Tangled quantum vortices

Carlo F. Barenghi

(http://research.ncl.ac.uk/quantum-fluids/)

Newcastle University

Warning: vortices will interact !

Wingtip vortices move down
(Helmholtz Theorem: Vortex lines move with the fluid)

Summary

Part 1: quantum vortices

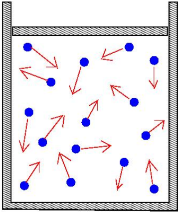
- Absolute zero
- Quantum fluids, Bose-Einstein condensation
- Quantum vortices
- Vortex reconnections, friction
- Gross-Pitaevskii eq, Vortex Filament models
- Vortex ring, Kelvin waves, vortex knots

Part 2: quantum turbulence

Absolute zero: XVII-XVIII centuries

Guillaume Amontons (1663-1705)

Daniel Bernoulli
(1700-1782)



- Amontons measured pressure changes induced by temperature changes. He argued that pressure cannot become negative, so there is a minimum temperature (estimated at -240 C) corresponding to no motion.
- Daniel Bernoulli's kinetic theory.

Absolute zero: XVIII-XIX century

• Side track:
Heat is a fluid called **caloric**

Antoine Lavoisier (1743-1794)

• Back on track:
Thermodynamics

Lord Kelvin (1824-1907)

Absolute zero: XX century

1908	Liquid helium (K. Onnes)
1911	Superconductivity (K. Onnes)
1924	Bose-Einstein condensation (BEC) (Bose & and Einstein)
1937	Superfluid ^{4}He (Kapitza, Allen & Misener)
1938	Link between BEC and superfludity (F. London)
1940s	Two-fluid model (Landau, Tisza)
1950s	Vortex lines (Onsager, Feynman, Hall & Vinen)
1970s	Superfluid ^{3}He (Lee, Osheroff, & Richardson)
1995	BEC of atomic gases (Cornell, Ketterle, & Weiman)

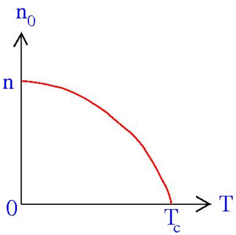
Kelvin temperature scale

273.15 K	Melting point of water (0 C)
210 K	Mean temperature on Mars (−63 C)
92 K	Superconducting point of Y-Ba-Cu-oxide (YBCO)
77 K	Boiling point of nitrogen
63 K	Melting point of nitrogen
20 K	Boiling point of hydrogen
14 K	Melting point of hydrogen
4.2 K	Boiling point of helium
4.1 K	Superconducting point of mercury
2.725 K	Cosmic microwave background radiation
2.1768 K	Superfluid ^{4}He
10^{-3} K	Superfluid ^{3}He
10^{-7} K	Bose-Einstein condensation in atomic gases
10^{-10} K	Rhodium
...	
0 K	Absolute zero (−273.15 C)

Quantum fluids

- Liquid ^{4}He ($T_c = 2.17\mathrm{K}$)
- Liquid ^{3}He ($T_c \approx 10^{-3}\mathrm{K}$)
- atomic condensates: ^{87}Rb, ^{23}Na, ^{7}Li, etc ($T_c \approx 10^{-7}\mathrm{K}$)
- two-components condensates, etc

Underlying physics:
- quantum statistics (bosons)
- Bose-Einstein condensation



n_0 = condensate density

Ideal Bose gas: at $T < T_c$ a finite fraction of the bosons falls into the lowest accessible state

$$\frac{n_0}{n} = 1 - \left(\frac{T}{T_c}\right)^{3/2}$$

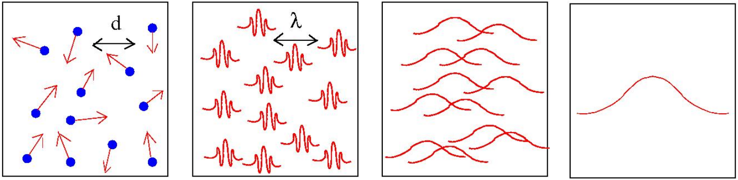
T_c = critical temperature

Bose-Einstein condensation

- Atom with momentum $p = mv$ has wavelength $\lambda = h/p$
- Average kinetic energy $mv^2/2 \approx k_B T$
- Wavelength increases with decreasing T:

$$\lambda \approx \frac{h}{\sqrt{mk_B T}}$$

- Compare λ against the average distance between atoms, d:



