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fundamental quantum tests

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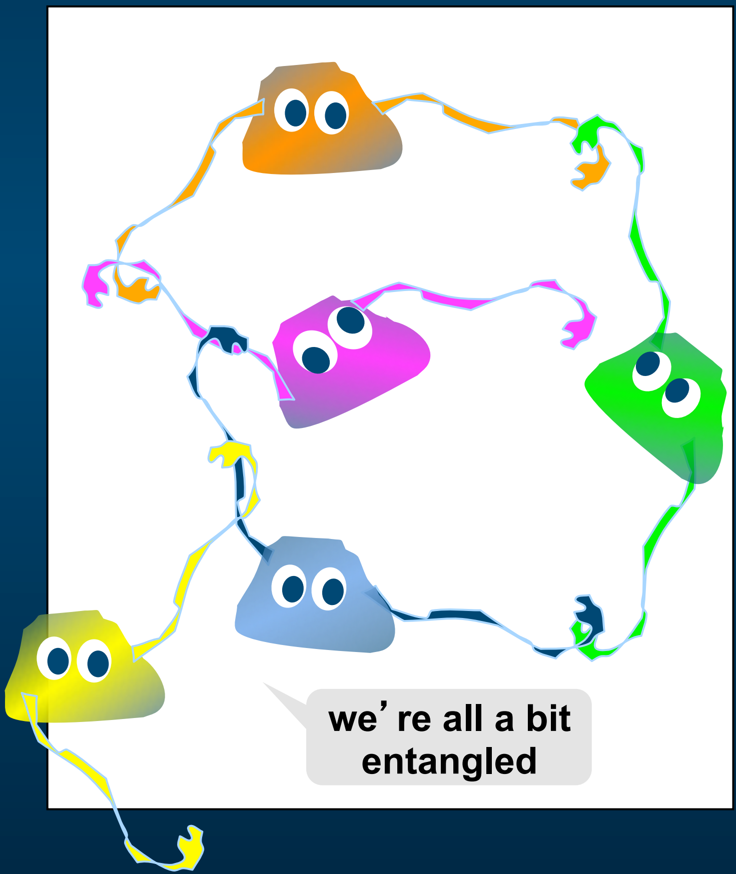
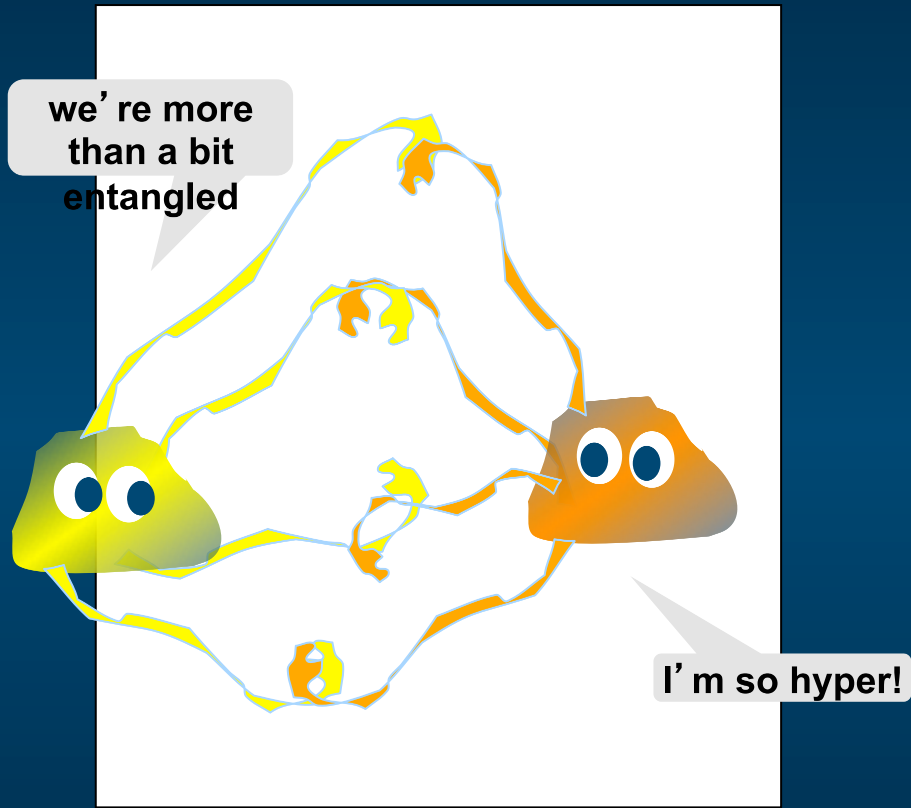
intro

1. high dimensional entanglement
   - entanglement
   - OAM state space – to infinity and beyond?
   - when OAM entered the quantum world
2. describing quantum states
   - Poincaré sphere
   - density matrices
3. quantum tests in a 2 dimensional OAM subspace
   - nonlocal hidden variable theories and Bell's inequality
   - local hidden variable theories and Leggett's inequality
4. more than 2 dimensions
   - Bell in 3D
   - EPR and entropic uncertainty relations

High dimensional entanglement

multipartite entanglement





many dimensional entanglement

1. High dimensional entanglement

- Photons, or more generally electromagnetic fields, play an important role in communication and computation.
  - In classical communication/computation the presence (1) or absence (0) of a photon may represent a bit.

- Can we do better? Or at least differently?

  - Go quantum → qubits
  - Add dimensions → qudits

Qubits

- Quantum theory allows superposition states, e.g.

  - Nb: This is different from a classical dim light bulb. For the quantum light bulb, the brightness cannot be determined without collapsing the state of the light bulb to either “bright” or “dark.”

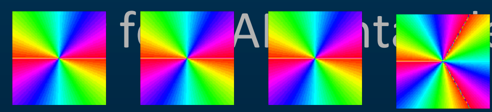
- Quantum theory also allows entanglement, e.g.

  - A quantum bit (qubit) could be inscribed in the polarisation states of a photon, or in path information in an interferometer.

High dimensional entanglement

- The information that can be carried by a qubit is still only one bit (only the processing differs).
- A picture says more than 1000 ~~words~~. bits (usually)
  - Can we use "quantum images" as a generalisation of "quantum bits"?
- We could encode any picture in the OAM state basis.
- OAM offers an infinite dimensio…

  - Even just using a very moderate 7 of these means that less photons are needed to encode the same amount of information.
  - Feel free to decode this important f… A… ta…ement:



Laguerre Gauss modes

- any light field can be written as a superposition of e.g. Laguerre Gauss modes

$$\vec{E}(r,\varphi,z) = \sum_{p,\ell} u_p^{\ell}(r,\varphi,z)\,\hat{e}$$

intensity profile of LG modes

propagation

wavefront curvature and Gouy phase

azimuthal phase

$$u_p^l(r,\varphi,z) = \sqrt{\frac{p!}{(p+\ell)!}}\,\frac{w_0}{w}\, e^{-r^2/w^2}\, L_p^{|\ell|}\left(\frac{2r^2}{w^2}\right) e^{ik(z-\omega t)}\, e^{-ik\frac{r^2}{2R}}\, e^{-i\zeta}\, e^{-il\varphi}.$$

$$u_p^l A(r,\varphi,z) = f_p^l(r,z)\, e^{il\varphi}$$

Rayleigh range

$$z_R = \frac{\pi w_0^2}{\lambda}$$

beam radius

$$w(z) = w_0\sqrt{1+\left(\frac{z}{z_R}\right)^2}$$

radius of curvature

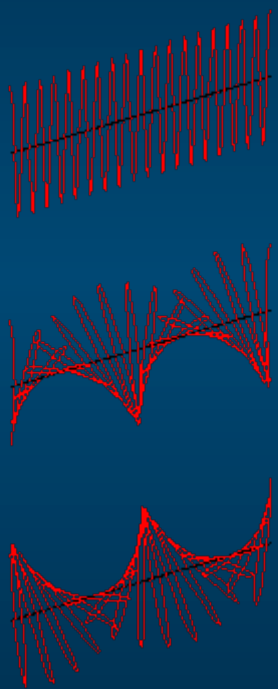
$$R(z) = z\left[1+\left(\frac{z_R}{z}\right)^2\right]$$

