# Representation of a Spiral Phase Plate as a two mode Quantum Phase Operator

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*We introduce a quantum-like representation of a Spiral Phase Plate as a two mode phase operator. The representation is based on the Newton binomial expansion and on properties of rational power of quantum operators.*

## 1. Introduction

Optical vortices (OV) in light beams are tightly bound to phase dislocations (or singularities): due to the continuous spatial nature of a field, the presence of these defects implies the vanishing of the field's amplitude in the singularity. In some types of dislocations the phase circulates around the singularity and creates a vortex [1]. Nye and Berry [2] used the term "phase dislocation" to define the locus of the zero amplitude of a field. Similarly to crystallography, phase dislocations can be classified in edge, screw and mixed screw-edge [3]. Recently the interest in OV increased because of the fact that fields in which they are included show an helical wave-front structure - developing around the screw dislocation line – entailing the presence of an orbital angular momentum (OAM). This is an important feature, which can be exploited in several applications, from optical tweezers to the generation of N–dimensional quantum states (q-nits) for quantum information applications.

A screw wave dislocation can be defined by means of the integer topological charge Q, which represents the phase winding number, and can be found through a circulation integral around the dislocation line:

$$Q = \frac{1}{2\pi}\oint df \tag{1}$$

where $f$ is the phase of the field.

The most used method of generating OAM in an optical beam is by imprinting one or more vortices on its transverse field distribution, in practice by passing a fundamental Gaussian beam through a device that modifies only the phase, such as a Spiral Phase Plate (SPP). When a Gaussian beam is diffracted off such an SPP, the resulting mode can be viewed as a superposition of Laguerre–Gaussian (LG) modes. It is known that a Laguerre-Gaussian beam of paraxial light has a well-defined orbital angular momentum [4–7].

In Ref. [6] Nienhuis and Allen employed operator algebra to describe the Laguerre-Gaussian beam, and noticed that Laguerre-Gaussian modes are laser mode analog of the angular momentum eigenstates of the isotropic 2-d harmonic oscillator. In Ref. [10] Simon and Agarwal presented a phase-space description (the Wigner function) of the LG mode by exploiting the underlying phase-space symmetry. Li-Yun Hu and Hong-yi Fan have shown that LG mode is just the wave function of the common eigenvector of the orbital angular momentum operator and the total photon number operator of 2-d oscillator in the entangled state representation [11, 12], which is based on the first formulation of Einstein-Podolsky-Rosen quantum entanglement (EPR paradox).

In the Quantum Mechanics picture it is useful to search a quantum operational representation of a SPP because, according to Dirac, there should exists a formal correspondence between quantum optics operators and classical optics transformations. Aiello et al [13] proposed a quantum operator representing the action of the SSP on a Gaussian Beam: it is the analog of the quantum phase operator.

The issue of defining a quantum phase operator for the electromagnetic field is a great challenge in quantum mechanics and quantum optics. Quite few proposals were made to define quantum phase operators consistent with quantum mechanics and coinciding with an experimental measurement. The most frequently discussed are those proposed by Susskind and Glogower [14], Pegg and Barnett [15], Paul [16], Noh, Fougires and Mandel [17], and Shapiro and Wagner (SW) [18]. The last ones are based on the simultaneous detection of quadrature components using a heterodyne detection method. In this paper we express the quantum phase operator related to a SPP in a novel way, which allows its application directly to Fock States without using Phase Space representation.

