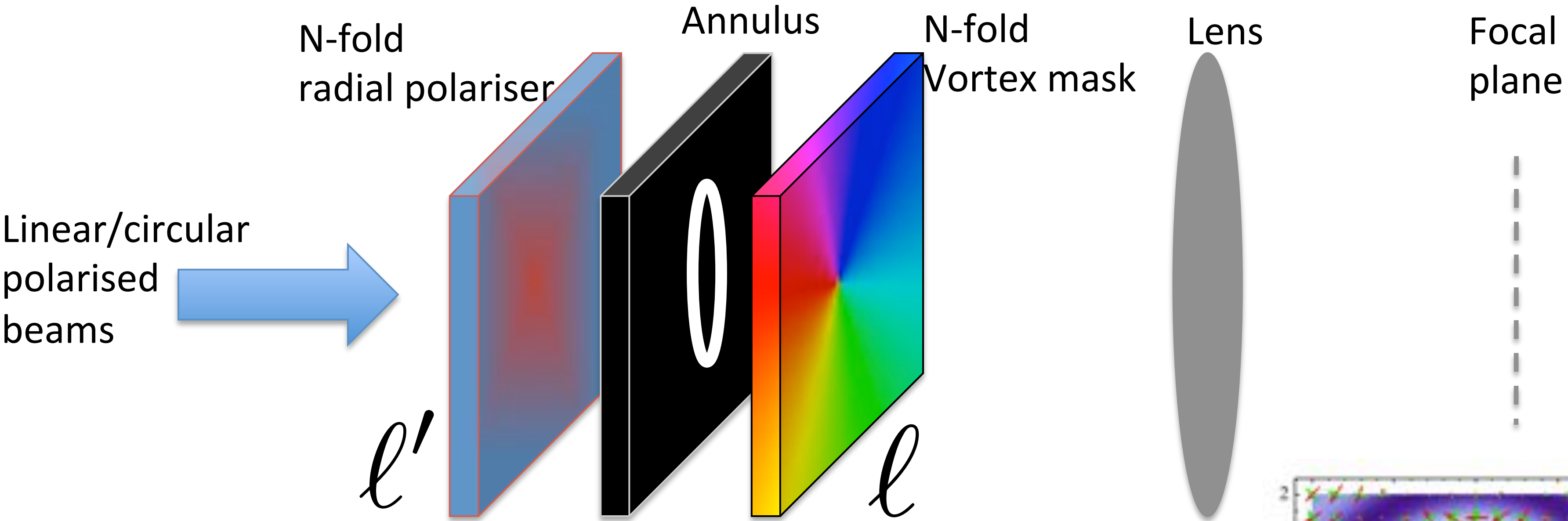
$$\begin{aligned} e_x &= a_x J_\ell(k_t\rho) \\ e_y &= a_y J_\ell(k_t\rho) \\ e_z &= \frac{1}{2}(-ia_x + a_y)J_{\ell-1}(k_t\rho)\tan(\gamma)e^{-i\phi} + \frac{1}{2}(ia_x + a_y)J_{\ell+1}(k_t\rho)\tan(\gamma)e^{i\phi} \end{aligned}$$

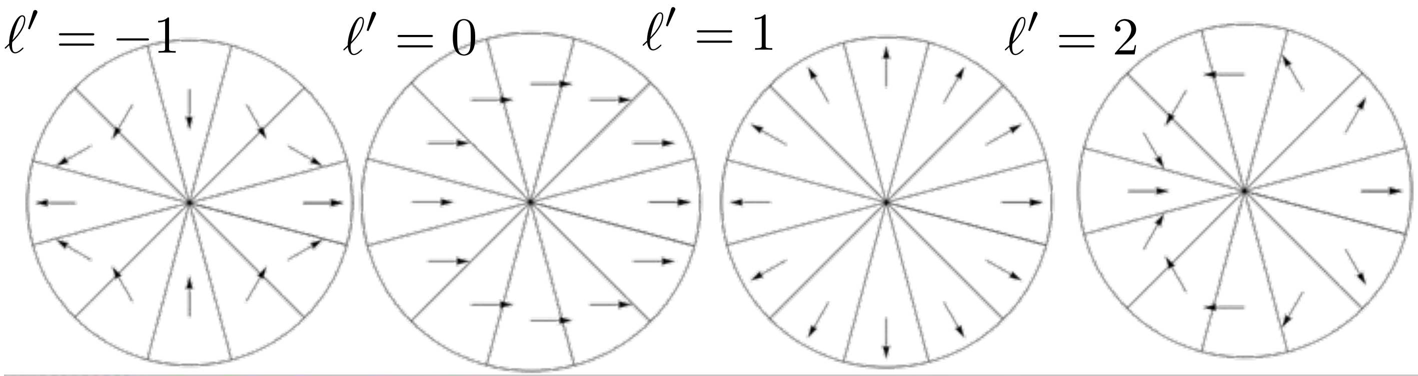
$$\mathbf{E} = \exp(i\omega t - ik_z z + \ell\phi)\begin{pmatrix} e_x \\ e_y \\ e_z \end{pmatrix}$$