Gouy phase

$$\zeta(z) = (2|l| + p + 1)\arctan\frac{z}{z_R}$$

qubits and quNits

- spin angular momentum

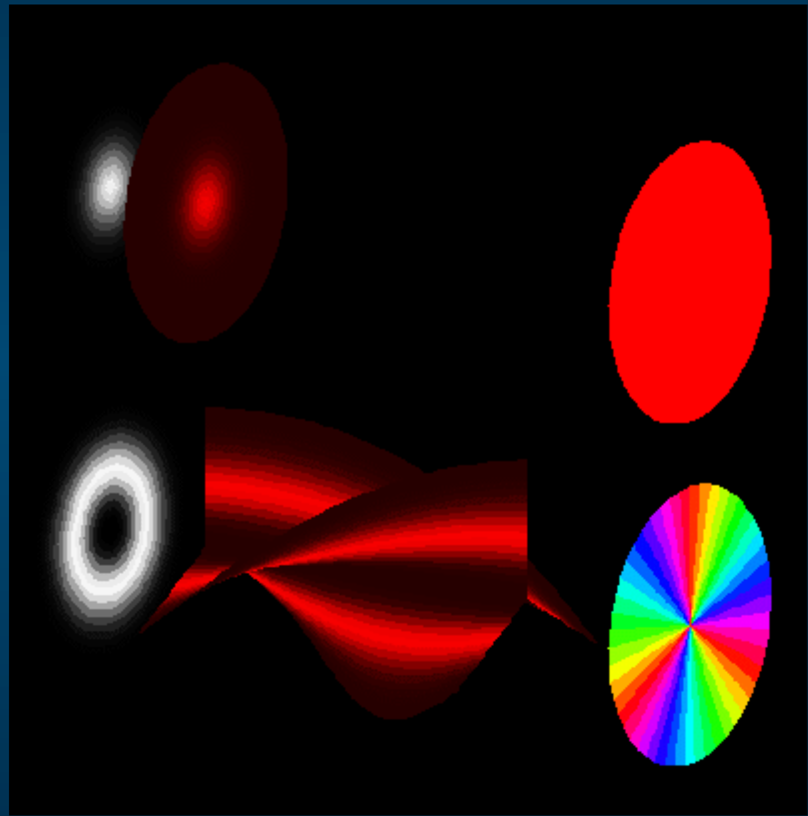


s=-1

s=+1

SAM inhabits a 2-dimensional Hilbert space and is a suitable physical realisation of a qubit.

- orbital angular momentum



⋮

ℓ=-1

ℓ=0

ℓ=+1

ℓ=+2

ℓ=+3

ℓ=+4

⋮

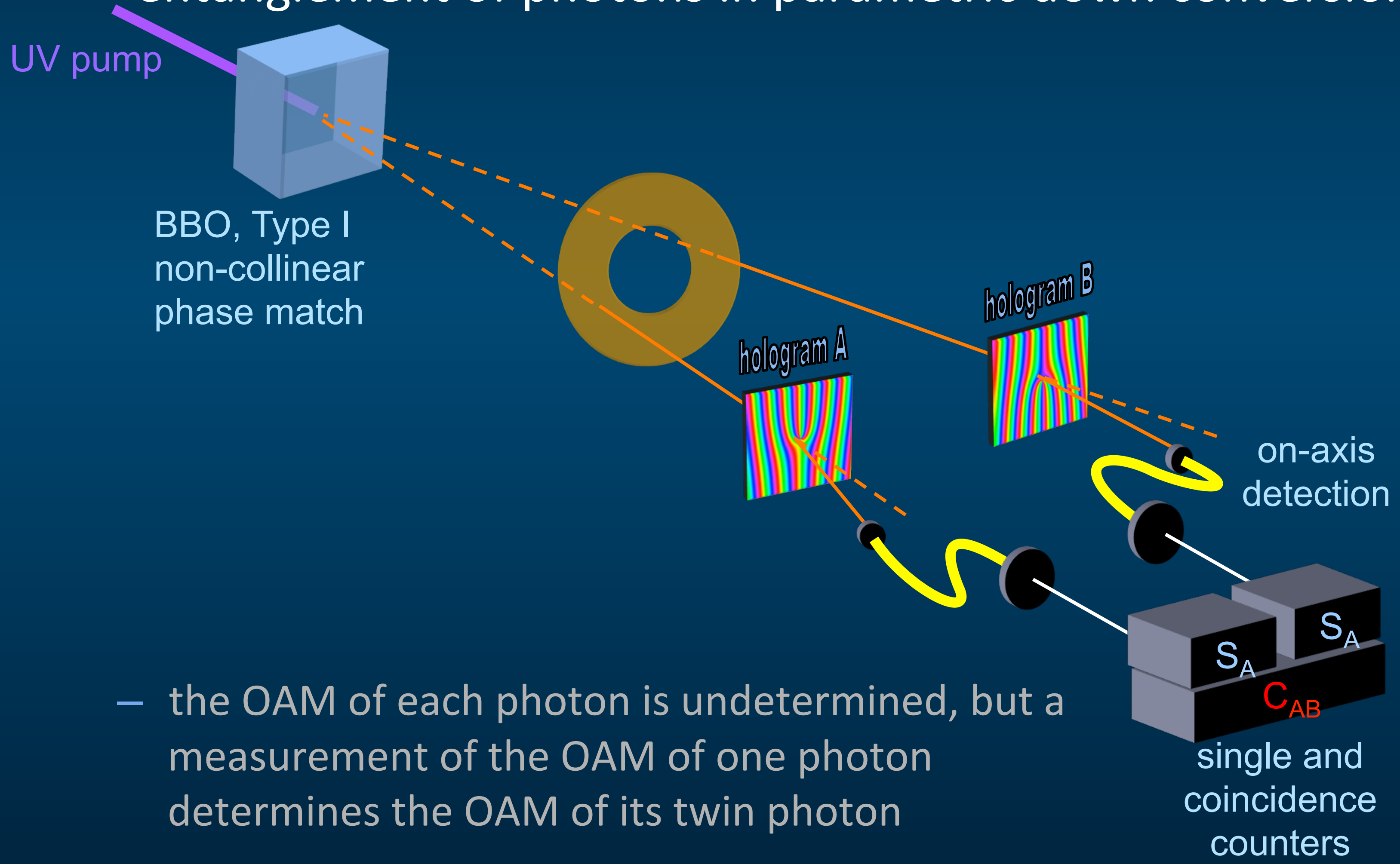
OAM can take (in principle) an unlimited number of values – and represent qudits.

Towards high D quantum applications?

- If we want to utilise the high dimensionality of OAM we need to
  - be able to generate, manipulate and detect OAM with high efficiency and reliability
- If we want to utilise quantum features of OAM we need to
  - produce superpositions and entangled states with high accuracy
  - test the quality of OAM entanglement
  - realise existing and develop new quantum protocols for high-dimensional OAM processing