## 2. The Spiral Phase Plate Quantum Operator

A unitary SPP is a transparent dielectric plate with an edge dislocation that can be freely rotated around the plate axis. Let $z$ be the axis of the plate, $\varphi$ the azimuth angle and without loss of generality, let us suppose that the edge dislocation is at $\varphi = 0$. When a light beam with transverse profile $V(z\bar{z})$ crosses such a SPP, it acquires an azimuthal-dependent phase $e^{i\varphi}$, i.e., in terms of wave function :

$$\Psi(z\bar{z}) \Rightarrow e^{iq\varphi} \langle z, \bar{z} \mid \Psi \rangle \qquad (2).$$

Let us introduce the quantum operator

$$\hat{e}^{i\varphi} = \frac{z}{|z|} = \sqrt{\frac{\hat{a}_+^\dagger + \hat{a}_-}{\hat{a}_+ + \hat{a}_-^\dagger}} = \sqrt{\frac{\hat{Y}}{\hat{Y}^\dagger}}, \qquad (3)$$

with the commutator $\left[\hat{Y}, \hat{Y}^\dagger\right] = 0$. The operator associated to a generic SPP, $\hat{e}^{iq\varphi}$, with $q \in \mathbb{R}$, can be expanded in the form

$$\hat{e}^{iq\varphi} = \sum_{k=-\infty}^{\infty} C_{qk} \hat{e}^{ik\varphi}, \qquad (4)$$

where

$$C_{qk} = \frac{1}{2\pi} \int_0^{2\pi} e^{i(q-k)\varphi} d\varphi = \frac{e^{i(q-k)\pi} Sin\left[(q-k)\pi\right]}{(q-k)\pi} \qquad (5)$$

and

$$\hat{e}^{ik\varphi} = \left( \frac{\hat{a}_+^\dagger + \hat{a}_-}{\hat{a}_+ + \hat{a}_-^\dagger} \right)^{\frac{k}{2}} \qquad (6)$$

with $k \in \mathbb{N}$. Following [19], or better, Isaac Newton (1665), the general operator $\hat{e}^{ik\varphi}$ can be formally expanded as

$$\hat{e}^{ik\varphi} = \left(\hat{a}_+^\dagger + \hat{a}_-\right)^{\frac{k}{2}} \left(\hat{a}_+ + \hat{a}_-^\dagger\right)^{-\frac{k}{2}} = \left(\hat{a}_+ + \hat{a}_-^\dagger\right)^{-\frac{k}{2}} \left(\hat{a}_+^\dagger + \hat{a}_-\right)^{\frac{k}{2}}$$
$$= \sum_{m=0}^{\infty} \sum_{h=0}^{\infty} \binom{k/2}{m} \binom{-k/2}{h} \hat{a}_+^{\dagger m} \hat{a}_-^{\frac{k}{2}-m} \hat{a}_+^{h} \hat{a}_-^{\dagger -\frac{k}{2}-h}, \qquad (7)$$

where

$$\binom{k/2}{h} = \frac{(k/2)_h}{h!},$$

and $(r)_h$ is the Pochhammer symbol. Moreover, to provide the application of the high non-linear operator directly to Fock States, without using Phase Space representations, we introduce the following selection rules:

$$\hat{a}^{\alpha}\hat{a}^{\dagger\beta}\left|n\right\rangle = \frac{\Gamma(1+n+\beta)}{\sqrt{n!\Gamma(1+n+\beta-\alpha)}}\left|n+\beta-\alpha\right\rangle, \tag{8}$$

and

$$\hat{a}^{\dagger\beta}\hat{a}^{\alpha}\left|n\right\rangle = \frac{\sqrt{n!\Gamma(1+n+\beta-\alpha)}}{\Gamma(1+n-\alpha)}\left|n+\beta-\alpha\right\rangle, \tag{9}$$

where $\alpha,\beta\in\mathbb{R}$, with the constrain $n+\beta\in\mathbb{N}^{+}$, $n+\beta-\alpha\in\mathbb{N}^{+}$. As an example, we apply the quantum operator $\hat{e}^{ik\varphi}$, corresponding to a SPP with topological charge $q=2.5$ to the displaced vacuum state of a 2-d Harmonic Oscillator, namely $\hat{D}(\alpha_{+})\hat{D}(\alpha_{-})\left|0,0\right\rangle = e^{\alpha\hat{a}_{+}^{\dagger}}e^{\bar{\alpha}\hat{a}_{-}^{\dagger}}\left|0,0\right\rangle$. That a state is the analogue of a Displaced Gaussian beam in paraxial approximation with $r_0^2=|\alpha|^2$. In terms of wavefunction we obtain