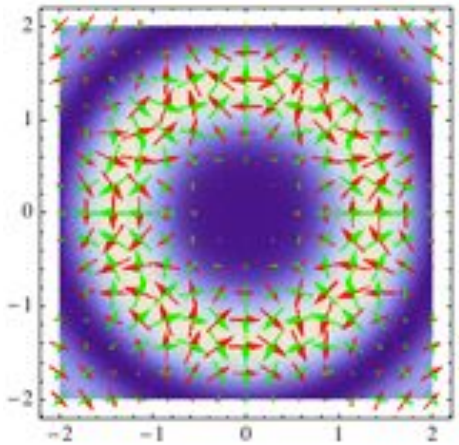
Invariant operators
are the same as in QM

$$\mathbb{A}_\phi, \mathbb{A}_z, \mathbb{A}_t$$

Locally polarised VBB







AOM and SAM split for VBB

$$
\begin{aligned}
e_x(\ell,\ell') &= ((a_x - ia_y)B_{\ell-\ell'} + (a_x + ia_y)B_{\ell+\ell'})\cos^2(\gamma/2) - ((a_x + ia_y)B_{\ell+\ell'-2} + (a_x - ia_y)B_{\ell-\ell} \\
e_y(\ell,\ell') &= ((a_y + ia_x)B_{\ell-\ell'} + (a_y - ia_x)B_{\ell+\ell'})\cos^2(\gamma/2) + ((a_y - ia_x)B_{\ell+\ell'-2} + (a_y + ia_x)B_{\ell-\ell} \\
e_z(\ell,\ell') &= (-a_x - ia_y)B_{\ell+\ell'-1}\sin(\gamma) + (-a_x + ia_y)B_{\ell-\ell'+1}\sin(\gamma)
\end{aligned}
$$

with: $B_\ell = (-i)^\ell J_\ell(k_t\rho)e^{i\ell\phi}$

Polarisation & phase equivalence

$$
\begin{aligned}
\mathbf{E}_{(\ell,\ell')}(1,i) &= \mathbf{E}_{(\ell+\Delta\ell,\ell'+\Delta\ell)}(1,i) \\
\mathbf{E}_{(\ell,\ell')}(1,-i) &= \mathbf{E}_{(\ell-\Delta\ell,\ell'+\Delta\ell)}(1,-i)
\end{aligned}
$$

variable: (a_x, a_y)

Definition:

$$
\begin{aligned}
\mathbf{E}_\ell^+ &= \mathbf{E}_{(\ell,0)}(1,i) = \mathbf{E}_{(\ell+\Delta\ell,\Delta\ell)}(1,i) \\
\mathbf{E}_\ell^- &= \mathbf{E}_{(\ell,0)}(1,-i) = \mathbf{E}_{(\ell-\Delta\ell,\Delta\ell)}(1,-i)
\end{aligned}
$$

Eigenmode

$$
\begin{aligned}
\mathbf{n}\times\mathbf{E}_\ell^+ + (\mathbf{r}\times\nabla)_\mathbf{n}\,\mathbf{E}_\ell^+ &= (\ell+1)\mathbf{E}_\ell^+ \\
\mathbf{n}\times\mathbf{E}_\ell^- + (\mathbf{r}\times\nabla)_\mathbf{n}\,\mathbf{E}_\ell^- &= (\ell-1)\mathbf{E}_\ell^- \\
\mathbf{n}\times\mathbf{H}_\ell^+ + (\mathbf{r}\times\nabla)_\mathbf{n}\,\mathbf{H}_\ell^+ &= (\ell+1)\mathbf{H}_\ell^+ \\
\mathbf{n}\times\mathbf{H}_\ell^- + (\mathbf{r}\times\nabla)_\mathbf{n}\,\mathbf{H}_\ell^- &= (\ell-1)\mathbf{H}_\ell^-
\end{aligned}
$$