Entering the quantum world

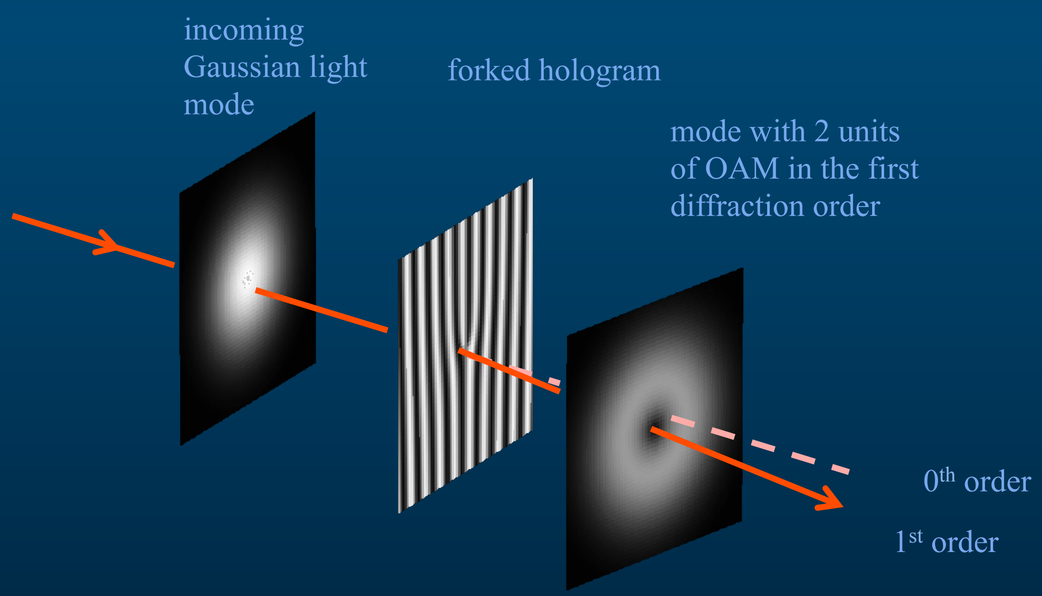
- In 2001, Alois Mair (with Anton Zeilinger) reported on OAM entanglement of photons in parametric down conversion.



  – the OAM of each photon is undetermined, but a measurement of the OAM of one photon determines the OAM of its twin photon

Aside: Generation of OAM states

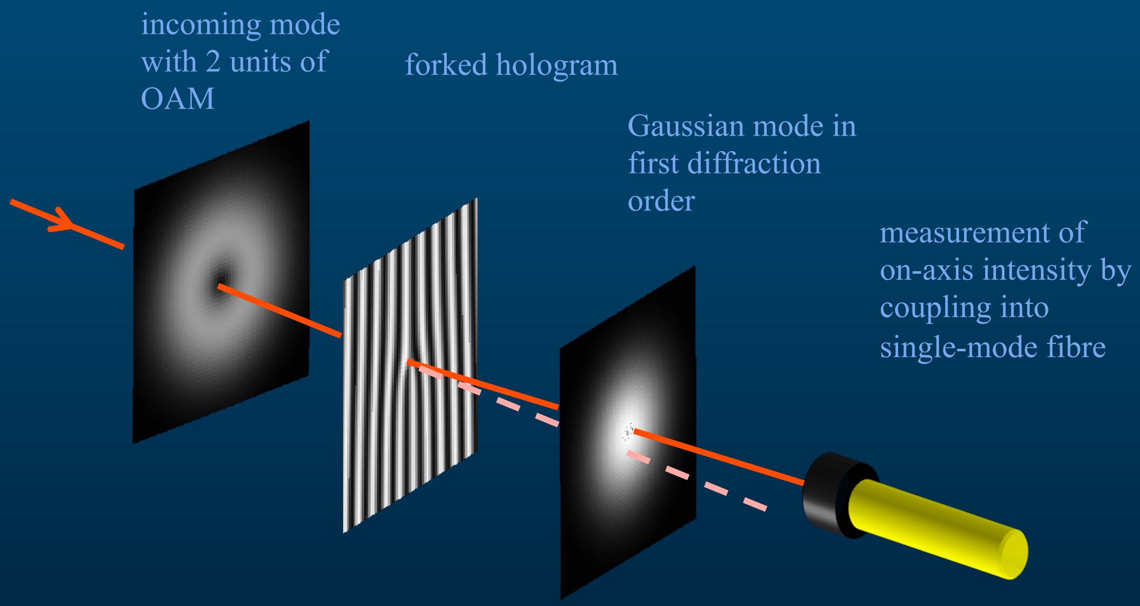
- light with a particular OAM can be generated from a Gaussian laser beam by changing its phase, e.g. via a hologram (Bashenov et al, Heckenberg et al, 1990s)
  - suitable shaping of the intensity favours particular LG modes with a given beam waist and radial p value.



Holograms can conveniently be displayed on SLMs (spatial light modulators) – programmable liquid crystal devices producing a spatially dependent phase delay on reflected beam.
Video resolution and refresh rate.

Detection of OAM states

- operated in reverse, such holograms can be used to identify a particular OAM state or superposition thereof



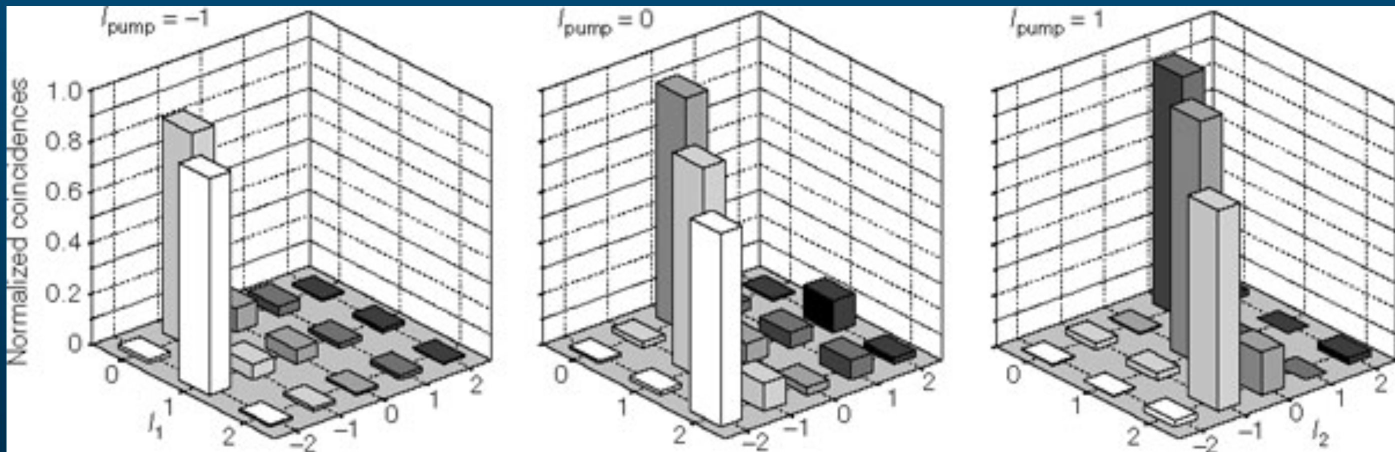
Zeilinger 2001

- The experiments indicated that OAM is conserved in the down-conversion process:
  - A pump beam without OAM generated frequent coincidences of

$$\ell_A + \ell_B = 0, \text{e.g.}\ \ell_A = 1 \text{ and } \ell_B = -1$$

  - more generally, a pump beam with given OAM

$$\ell_A + \ell_B = \ell_{\text{pump}}$$



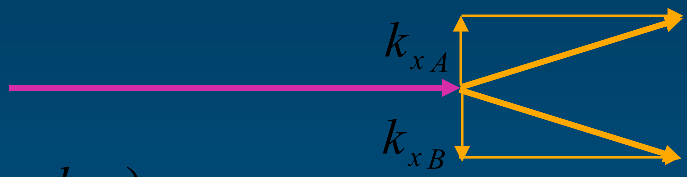
A. Mair et al. ***Nature*** **412**, 313-316 (19 July 2001)
also e.g. Monken et al. PRA **57**, 3123 (1998), Walborn et al PRA **69**, 23811 (2004)

OAM conservation

- Simple explanation OAM correlation from phase matching
  - phase matching requires momentum conservation in the transverse plane → anticorrelated transverse k vectors

$$\vec{k}_{pump} = \vec{k}_A + \vec{k}_B$$

$$\omega_{pump} = \omega_A + \omega_B$$



$$\delta(k_{xA} + k_{xB})\delta(k_{yA} + k_{yB})$$

  - Fourier transform : position correlation

$$\delta(x_A - x_B)\delta(y_A - y_B) = \frac{\delta(r_A - r_B)}{r}\delta(\theta_A - \theta_B)$$

correlated angle

$$= \frac{\delta(r_A - r_B)}{r}\sum_m \mathrm{e}^{im\theta_A}\mathrm{e}^{-im\theta_B}$$

anti-correlated OAM

S. Franke-Arnold et al Phys. Rev. A, 65:033823 (2002)

more detailled descriptions Torres et al, very recently A. Yao et al, ...