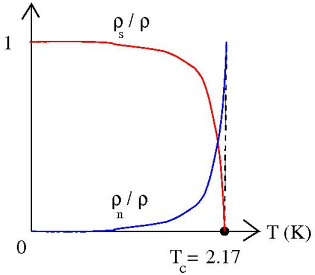
BEC occurs when $\lambda \approx d$

Two-fluid model

Landau & Tisza:

- Quantum ground state $\Rightarrow$ superfluid component (careful)
- Thermal excitations $\Rightarrow$ normal fluid component

Accounts for unusual behaviour of ^{4}He for $T < T_c$
(thermal/mechanical effects, second sound, counterflow)



	Normal fluid	Superfluid
Density	ρ_n	ρ_s
Velocity	$\mathbf{v}_n$	$\mathbf{v}_s$
Entropy	S	0
Viscosity	η	0

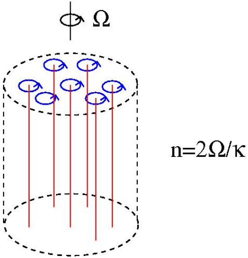
$$\rho = \rho_n + \rho_s$$

- Macroscopic $\Psi = Ae^{i\phi}$
- $\mathbf{v}_s = (\hbar/m)\nabla\phi$
- Quantum of circulation

(Onsager 1948, Feynman 1954, Vinen 1961)

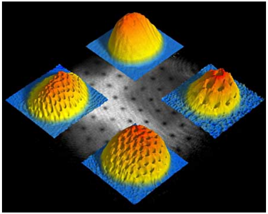
$$\oint_C \mathbf{v}_s \cdot \mathbf{dr} = \frac{\hbar}{m} = \kappa$$

Vortex lattice in rotating helium



Packard et al 1982

Bewley, Lathrop & Sreenivasan 2006



Ketterle et al 2001

Gross Pitaevskii Equation (GPE)

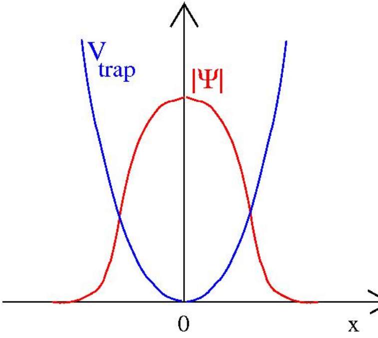
Assume weak interactions (realistic for atomic gases, idealised for helium)

$$V(\mathbf{x}-\mathbf{x}') = g\delta(\mathbf{x}-\mathbf{x}')$$
$$g = 4\pi a_s \hbar^2/m$$
$$a_s = \text{scattering length}$$

$$i\hbar\frac{\partial \Psi}{\partial t} = -\frac{\hbar^2}{2m}\nabla^2\Psi + g\Psi|\Psi|^2 + V_{trap}\Psi$$

$$\int |\Psi(\mathbf{r})|^2 d^3\mathbf{r} = N$$

$$V_{trap} = \frac{1}{2}\omega^2 x^2 \quad \text{trapping potential}$$



Fluid dynamics interpretation of GPE

Substitute $\Psi = Ae^{i\phi}$ into GPE, define

density $\rho = |\psi|^2$
velocity $\mathbf{v} = (\hbar/m)\nabla\phi$

and get

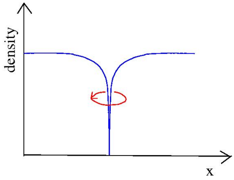
$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{v}) = 0 \qquad \text{(Continuity eq.)}$$

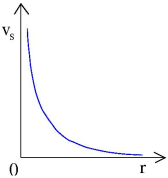
$$\rho \left(\frac{\partial v_j}{\partial t} + v_k \frac{\partial v_j}{\partial x_k} \right) = -\frac{\partial p}{\partial x_j} + \frac{\partial \Sigma_{jk}}{\partial x_k} \qquad \text{(Quasi} - \text{Euler eq.)}$$

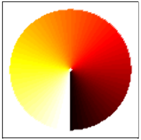
$$\text{Pressure } p = \frac{g}{2m^2}\rho^2 \qquad \text{Quantum stress } \Sigma_{jk} = \left(\frac{\hbar}{2m}\right)^2 \rho \frac{\partial^2 \ln \rho}{\partial x_j \partial x_k}$$

- At scales larger than $\xi = \hbar/\sqrt{m\mu} \Rightarrow \Sigma_{jk}$ negligible $\Rightarrow$ Euler eq.

