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Grand Unification and Proton Decay - I

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0 Reminder

This is written for a series of 4 lectures at ICTP Summer School 2011. The choice of topics and the references are biased. This is not a review on the subject or a correct historical overview. The quotations I mention are incomplete and chosen merely for further reading.

There are some good books and reviews on the market. Among others I would mention [1, 2, 3, 4].

1 Introduction to grand unification

The (MS)SM has 3 gauge interactions described by the corresponding carriers

$$g_\mu^a \; (a=1\ldots 8) \quad , \quad W_\mu^i \; (i=1\ldots 3) \quad , \quad B_\mu \tag{1}$$

and 5 different matter representations (with a total of 15 Weyl fermions) for each generation

$$Q \, , \, L \, , \, u^c \, , \, d^c \, , \, e^c \tag{2}$$

It has also three types of $N_g \times N_g$ (N_g is the number of generations, at the moment believed to be 3) Yukawa matrices

$$\mathcal{L}_Y = u^c Y_U Q H + d^c Y_D Q H^* + e^c Y_E L H^* + h.c. \tag{3}$$

This notation is highly symbolic. It means actually

$$\mathcal{L}_Y = u^{cT}_{\alpha k} i\sigma_2 \left(Y_U\right)_{kl} Q_l^{\alpha a} \epsilon_{ab} H^b + d^{cT}_{\alpha k} i\sigma_2 \left(Y_D\right)_{kl} Q_l^{\alpha a} H_a^* + e^{cT}_k i\sigma_2 \left(Y_E\right)_{kl} L_l^a H_a^* + h.c. \tag{4}$$

where we denoted by $a, b = 1, 2$ the $\mathrm{SU}(2)_L$ indices, by $\alpha, \beta = 1 \ldots 3$ the $\mathrm{SU}(3)_C$ indices, by $k, l = 1, \ldots N_g$ the generation indices, and where $i\sigma_2$ provides Lorentz invariants between two spinors.

The SM at the renormalizable level predicts a massless neutrino (there is no right-handed neutrino ν^c), while a massive neutrino can be incorporated in a R-parity violating MSSM (we will not consider this option in these lectures).

Finally, there is no real explanation of the quantization of the electric charge. Although anomaly cancellation constraints do predict the electric charge quantization in the SM, this does not have any further experimental consequences. Also, any addition to it could involve non quantized charges.

The idea of grand unification theories (GUT) is to reduce all the gauge interactions to one single gauge group and all the fermionic multiplets into one or two different representations for each generation of matter. Of course our SM gauge group should then be a subgroup of the grand unified gauge group, and the SM fermions included in the GUT matter representations. The electric charge operator is in a GUT made out of a linear combination of non-abelian gauge algebra generators, and its eigenvalues are obviously quantized. Finally, GUTs can be or not theories of neutrino mass. In some cases - for example in SU(5) - one can adjust the theory to give a nonzero neutrino mass (similarly as adding right-handed neutrinos in the SM), while some other GUTs - typically SO(10) - can be more predictive, and connect it to charged fermion Yukawas.

1.1 The renormalization group equations (RGE)

But what does unification really mean? That we put for example all SM gauge fields together in a bigger adjoint representation of a simple group is clear, but we know that the gauge couplings of the three SM gauge interactions are numerically different. So in which sense they can unify? Here it is crucial the notion of running coupling constants. We know that the gauge (and other) couplings run with energy. So what we have to do, is to let them run and check if they meet all three together [5]. And if they do, the scale at which this happens will be the scale of (the spontaneous breaking of) grand unification. Fortunately this is easy to do at the 1-loop level, all we need is to solve the renormalization group equations (RGEs):

$$\frac{dg_i}{d\log\mu} = -\frac{b_i}{(4\pi)^2} g_i^3 \qquad i = 1, 2, 3 \tag{5}$$