angle and angular momentum

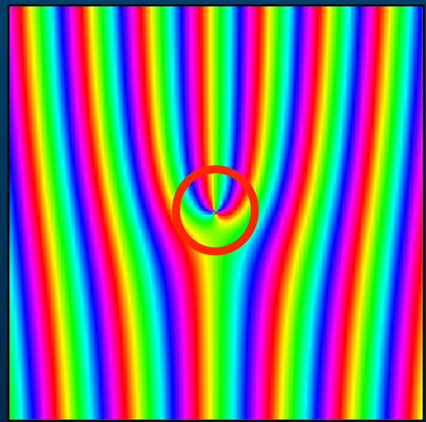
- Angular momentum and angle are related by Fourier transform.
  - Correlation between angle and OAM arises from a Fourier relation *i.e.* is a classical effect...
  - ... but it prevails on the single photon level
  - ... and for entangled states.

$$\psi(\theta) = \frac{1}{\sqrt{2\pi}} \sum_{\ell=-\infty}^{+\infty} e^{i\ell\theta} A(\ell)$$

$$A(\ell) = \frac{1}{\sqrt{2\pi}} \int_0^{2\pi} d\phi \, e^{-i\ell\theta} \psi(\phi)$$

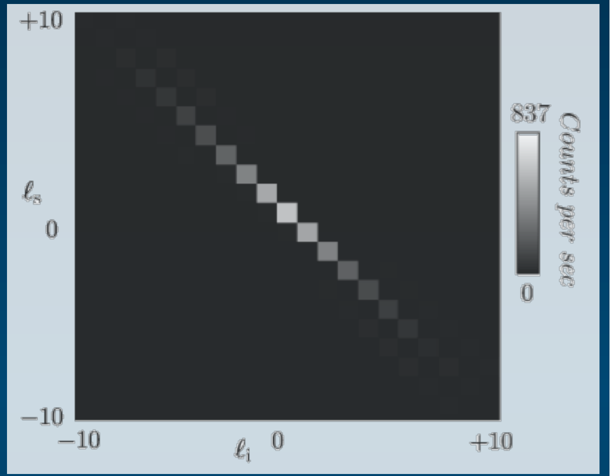
OAM entanglement

- The 2001 experiment
  - measured high coincidence countrates for anticorrelated OAM of partner photons,
  - showed conservation of OAM in downconversion.
- Entanglement?
  - Unlike classical correlations, quantum correlations relate the complex amplitudes of states. This can be tested by observing quantum correlations also for superpositions.
  - For OAM, superposition states can be created by deliberately moving the hologram so that the fork dislocation is no longer centred with the beam.
- The experiment found correlations between superposition modes - an indication of true entanglement!



entangled OAM states

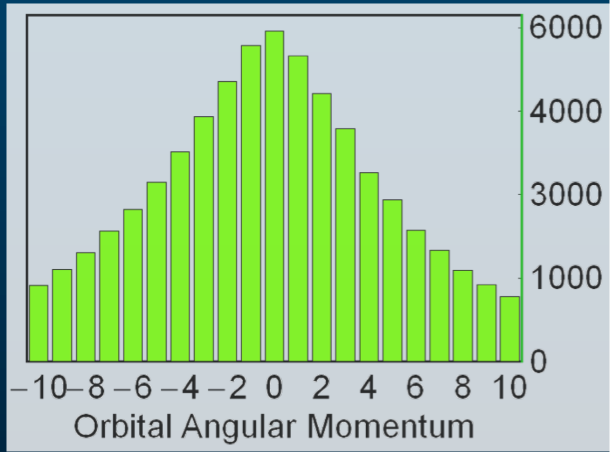
- The two photon wavefunction is an entangled state...
  - Whenever a photon is measured with a particular OAM, the partner photon has the opposite OAM.



$$|\Psi\rangle_{2\,\text{photon}} = \sum_{\ell=-\infty}^{+\infty} c_\ell |\ell\rangle_A |-\ell\rangle_B, \quad \text{where } c_\ell = c_{-\ell}$$

$$\rho_{2\,\text{photon}} = |\Psi\rangle\langle\Psi|_{2\,\text{photon}} \qquad (\text{pure as } \rho = \rho^2)$$

- ..., despite the fact that the mode in each individual down-converted beam is incoherent!
  - Mathematically this is expressed by the mixed density operator of the individual beams:



$$\rho_A = Tr_B |\Psi\rangle\langle\Psi|_{2\,\text{photon}} = \sum_m \langle m|_B |\Psi\rangle\langle\Psi|_{2\,\text{photon}} |m\rangle_B$$

$$= \sum_{m=-\infty}^{\infty} |c_m|^2 |m\rangle_A \langle m|_A \qquad (\rho \neq \rho^2)$$

single and coincidence measurements

- Formal description of measurements:
  - ideal probability operator of OAM detector

$$\hat{\pi}_\ell = |\ell\rangle\langle\ell| \qquad \text{detector fires}$$

$$\hat{\pi}_{\bar{\ell}} = 1 - |\ell\rangle\langle\ell| \qquad \text{detector does not fire}$$

- Ideal coincidence countrate

$$C_{\ell_A m_B} = Tr\left(\rho_{2\text{photon}} \hat{\pi}_{\ell\,A} \hat{\pi}_{m\,B}\right)$$

$$= \left| \langle\Psi|_{2\text{photon}} \hat{\pi}_{\ell\,A} \hat{\pi}_{m\,B} |\Psi\rangle_{2\text{photon}} \right|^2 = |c_\ell|^4 \delta_{\ell,-m}$$

for the two-photon state $|\Psi\rangle_{2\,\text{photon}} = \sum_{\ell=-\infty}^{+\infty} c_\ell |\ell\rangle_A |-\ell\rangle_B$

- Ideal single countrate

$$P_{\ell_A} = Tr(\rho_A \hat{\pi}_\ell) = |c_\ell|^2$$

for the single-photon density matrix $\rho_A = \sum_{m=-\infty}^{\infty} |c_m|^2 |m\rangle_A \langle m|_A$

The two-photon coincidences fall off faster than the single photon counts.

a note on purity

- Of course we have been somewhat generous claiming a two-photon wavefunction.
  - Quantum states in optics are always mixed to some extent, i.e. there are off-diagonal elements to the density operator.

- In particular, there's always a finite chance to find partner photons that do not fulfil OAM conservation.
  - detector noise can produce a count in one arm that does not have a "partner" in the other arm
  - inefficient detection may keep only one of the counts
  - the production of two photon pairs within the detection window may lead to wrong pairings

aside

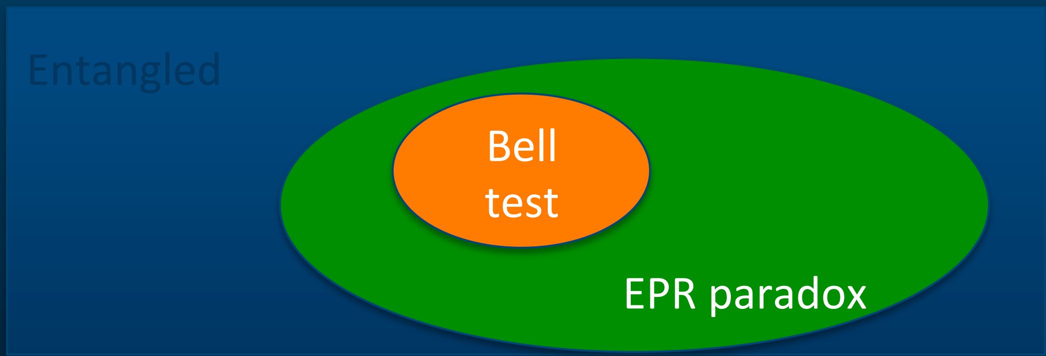
- What is entanglement?
  - There's only a negative definition:

A state is entangled if it cannot be expressed as a product state of its constituent systems.