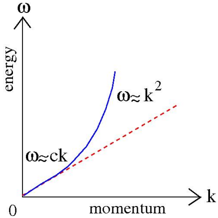
- Vortex is a hole of radius $\approx \xi$ around which the phase of ψ changes by 2π

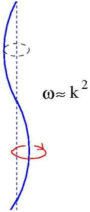






Phase

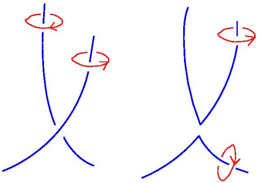


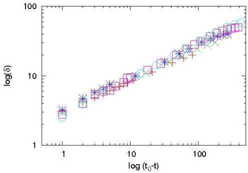


- Perturb uniform solution: waves
- Perturb straight vortex: Kelvin waves

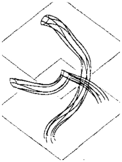
Vortex reconnections

- Conjectured (Schwarz 1988); • Proved (Koplik & Levine 1993)
- For bundles (Alamri, Youd & Barenghi PRL 2008)
- Observed (Paoletti et al 2008) using $\delta(t) = A\sqrt{\kappa(t_0 - t)}$





(Tebbs & CFB, JLTP 2011)



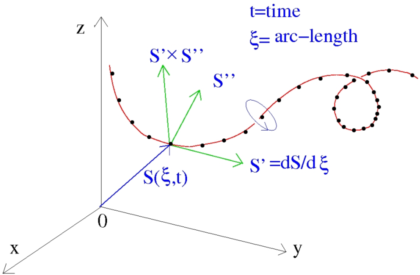
(Kida & Takaoka 1988)

Classical reconnections:

- rely on viscosity
- Moffatt's 1969 helicity theorem
- blow-up ?

Vortex filament model

- ℓ = typical distance between vortex lines
- Schwarz 1988:
  $\xi << \ell \Rightarrow$ model vortex line as reconnecting space curve $\mathbf{s}(\xi, t)$



Biot-Savart (BS) law:

$$\frac{d\mathbf{s}}{dt} = \frac{\kappa}{4\pi} \oint \frac{(\mathbf{z}-\mathbf{s}) \times \mathbf{dz}}{|\mathbf{z}-\mathbf{s}|^3}$$

Local induction approx. (LIA):

$$\frac{d\mathbf{s}}{dt} \approx \beta \mathbf{s}' \times \mathbf{s}''$$

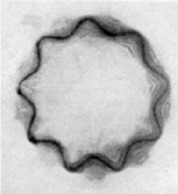
- N = number of discretization points
- Biot-Savart is slow: CPU $\sim N^2$
- Tree algorithm is faster: CPU $\sim N \log N$ (Baggaley & CFB 2011)

Vortex rings and Kelvin waves

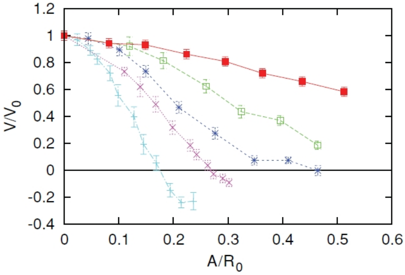
- Kelvin waves: Thomson (1880), Pocklington (1895)

- Slow-down effect noticed much later:

LIA: Kiknadze & Mamaladze (JLTP 2002)
BS: Hiänninen, CFB & Tsubota (PRE 2006)
GPE: Helm, CFB & Youd (PRA 2011)