SAM + OAM = AM

Polarisation spin

QM: Spin operator

Schrödinger's equations including spin:

$$i\hbar\frac{\partial}{\partial t}\Psi_+(\mathbf{r},\,t) \;=\; -\frac{\hbar^2}{2m}\nabla^2\Psi_+(\mathbf{r},\,t) + V(\mathbf{r})\Psi_+(\mathbf{r},\,t)$$

$$i\hbar\frac{\partial}{\partial t}\Psi_-(\mathbf{r},\,t) \;=\; -\frac{\hbar^2}{2m}\nabla^2\Psi_-(\mathbf{r},\,t) + V(\mathbf{r})\Psi_-(\mathbf{r},\,t)$$

Using the spinor notation $\chi = (\Psi_+, \Psi_-)$ we have

$$i\hbar\frac{\partial}{\partial t}\chi \;=\; -\frac{\hbar^2}{2m}\nabla^2\chi + V\chi$$

Conservation relations:

$$\begin{aligned} 0 &= \nabla\cdot\mathbf{j} + \partial_t\rho \\ \mathbf{j} &= i\frac{\hbar}{2m}(\chi\nabla\chi^* - \chi^*\nabla\chi) \\ \rho &= \chi^*\chi \end{aligned}$$

Interference conservation

$$\begin{aligned} 0 &= \nabla\cdot\mathbf{j}_{12} + \partial_t\rho_{12} \\ \mathbf{j}_{12} &= i\frac{\hbar}{2m}(\chi_1\nabla\chi_2^* - \chi_2^*\nabla\chi_1) \\ \rho_{12} &= \chi_2^*\chi_1 \end{aligned}$$

Invariant symmetry operators
(Pauli matrices)

$$\sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \qquad \sigma_y = \begin{pmatrix} 0 & i \\ -i & 0 \end{pmatrix} \qquad \sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

EM in the Riemann-Silberstein form

Maxwell's equations in vacuum (complex with ω>0):

$$\nabla \cdot \mathbf{E} = 0 \qquad \nabla \times \mathbf{E} = -\frac{1}{c}\partial_t \mathbf{B}$$

$$\nabla \cdot \mathbf{B} = 0 \qquad \nabla \times \mathbf{B} = \frac{1}{c}\partial_t \mathbf{E}$$

Normalisation such that the energy density is:

$$\hbar\omega(\mathbf{E}^* \cdot \mathbf{E} + \mathbf{B}^* \cdot \mathbf{B})$$

$$\mathbf{E} = \mathbf{E}_r + i\mathbf{E}_i \text{ and } \mathbf{B} = \mathbf{B}_r + i\mathbf{B}_i$$

$$\mathbf{F}_r = \mathbf{E}_r + i\mathbf{B}_r \text{ and } \mathbf{F}_i = \mathbf{E}_i + i\mathbf{B}_i$$

The ancillary field $\mathbf{F}_i$ facilitates the representation of the polarisation of the light field.

$$\nabla \cdot \mathbf{F}_r = 0 \qquad \nabla \times \mathbf{F}_r = \frac{i}{c}\partial_t \mathbf{F}_r$$

$$\nabla \cdot \mathbf{F}_i = 0 \qquad \nabla \times \mathbf{F}_i = \frac{i}{c}\partial_t \mathbf{F}_i$$

RS spinor equation

$$\boldsymbol{F} = (\mathbf{F}_r, \mathbf{F}_i) \qquad \nabla \cdot \boldsymbol{F} = 0 \qquad \nabla \times \boldsymbol{F} = \frac{i}{c}\partial_t \boldsymbol{F}$$

Interference conservation:

$$0 = \nabla \cdot \mathbf{j}_{12} + \partial_t \rho_{12}$$

$$\mathbf{j}_{12} = -ic\boldsymbol{F}_2^* \times \boldsymbol{F}_1 = -ic\mathbf{F}_{2r}^* \times \mathbf{F}_{1r} - ic\mathbf{F}_{2i}^* \times \mathbf{F}_{1i}$$

$$\rho_{12} = \boldsymbol{F}_2^* \cdot \boldsymbol{F}_1 = \mathbf{F}_{2r}^* \cdot \mathbf{F}_{1r} + \mathbf{F}_{2i}^* \cdot \mathbf{F}_{1i}$$