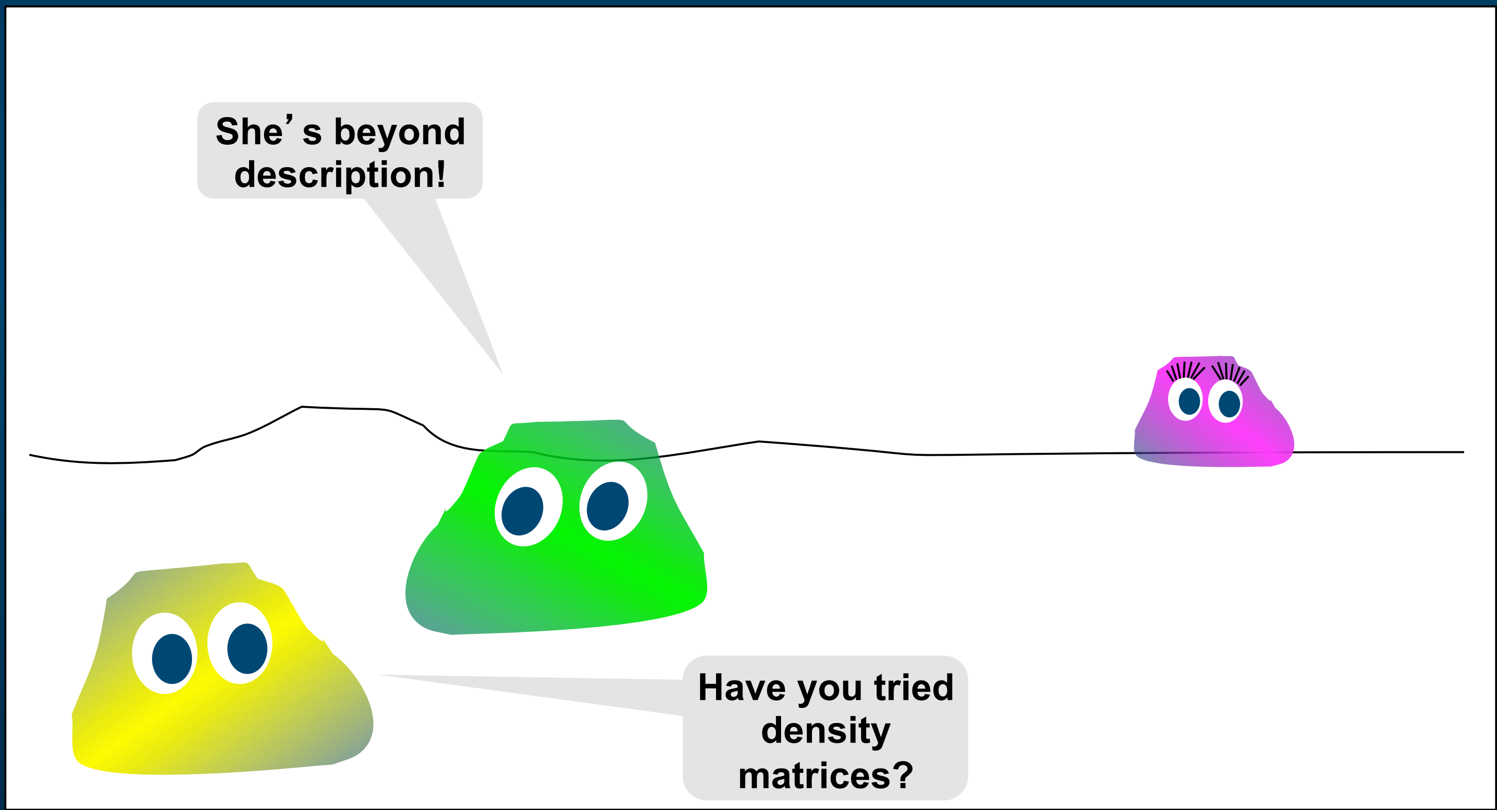
$$|\Psi\rangle = |\Psi\rangle_A |\Psi\rangle_B \quad \leftrightarrow \quad \text{pure state } |\Psi\rangle \text{ is not entangled}$$

$$\hat{\rho} = \sum_{i,j} P(A_i, B_j) \hat{\rho}_A{}^i \otimes \hat{\rho}_B{}^j \quad \leftrightarrow \quad \text{mixed state is not entangled}$$

  - If one knows the density matrix of a state one can test this criterion and identify "measures of entanglement".
  - A variety of tests identify certain classes of entangled states, but they do not cover all entangled states.

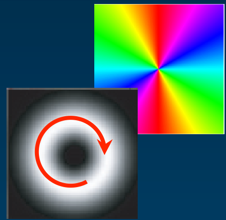
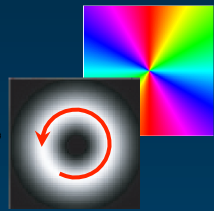


2. Describing OAM states

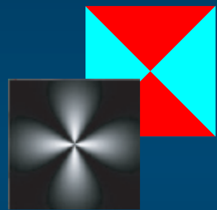


2. Describing quantum states

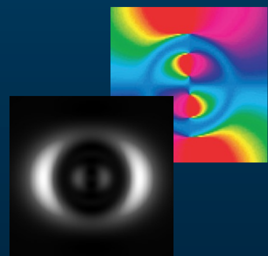
- OAM states

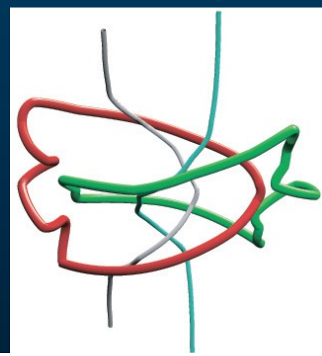
$|\ell = 2\rangle$  $|\ell = -2\rangle$ 

- OAM superposition states
  - of few states $\frac{1}{\sqrt{2}}(|2\rangle + |-2\rangle)$



  - superposition of infinitely many OAM states with Gaussian envelope
  - by displacing the hologram $p|0\rangle + (1-p)|1\rangle$
  - to generate vortex links, by superimposing 3 LG components with prescribed amplitudes