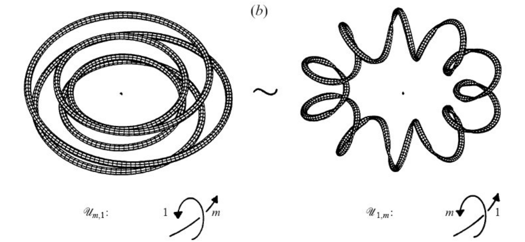


Krutzsch
Annalen der Physik 1939

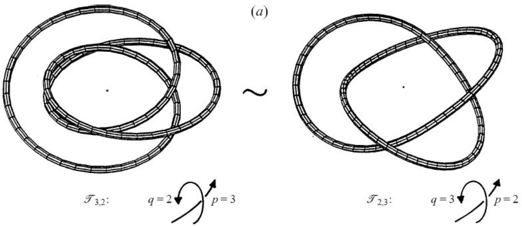


Helm, CFB & Youd 2011

Vortex knots and unknots



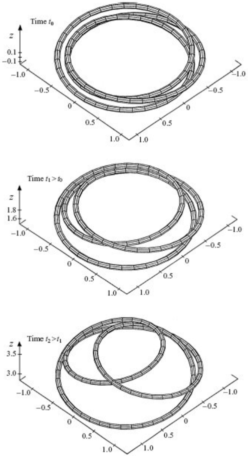
Toroidal and poloidal vortex coils



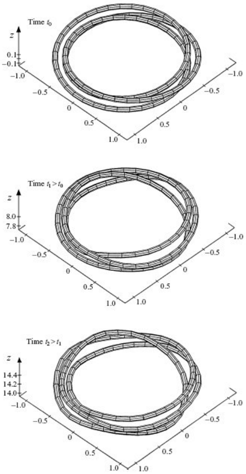
Torus vortex knot T_{pq}

Evolution of torus knot T_{32}

LIA evolution



Biot-Savart evolution



(Ricca, Samuels & CFB, JFM 1999;
Maggioni, Alamri, CFB & Ricca PRE 2010)

Friction

- At $T > 0$ thermal excitations interact with vortex fields/cores: mutual friction force between normal fluid and superfluid

$$\frac{d\mathbf{s}}{dt} = \mathbf{v}_s + \alpha \mathbf{s}' \times (\mathbf{v}_n - \mathbf{v}_s) - \alpha' \mathbf{s}' \times [\mathbf{s}' \times (\mathbf{v}_n - \mathbf{v}_s)]$$

α, α' known from experiments

- Hot topic for atomic BECs (Jackson & CFB, PRA 2009):

$$i\hbar \frac{\partial \psi}{\partial t} = \left(-\frac{\hbar^2}{2m} \nabla^2 + V + g n_c + 2g\tilde{n} - iR \right) \psi$$

$$\frac{\partial f}{\partial t} + \frac{\mathbf{p}}{m} \cdot \nabla f - \nabla U \cdot \nabla_{\mathbf{p}} f = C_{22} + C_{12}$$

Condensate and thermal cloud densities:

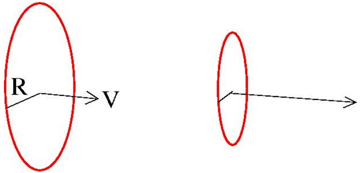
$$n_c(\mathbf{r}, t) = |\psi(\mathbf{r}, t)|^2, \qquad \tilde{n}(\mathbf{r}, t) = \int \frac{d\mathbf{p}}{(2\pi\hbar)^3} f(\mathbf{p}, \mathbf{r}, t)$$

Friction

At $T = 0$ vortex ring of radius R has self-induced velocity

$$V = \frac{\kappa}{4\pi R}(\ln(8R/\xi) - 1/2)$$

At $T > 0$ the vortex ring loses energy, shrinks and speeds up



$$\frac{dR}{dt} = -\frac{\gamma\kappa}{\rho_s}V$$

$\gamma =$ friction coeff

Summary

Part 1: quantum vortices

Part 2: quantum turbulence

- Fractal geometry, topology
- Classical turbulence:
  Richardson cascade, Kolmogorov spectrum
- Turbulence at (relatively) high temperature:
  Spectrum, vorticity aliognment
- Turbulence near absolute zero: Kelvin waves cascade
- Velocity statistics

Experimentally, there are many ways to create a tangle of vortices:

- Heat current (Vinen 1957): Prague, Florida, Maryland
- Ultra sound (Schwarz & Smith 1981)
- Towed grid (Donnelly, Vinen & 1993)
- Ions: Manchester
- Vibrating sphere, grid, fork, wire: Regensburg, Osaka, Lancaster, Prague
- Propellers (Tabeling 1998): Grenoble
- Instabilities following rotation: Helsinki
- Laser spoon, shaking the trap (BEC): Sao Paulo

Vortex line density L (vortex length per unit volume)
Average distance between vortices $\ell \approx L^{-1/2}$

Quantum turbulence

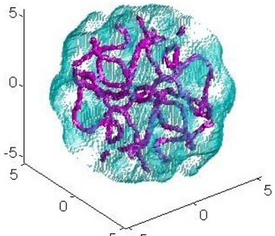
ξ = vortex core, ℓ = average vortex spacing, D = system size

Homogeneous: (^{3}He, ^{4}He)

- constant density,
- $\xi << \ell << D$
- parameters fixed by nature

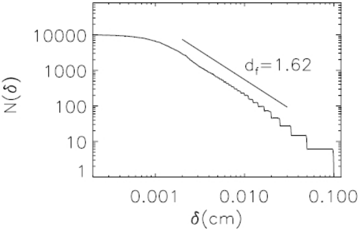
Non-Homogeneous: (BECs)

- non-uniform density,
- $\xi < \ell < D$
- control geometry, dimensions, strength and type of interactions



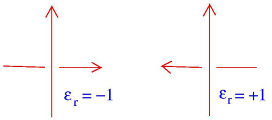
Geometry of quantum turbulence

- Quantum turbulence is fractal:



(Kivotides, CFB & Samuels PRL 87, 155301, 2001)

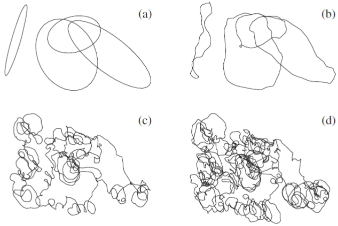
- Quantum turbulence is topologically nontrivial:

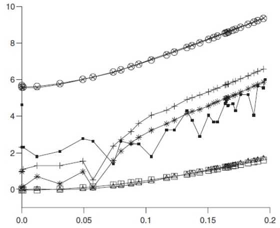


Crossing number $\epsilon_r = \pm 1$

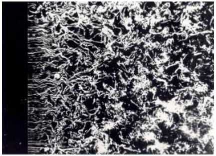
$L_k(\ell_i, \ell_j) = \frac{1}{2} \sum_{r \in \ell_i \cap \ell_j} \epsilon_r$

$L_k = \sum_{i \neq j} |L_k(\ell_i, \ell_j)|$





Big whorls have little whorls
That feed on their velocity,
And little whorls have lesser whorls
And so on to viscosity.
(L.F. Richardson)

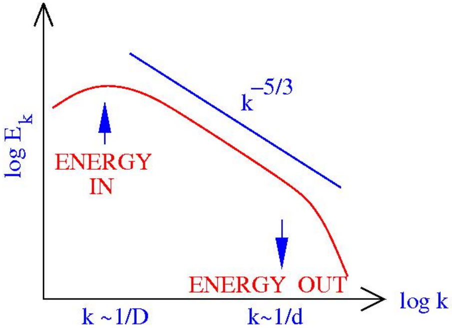


Grid turbulence

Eddies in turbulent water
(Leonardo da Vinci 1452-1519)

Classical turbulence: Kolmogorov scaling

- Fourier trasform the velocity $\mathbf{u}$
- Energy is injected at wavenumber $k \approx 1/D$ and dissipated at $k \approx 1/d$



$$E = \frac{1}{V}\int \frac{1}{2}|\mathbf{u}|^2 dV = \int_0^\infty E_k dk$$

E_k = energy spectrum
$k = |\mathbf{k}|$
ϵ = energy dissipation rate

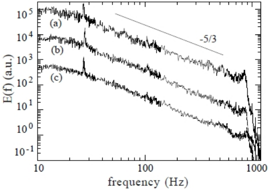
- In the inertial range $1/D << k << 1/d$

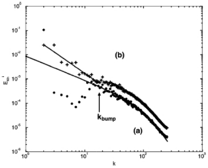
$E_k = C\epsilon^{2/3}k^{-5/3}$ (Kolmogorov 5/3 law)