Some conserving densities

Operator	$\mathcal{T}$	$\rho_{12}(\mathcal{T}) = \boldsymbol{F}^* \cdot \mathcal{T}\boldsymbol{F}$
ρ	$\boldsymbol{\sigma}_0$	$\mathbf{E}^* \cdot \mathbf{E} + \mathbf{B}^* \cdot \mathbf{B}$
$\mathcal{E}$	$i\hbar\partial_t\boldsymbol{\sigma}_0$	$\mathbf{E}^* \cdot (i\hbar\partial_t)\mathbf{E} + \mathbf{B}^* \cdot (i\hbar\partial_t)\mathbf{B}$
P_x	$-i\hbar\partial_x\boldsymbol{\sigma}_0$	$-\mathbf{E}^* \cdot (i\hbar\partial_x)\mathbf{E} - \mathbf{B}^* \cdot (i\hbar\partial_x)\mathbf{B}$
P_y	$-i\hbar\partial_y\boldsymbol{\sigma}_0$	$-\mathbf{E}^* \cdot (i\hbar\partial_y)\mathbf{E} - \mathbf{B}^* \cdot (i\hbar\partial_y)\mathbf{B}$
P_z	$-i\hbar\partial_z\boldsymbol{\sigma}_0$	$-\mathbf{E}^* \cdot (i\hbar\partial_z)\mathbf{E} - \mathbf{B}^* \cdot (i\hbar\partial_z)\mathbf{B}$
L_x	$-(i\hbar\mathbf{r} \times \nabla)_x\boldsymbol{\sigma}_0$	$-\mathbf{E}^* \cdot (i\hbar\mathbf{r} \times \nabla)_x\mathbf{E} - \mathbf{B}^* \cdot (i\hbar\mathbf{r} \times \nabla)_x\mathbf{B}$
L_y	$-(i\hbar\mathbf{r} \times \nabla)_y\boldsymbol{\sigma}_0$	$-\mathbf{E}^* \cdot (i\hbar\mathbf{r} \times \nabla)_y\mathbf{E} - \mathbf{B}^* \cdot (i\hbar\mathbf{r} \times \nabla)_y\mathbf{B}$
L_z	$-(i\hbar\mathbf{r} \times \nabla)_z\boldsymbol{\sigma}_0$	$-\mathbf{E}^* \cdot (i\hbar\mathbf{r} \times \nabla)_z\mathbf{E} - \mathbf{B}^* \cdot (i\hbar\mathbf{r} \times \nabla)_z\mathbf{B}$

Number of photons, energy, linear momentum and orbital angular momentum

$$\boldsymbol{\sigma}_0 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

Spin Vector

Additional invariant operators.

$$\boldsymbol{\sigma}_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad \boldsymbol{\sigma}_2 = \begin{pmatrix} 0 & i \\ -i & 0 \end{pmatrix} \quad \boldsymbol{\sigma}_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

Conserving density and currents

Operator	$\mathcal{T}$	$\rho_{12}(\mathcal{T}) = \boldsymbol{F}^* \cdot \mathcal{T}\boldsymbol{F}$	$\mathbf{j}_{12}(\mathcal{T}) = -ic\boldsymbol{F}^* \times \mathcal{T}\boldsymbol{F}$
S_0	$\hbar\boldsymbol{\sigma}_0$	$\hbar(\mathbf{E}^* \cdot \mathbf{E} + \mathbf{B}^* \cdot \mathbf{B})$	$\hbar c(\mathbf{E}^* \times \mathbf{B} - \mathbf{B}^* \times \mathbf{E})$
S_1	$\hbar\boldsymbol{\sigma}_1$	$2\hbar(\mathbf{E}_i \cdot \mathbf{E}_r + \mathbf{B}_i \cdot \mathbf{B}_r)$	$2\hbar c(\mathbf{E}_r \times \mathbf{B}_i + \mathbf{E}_i \times \mathbf{B}_r)$
S_2	$\hbar\boldsymbol{\sigma}_2$	$2\hbar(\mathbf{E}_i \cdot \mathbf{B}_r - \mathbf{E}_r \cdot \mathbf{B}_i)$	$2\hbar c(\mathbf{E}_r \times \mathbf{E}_i + \mathbf{B}_r \times \mathbf{B}_i)$
S_3	$\hbar\boldsymbol{\sigma}_3$	$\hbar(\mathbf{E}_r \cdot \mathbf{E}_r - \mathbf{E}_i \cdot \mathbf{E}_i - \mathbf{B}_i \cdot \mathbf{B}_i + \mathbf{B}_r \cdot \mathbf{B}_r)$	$2\hbar c(\mathbf{E}_r \times \mathbf{B}_r + \mathbf{B}_i \times \mathbf{E}_i)$