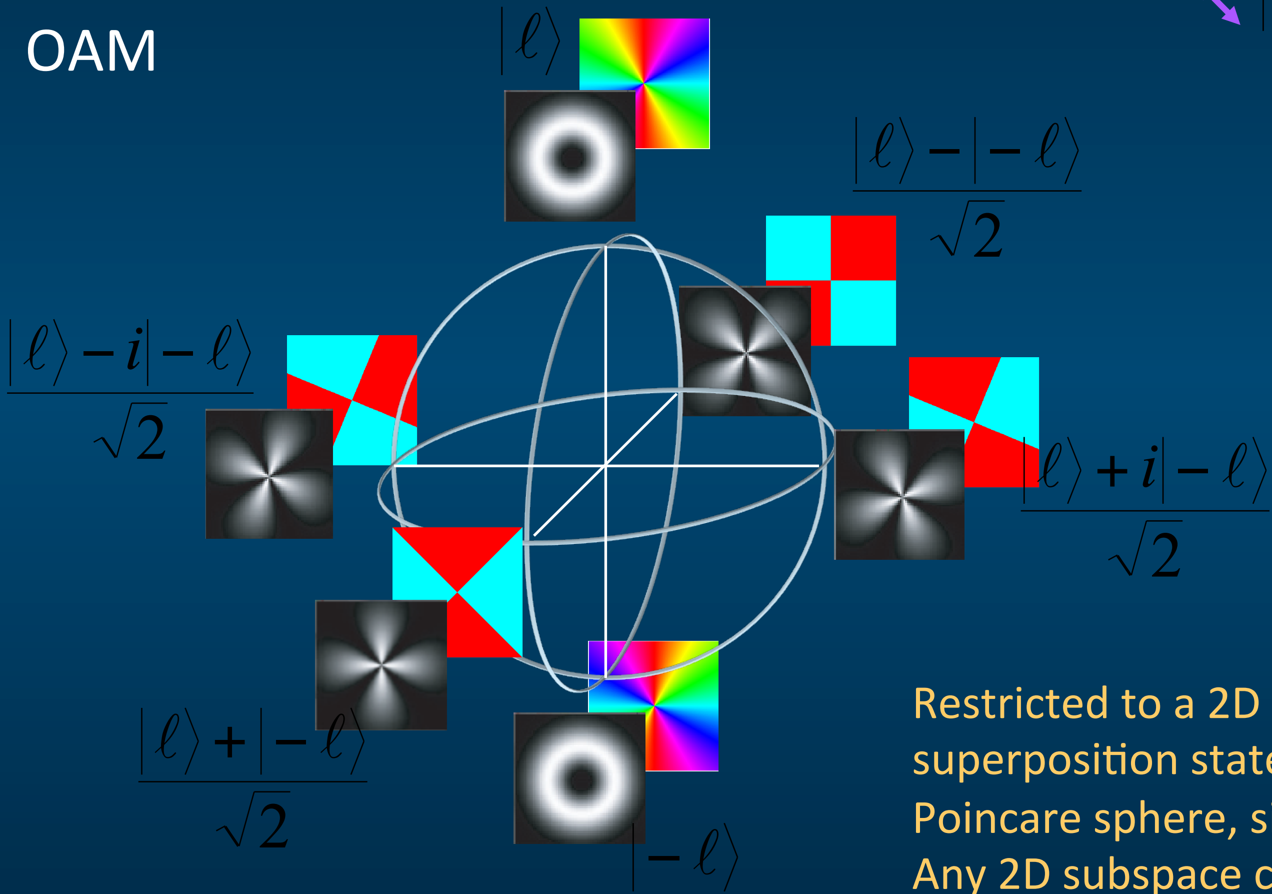


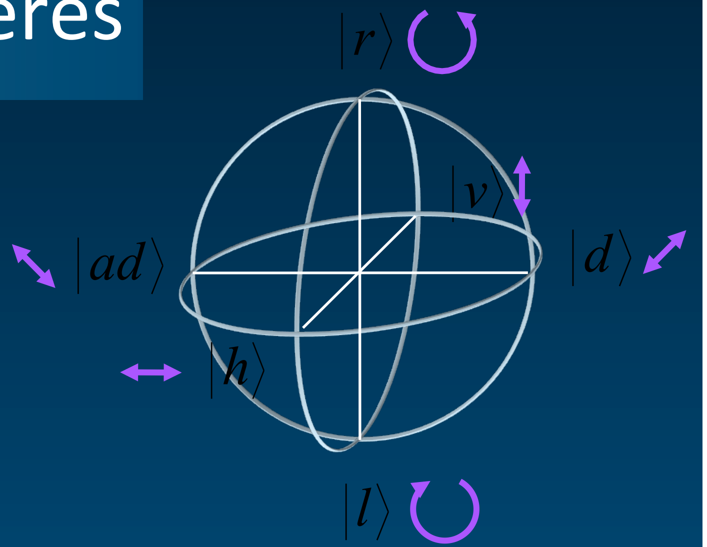


Leach et al NJP 7 (2004)

Poincaré spheres

OAM



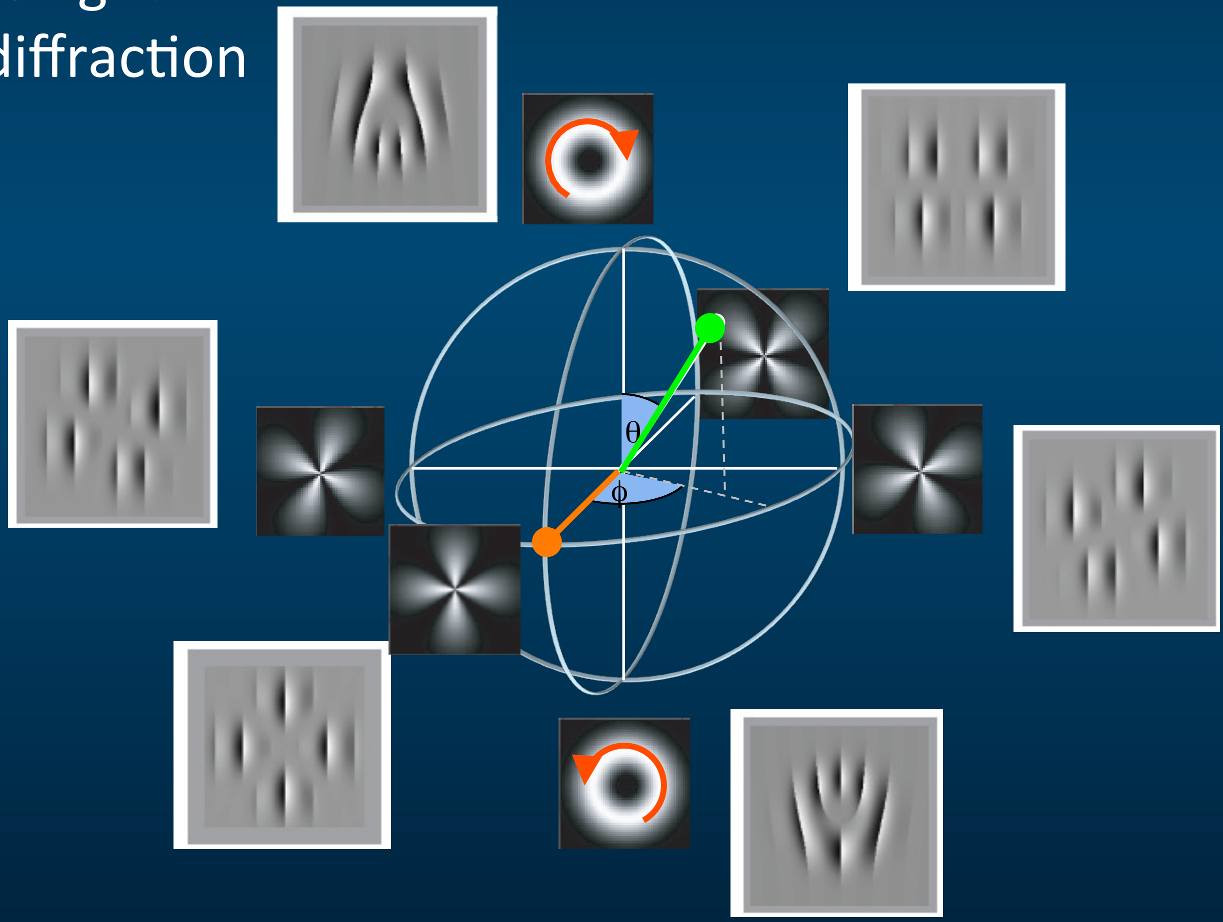


polarisation

Restricted to a 2D OAM subspace, all possible superposition states can be represented on a Poincare sphere, similar to polarisation states. Any 2D subspace can be parametrised like this! For higher dimensions the resulting Riemann spheres become hard to picture.

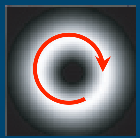
generating arbitrary states

- In Glasgow, we use SLMs to display holograms that generate the desired light pattern in the first diffraction order.



identifying single photon OAM

- given intensity/phase pattern, can identify OAM components via Fourier decomposition
- can identify a density matrix in e.g. an OAM basis $\rho = \begin{pmatrix} \rho_{11} & \rho_{12} \\ \rho_{21} & \rho_{22} \end{pmatrix}$

$|\ell = 2\rangle$  $\rho = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$

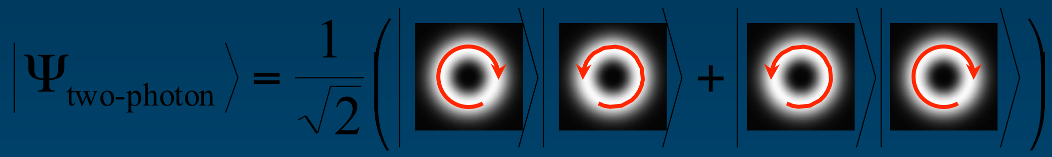
$\frac{1}{\sqrt{2}}(|2\rangle + |-2\rangle)$  $\rho = \begin{pmatrix} 0.5 & 0 \\ 0 & 0.5 \end{pmatrix}$

In order to identify the 4 elements of the density matrix one needs to measure not only in the +2 and -2 basis, but also a rotated basis.