The 1-loop beta coefficient b_i can be straightforwardly calculated via (G,F,B stay for gauge bosons, fermions, bosons)

$$b = \frac{11}{3} T_G - \frac{2}{3} T_F - \frac{1}{3} T_B \; . \tag{6}$$

where at the scale μ one must take into account all the particles with mass lower than μ. The Dynkin index

$$T_R \delta^{ab} = Tr\left(T_R^a T_R^b\right) \tag{7}$$

depends on the choice of the gauge group and on the representation R involved. The indices a, b run over the generators of the group ($N^2 - 1$ in SU(N)). The normalization usually chosen is $T = 1/2$ for the fundamental representation (quarks and leptons in SM). Then one has in the SU(N) group for the adjoint $T = N$. To remember also that in SU(2) the generators in the fundamental are the Pauli matrices $T_a^{ij} = \tau_a^{ij}/2$, while in the adjoint representation are the Levi-Civita antisymmetric tensor $T_a^{ij} = -i\epsilon_{aij}$.

For supersymmetric theories we know that for each fermion (boson) there is a boson (fermion) in the same group representation, so (6) can be written more compactly as

$$b = 3T_G - T \; . \tag{8}$$

The beta coefficients in the SM are $b_i = (-41/10, 19/6, 7)$ (positive coefficients here mean asymptotic freedom). One knows the experimental values of g_i at M_Z and can evolve them towards larger scales μ using (5). It is now easy to check that there is no unification of couplings in the SM. Two loops will not help so the only possibility for unification is to add new particles in order to change the beta coefficients for energies above their mass. We will see in the next sections two such examples.

2 The Georgi-Glashow SU(5) model

The Georgi-Glashow SU(5) grand unified model [6] includes the SM three generations of fermions (the number of generations in GUTs are unfortunately not predicted, but put by hand, as in the SM) in the 10_F and 5^c_F representations

$$10_F = \begin{pmatrix} 0 & u^c_3 & -u^c_2 & u_1 & d_1 \\ -u^c_3 & 0 & u^c_1 & u_2 & d_2 \\ u^c_2 & -u^c_1 & 0 & u_3 & d_3 \\ -u_1 & -u_2 & -u_3 & 0 & e^c \\ -d_1 & -d_2 & -d_3 & -e^c & 0 \end{pmatrix} \quad , \quad 5^c_F = \begin{pmatrix} d^c_1 \\ d^c_2 \\ d^c_3 \\ e \\ -\nu \end{pmatrix} \tag{9}$$

The Higgs sector is made of an adjoint 24_H, which gets a vacuum expectation value (vev) to spontaneously break SU(5)→SU(3)$_C$×SU(2)$_W$×U(1)$_Y$:

$$\langle 24_H \rangle = \frac{v}{\sqrt{30}} \begin{pmatrix} 2 & 0 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 & 0 \\ 0 & 0 & 2 & 0 & 0 \\ 0 & 0 & 0 & -3 & 0 \\ 0 & 0 & 0 & 0 & -3 \end{pmatrix} \tag{10}$$

and of one fundamental representation, which contains also the SM Higgs doublet $H = (H^+, H^0)^T$:

$$5_H = (H^C_1, H^C_2, H^C_3, H^+, H^0)^T \tag{11}$$

Now we have the whole particle content. But how do we break SU(5)? For it we need the Lagrangian for the adjoint.

2.1 The Higgs sector

The adjoint of SU(5) is Hermitian and transforms as

$$\Sigma \to U \Sigma U^\dagger \tag{12}$$

Keeping in mind that it is traceless, the only invariants we can use up to fourth order are

$$Tr\Sigma^2 \qquad Tr\Sigma^3 \qquad Tr\Sigma^4 \tag{13}$$

The most general potential (with an additional Z_2 symmetry $\Sigma \to -\Sigma$ for simplicity) is thus

$$V = -\frac{\mu^2}{2} Tr\Sigma^2 + \frac{\lambda}{4} Tr\Sigma^4 + \frac{\lambda'}{4}\left(Tr\Sigma^2\right)^2 \tag{14}$$

The tracelessness is taken into account adding to the potential a Lagrange multiplier

$$\xi Tr\Sigma \tag{15}$$

The equations of motion are simply

$$\frac{\partial V}{\partial \Sigma_{ji}} = -\mu^2 \Sigma_{ij} + \lambda\left(\Sigma^3\right)_{ij} + \lambda' Tr\Sigma^2 \Sigma_{ij} + \xi\delta_{ij} = 0 \tag{16}$$