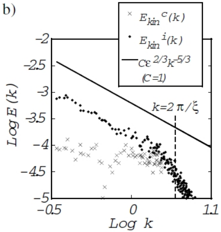
Kolmogorov energy spectrum

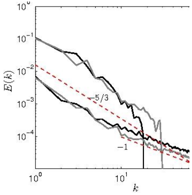
Kolmogorov spectrum $E_k \sim k^{-5/3}$ observed in ^{4}He (Tabeling 1998)

Computed by:
- -Nore & Brachet 1998
- -Kivotides & CFB 2002
- -Tsubota & al 2002, 2005
- -Kerr 2010
- -Baggaley & CFB 2011, etc







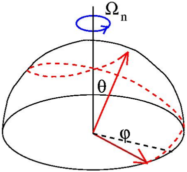


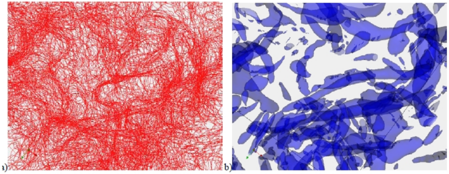
Turbulence at high temperature

- Superfluid vortices align with normal vorticity (Samuels, 1993; CFB, Bauer & al, PoF 1997)

$$\theta(t) = 2\tan^{-1}(e^{-\alpha\Omega t})$$
$$\phi(t) = \alpha'\Omega t$$

(CFB & Hulton, PRL 2002)



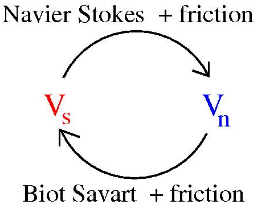


(Morris & Koplik PRL 2008)

- Polarization of vortices at large scale (Vinen & Niemela 2002)

Turbulence at high temperature

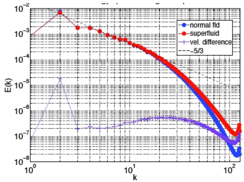
$\mathbf{v}_s$ and $\mathbf{v}_n$ are both turbulent: self-consistent approach is needed



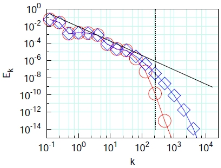
Done for vortex ring (Kivotides & CFB, Science 2000)

Simpler attempts:

- Two-fluid DNS model
- Two-fluid shell model



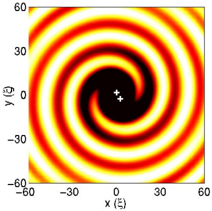
(Roche, Leveque & CFB, EPL 2009)

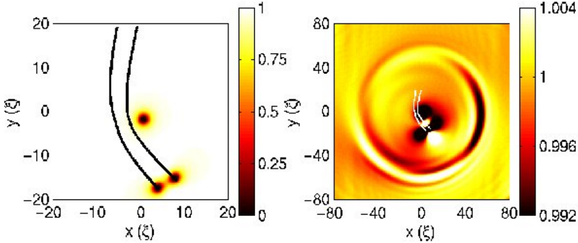


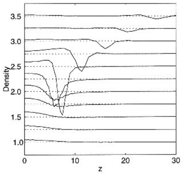
(Wacks & CFB 2011)

Turbulence near absolute zero

- Viscosity=0 but turbulence decays
- Sound = sink of kinetic energy (Vinen 2001)







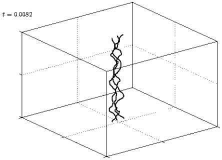
Moving vortices ⇒ sound waves

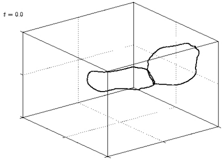
Reconnections ⇒ sound pulses

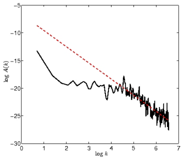
(Leadbeater, CFB, & al
PRL 2001, PRA 2002, JLTP 2005)

Kelvin waves cascade

Cascade to large k: energy radiated away
(Svistunov 1995, Kivotides & CFB 2001, Vinen & Tsubota 2003)