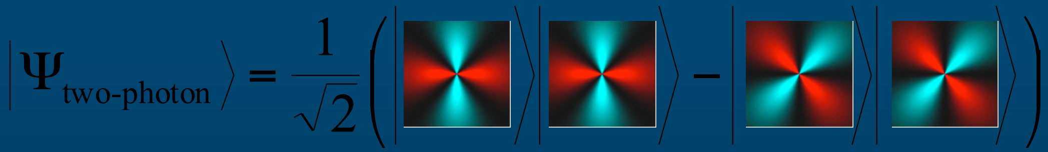
- Bipartite states, e.g. our entangled state is described by a 4 x 4 density matrix.

entangled states

- We'd like to have a two-photon state that is anticorrelated in OAM

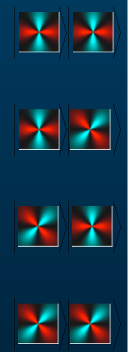


- and correlated in angle



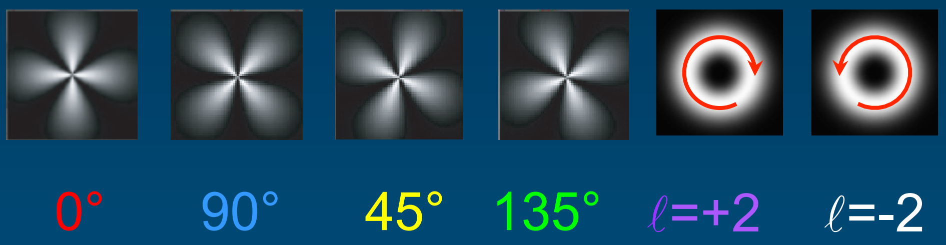
- This state would be described by the two-photon density matrix:

$$\rho = \begin{pmatrix} 0.5 & 0 & 0 & 0.5 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0.5 & 0 & 0 & 0.5 \end{pmatrix}$$

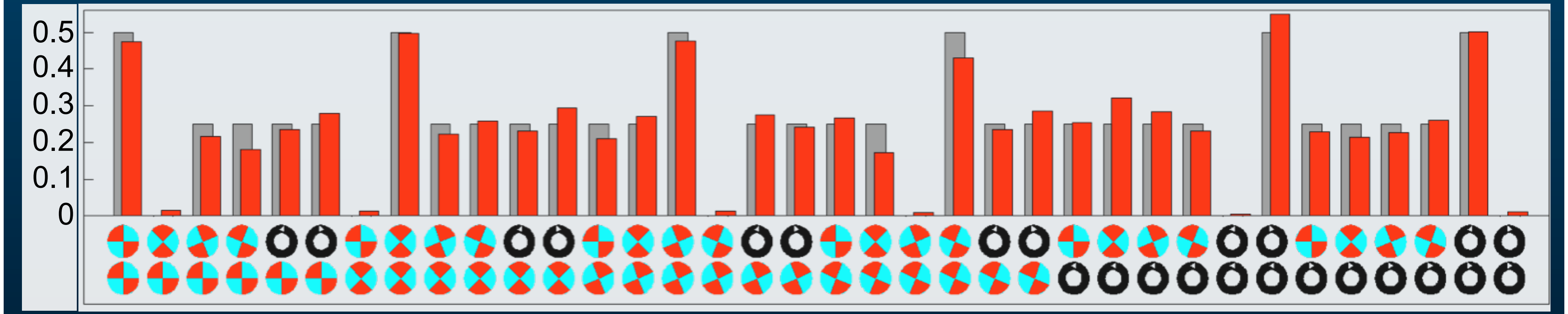


tomographic measurements

- Tomographic reconstruction requires measurements in 6 non-orthogonal basis states.



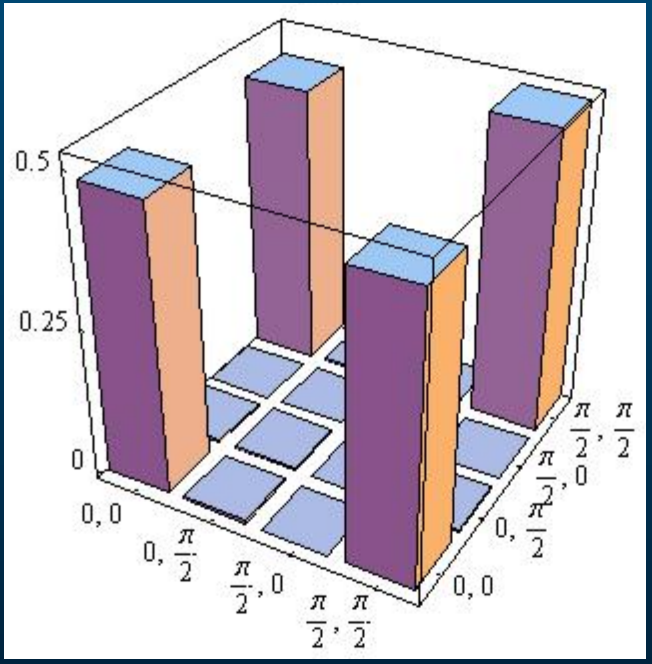
- Resulting in 36 coincidence probabilities:



density matrix

- 36 equations to determine 16 elements of two photon density matrix
- use least square approach or random search to "guess" allowed density matrix that best accounts for observations
- $Tr\left(\rho^2\right) = 0.948$

Density Matrix



B. Jack et al. New J Phys **11**, 103024 (2009)
earlier work by Langford et al, PRL **93,** 53601 (2004)

learning from the density matrix

- linear entropy: quantifies how mixed the measured state is

pure $0 \leq \quad \mathrm{SL} = \frac{4}{3}\left[1 - \mathrm{Tr}(\rho^2)\right] \quad \leq 1$ completely mixed

SL=0.050

- fidelity: quantifies how close a state is to a "target state" (e.g. the Bell state)

orthogonal $0 \leq \quad \mathrm{F} = \langle \Psi | \rho | \Psi \rangle \quad \leq 1$ the same

F=0.978

- entanglement: Find the partial transpose of the two-photon state, (transpose only one of the subsystems) and identify the eigenvalues λ_j. If the system was a product state, this process sorts between the subsystems and the partial transpose is a valid density matrix with positive eigenvalues. If not – not.