$$\Psi(r,\varphi)=\sum_{p=0}^{\infty}\sum_{l=-\infty}^{\infty}e^{-\frac{r_0^2}{2w_0^2}}\left(\frac{r_0^2}{2w_0^2}\right)^{p+\frac{|l|}{2}}\frac{(-1)^p e^{-il\varphi_0}}{\sqrt{p!(p+|l|)!}}\sum_{h=0}^{\infty}\sum_{k=-\infty}^{\infty}C_{plhk}LG_h^k(r,\varphi)e^{ik\varphi}, \tag{10}$$

with

$$C_{plhk}=e^{i(q+l-k)\pi}\sqrt{\frac{p!h!}{(p+|l|)!(h+|k|)!}}\frac{Sin\left[(q+l-k)\pi\right]}{\pi(q+l-k)}I_{p,h}(l,k), \tag{11}$$

and

$$I_{p,h}(l,k)=(-1)^{p+h}\Gamma\left(\frac{|l|+|k|}{2}+1\right)\sum_{r=0}^{Min\{p,h\}}\binom{\frac{|k|-|l|}{2}}{p-r}\binom{\frac{|l|-|k|}{2}}{h-r}\binom{\frac{|l|+|k|}{2}+r}{r}. \tag{12}$$

$LG_h^k$ are the generalized Laguerre-Gaussian modes.

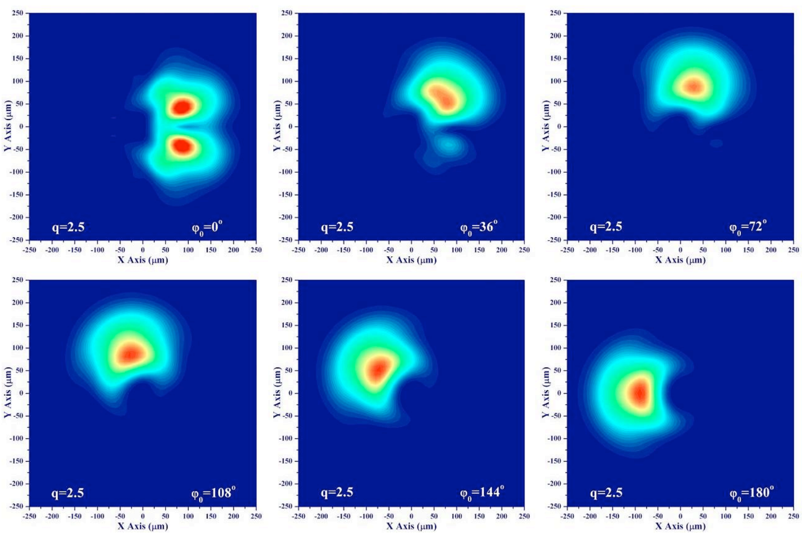


**Figure 1:** **Contour plot of the diffracted field generated by a Displaced Gaussian Beam impinging on a Spiral Phase Plate at different angles in respect to the phase dislocation. The SSP is characterized by a topological charge q=2.5. and, in the particular case, the displacement is equal to the waist, $w_0$=100μm.**

## 5. Conclusions

We identified the quantum operator, representing a SPP acting on an electromagnetic field, as the two mode phase operator and we introduced a novel representation based on the Newton binomial expansion and on the properties of rational power of lowering and raising operators of electromagnetic field. This method provides to handling nonlinear operators without using auxiliary picture such as P-representation in phase-space. Moreover, by exploiting the one-to-one correspondence between the state of the two-dimensional harmonic oscillator and a monochromatic paraxial beam of light, we considered the action of a Spiral Phase Plate on a displaced gaussian mode (a displaced vacuum state in quantum picture).

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