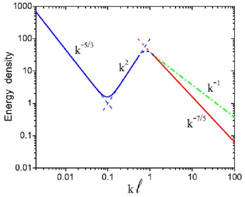




(Baggaley, CFB PRB 2011)

Two cascades:
$k << 1/\ell$: Kolmogorov cascade
$k >> 1/\ell$: Kelvin waves cascade

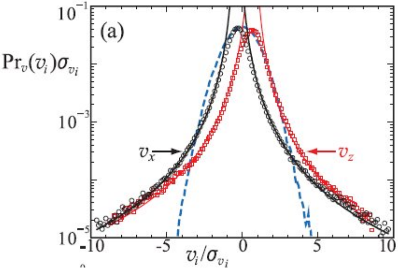
Bottleneck between the two cascades ?
L'vov & Nazarenko PRL 2010, PRB 2010
Kozik & Svistunov JLTP 2009, PRL 2008

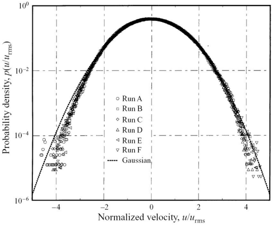


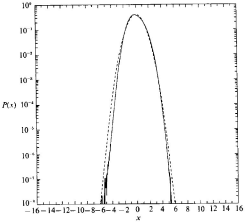
Velocity statistics

In ^{4}He:
$\mathrm{PDF}(v_x) \sim v_x^{-3}$
(Paoletti & al 2008)

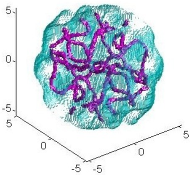
In ordinary fluids:
$\mathrm{PDF}(v_x) = \text{Gaussian}$

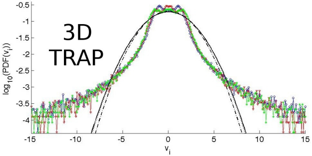


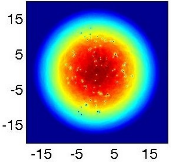


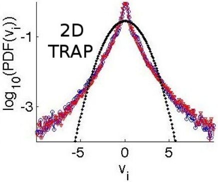


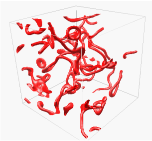
Experiment:
Noullez & al (JFM 1997)
Theory:
Vincent & Meneguzzi
(JFM 1991)

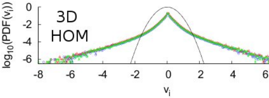






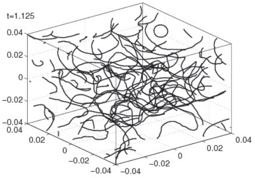


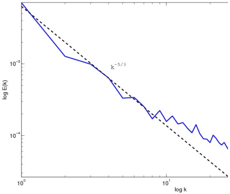


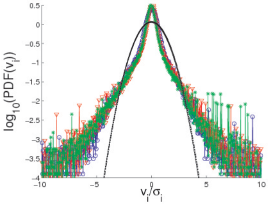


Non-Gaussian statistics arise from singular nature of vortex (White, CFB & al PRL 2010)

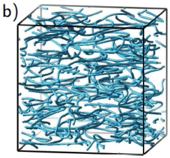
Velocity statistics

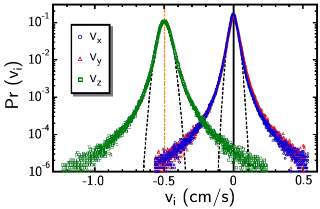






(Baggaley, CFB 2011)





(Adachi, Tsubota 2011)

Conclusions

- Quantum turbulence is simpler than classical turbulence: no viscosity, vorticity confined to filaments (fixed core and circulation)
- Quantum turbulence has many regimes:
  -low temperature ^{3}He and ^{4}He: one turbulent fluid
  -high temperature ^{3}He: one turbulent fluid + friction
  -high temperature ^{4}He: two coupled turbulent fluids
  -atomic BECs
- Similarities between quantum and classical turbulence: (Kolmogorov spectrum, pressure drops, drag crisis)
- There are also dissimilarities: (velocity statistics, pressure spectrum)
- How many quanta are needed to recover classical behaviour ?
- Dynamics vs geometry vs topology