not entangled

$0 \leq \quad \mathrm{Negativity} = |\min(\lambda_j)|/2 \quad \leq 0.25$

$0 \leq$ Concurrence ≤ 1

$0 \leq$ Entanglement of Formation ≤ 1

maximally entangled

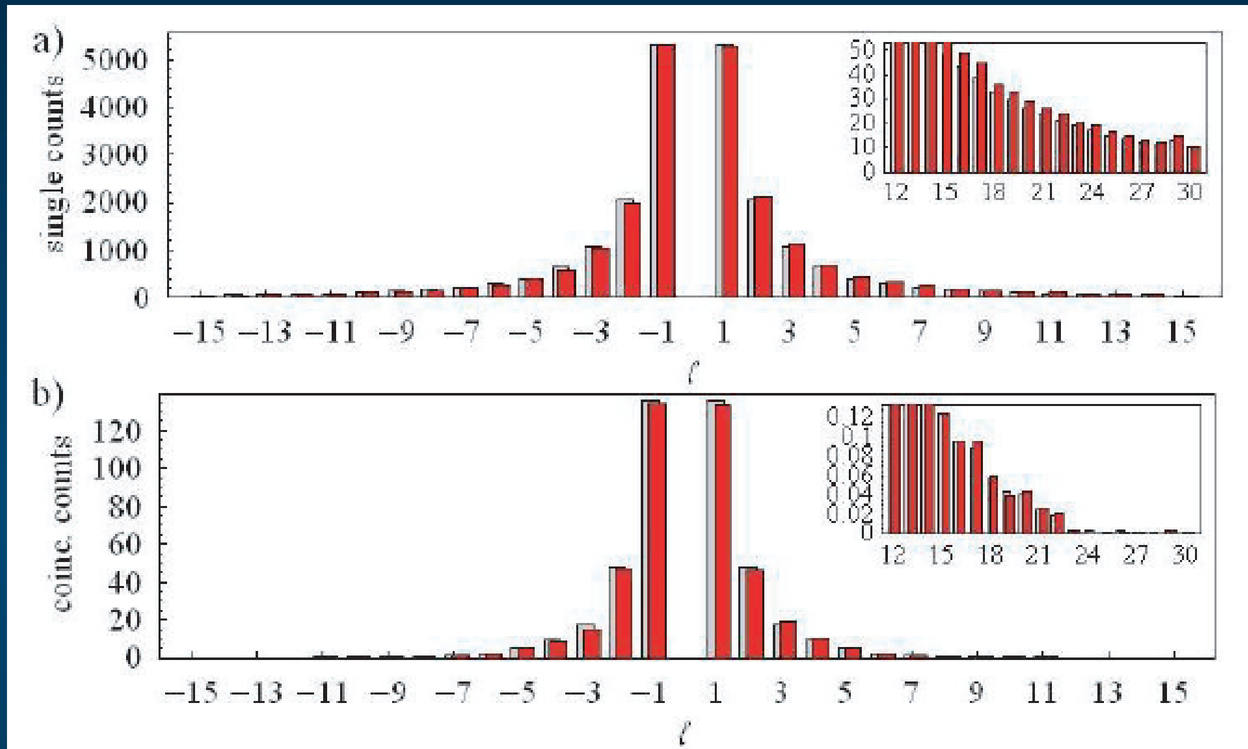
Neg=0.241

C=0.969

EoF=0.956

high dimensions

- OAM has infinite dimensions, can we actually use them?

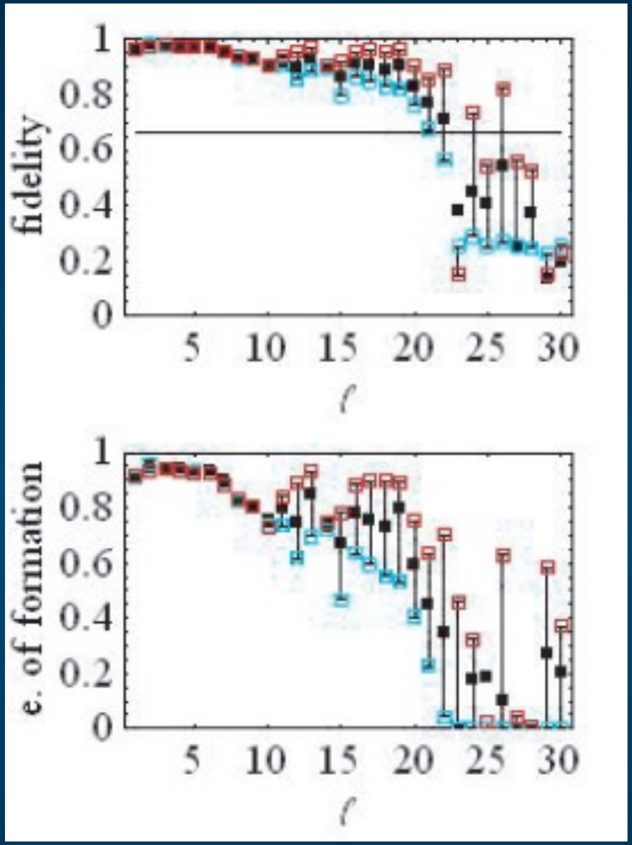


- Problem: the single countrate and therefore also the coincidence countrate diminishes with increasing ℓ.
- Reconstruct the 2 photon density matrix in two-dimensional state subspaces defined by the OAM states $\pm\ell$ up to $\ell = 30$.

tomography for high dimensions

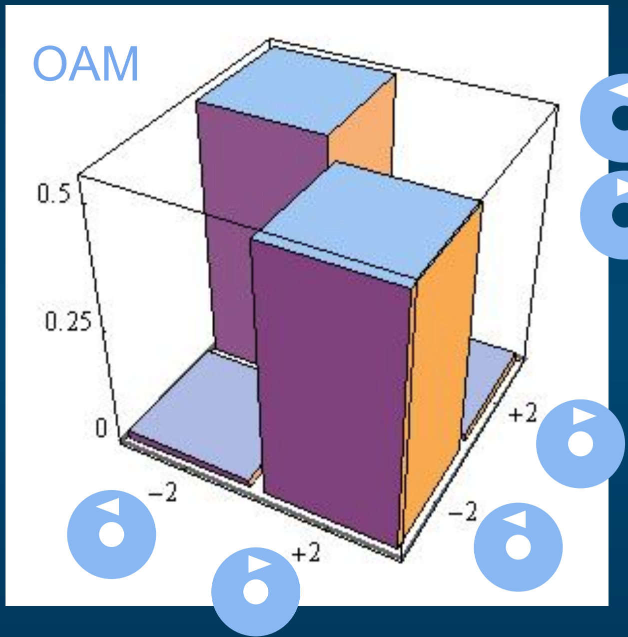
- The reconstructed density matrix, even for OAMs up to around ℓ = 20, is close to that of a maximally entangled Bell state with a fidelity in the range between *F* = 0.979 and *F* = 0.814.

- Although the single countrate diminishes with increasing ℓ, entanglement persists.



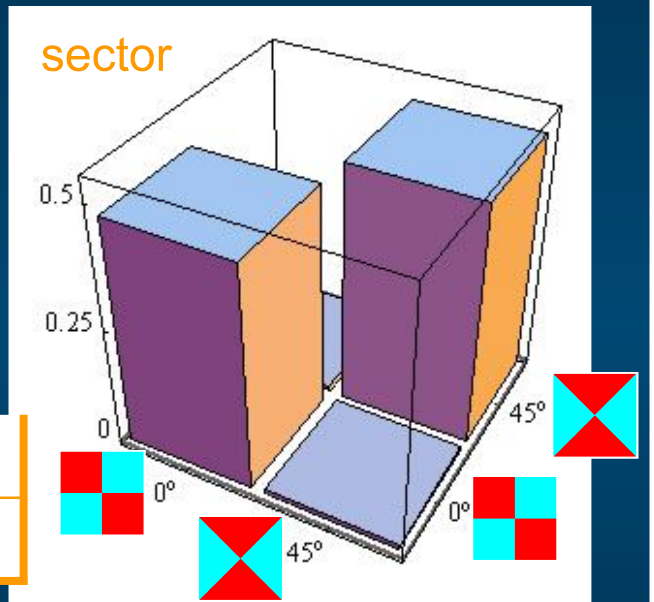
Angular EPR argument

- Infer OAM / sector state of one photon from an errorless measurement of the OAM / sector state of its twin photon.



0.005	0.515
0.469	0.011

0.475	0.015
0.013	0.497



In the Einstein sense, these inferred predictions "do not disturb the photon in any way" and therefore relate to simultaneous elements of reality for OAM and angle – forbidden by quantum mechanics.

theory: J. Götte *et al.*, J. Mod. Opt. **53,** 627 (2007)

Conclusions

- OAM is a high dimensional quantum system - rather convenient system for fundamental quantum tests.
- Down-converted photons are spatially incoherent but entangled (two-photon coherent) in OAM.
- The analogy between 2D OAM subspaces and polarisation allows to translate various fundamental quantum tests (more tomorrow).
- We can perform tomography on the bipartite state and find that entanglement persists up to high dimensions.