$$0 = \nabla \cdot \mathbf{j}_{12} + \partial_t \rho_{12}$$

optical manipulation

M. Mazilu, Journal of Optics, **13,** 064009 (2011)

University of St Andrews

Spin eigenmodes

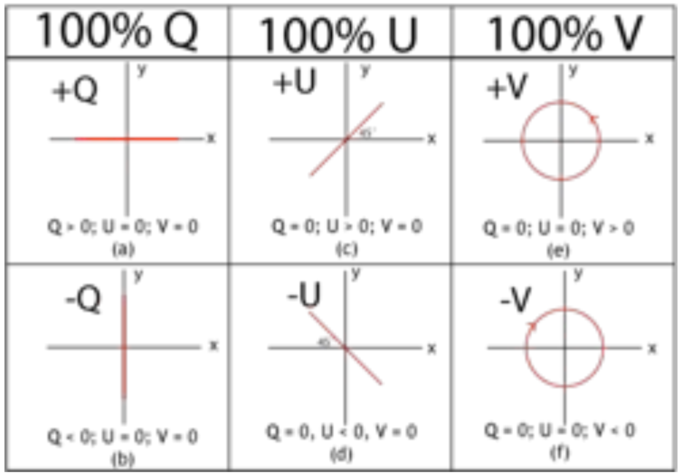
$$\boldsymbol{\sigma}_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \qquad \boldsymbol{\sigma}_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \qquad \boldsymbol{\sigma}_2 = \begin{pmatrix} 0 & i \\ -i & 0 \end{pmatrix}$$

$$\mathbf{E}_1 = \mathbf{u}_x \qquad \mathbf{E}_1 = -\mathbf{u}_x + \mathbf{u}_y \qquad \mathbf{E}_1 = -i\mathbf{u}_x + \mathbf{u}_y$$

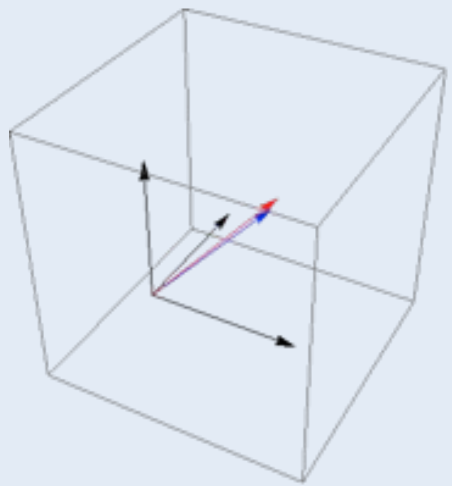
$$\mathbf{E}_2 = \mathbf{u}_y \qquad \mathbf{E}_2 = \mathbf{u}_x + \mathbf{u}_y \qquad \mathbf{E}_2 = i\mathbf{u}_x + \mathbf{u}_y$$

⇔ Stokes parameters

$$\begin{aligned} I &= |E_x|^2 + |E_y|^2, \\ Q &= |E_x|^2 - |E_y|^2, \\ U &= 2\mathrm{Re}(E_x E_y^*), \\ V &= 2\mathrm{Im}(E_x E_y^*), \end{aligned}$$



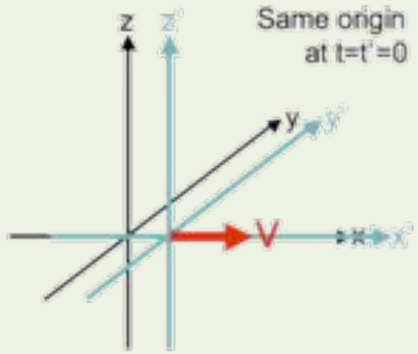
Rotation properties and Lorentz invariance



(u_1,u_2,u_3): Rotation axis
α: Rotation angle

$$\mathbf{R} = \exp(i(u_1\boldsymbol{\sigma}_1 + u_2\boldsymbol{\sigma}_2 + u_3\boldsymbol{\sigma}_3)\alpha/2).$$

A rotation of the coordinates implies the application of **R** to the spinor **F**.
=> Rotation of **S** around **u**.



In a Lorentz transformation **F** is transformed using:

$$\mathbf{R}_L = \exp(\boldsymbol{\sigma}_x\theta/2).$$

$$\tanh(\theta) = v_x/c$$

$$(S_0, S_1, S_2, S_3)$$

behaves like a 4-vector

For coherent monochromatic plane wave we have: $S_1^2 + S_2^2 + S_3^2 = S_0^2$

Summary

- Introduced optical eigenmodes mimicking QM
- Discussed their properties:
  - conservation
  - additivity of partial field superposition
  - symmetry ⇔ operator
- Possible link between SAM and OAM for vector beams
- Introduced a vector version of the optical spin

Next:

- Finite size optical eigenmodes
- Optical eigenmode imaging (indirect)
- Finite vortex eigenmodes
- Interactions with hard singularities