The Lagrange multiplier can be calculated by requiring the trace of this equation (and of Σ as well) to vanish:

$$\delta_{ij}\frac{\partial V}{\partial \Sigma_{ji}} = \lambda Tr\Sigma^3 + 5\xi = 0 \tag{17}$$

to give

$$-\mu^2 \Sigma_{ij} + \lambda\left(\Sigma^3\right)_{ij} + \lambda' Tr\Sigma^2 \Sigma_{ij} - \frac{\lambda}{5} Tr\Sigma^3 \delta_{ij} = 0 \tag{18}$$

The Hermitian and traceless adjoint of SU(5) has 24 (real) degrees of freedom. Due to the gauge freedom, we can however rotate away the non-diagonal elements, since any Hermitian matrix Σ can be put in a diagonal form Σ^d with a proper choice of a unitary matrix U:

$$U\Sigma U^\dagger = \Sigma^d \tag{19}$$

This is nothing else than a gauge transformation that we are free to choose at will. From now on we will work with a diagonal

$$\Sigma_{ij} = \sigma_i \delta_{ij} \tag{20}$$

Eq. (18) becomes

$$\sigma_i^3 - \left(\frac{\mu^2}{\lambda} - \frac{\lambda'}{\lambda} Tr\Sigma^2\right)\sigma_i - \frac{1}{5} Tr\Sigma^3 = 0 \tag{21}$$

For any fixed choice of the SU(5) invariants $Tr\Sigma^2$, $Tr\Sigma^3$ this equation is third order and we can thus have at most three different solution.

Accounting for the tracelessness and barring trivial renaming of indices we can think of the following possibilities:

$$diag(a, a, a, b, -3a - b) \tag{22}$$
$$diag(a, a, b, b, -2a - 2b) \tag{23}$$
$$diag(a, a, a, -3a/2, -3a/2) \tag{24}$$
$$diag(a, a, a, a, -4a) \tag{25}$$
$$diag(0, 0, 0, 0, 0) \tag{26}$$

The last three cases are actually special cases of the first two, which are the only independent ones. Furthermore, the absence of the quadratic term of σ_i^2 in eq. (21) tells us that the sum of the three solutions is zero. This means that the above reduce to (we skip here the vev)

$$diag(1, -1, 0, 0, 0) \tag{27}$$
$$diag(1, 1, -1, -1, 0) \tag{28}$$
$$diag(2, 2, 2, -3, -3) \tag{29}$$
$$diag(1, 1, 1, 1, -4) \tag{30}$$
$$diag(0, 0, 0, 0, 0) \tag{31}$$

The only case from the above that is not obviously unrealistic is case (29) which breaks SU(5) into the SM SU(3)×SU(2)×U(1). Let us now find the solution explicitly. Our ansatz will be (the normalization is only for convenience)

$$\langle \Sigma \rangle = \frac{v}{\sqrt{30}} diag(2, 2, 2, -3, -3) \tag{32}$$

The potential becomes

$$V(v) = -\frac{1}{2}\mu^2 v^2 + \left(\frac{7\lambda}{30} + \lambda'\right)\frac{v^4}{4} \tag{33}$$

and the equation of motion

$$\frac{\partial V}{\partial v} = v\left[-\mu^2 + \left(\frac{7\lambda}{30} + \lambda'\right)v^2\right] = 0 \tag{34}$$

The solution

$$v^2 = \frac{30\mu^2}{7\lambda + 30\lambda'} \tag{35}$$

is a minimum only if

$$7\lambda + 30\lambda' > 0 \quad , \quad \mu^2 > 0 \tag{36}$$

Let us now calculate the spectrum in this sector. In the breaking of SU(5), the adjoint gets decomposed into pieces with SM quantum numbers. We get

$$24 \to O(8,1,0) + T(1,3,0) + S(1,1,0) + X(3,2,-5/6) + \bar{X}(\bar{3},2,5/6) \tag{37}$$

The adjoint can be imagined in blocks

$$\Sigma = \begin{pmatrix} 3\times 3 & 3\times 2 \\ 2\times 3 & 2\times 2 \end{pmatrix} \tag{38}$$

so the above fields live schematically in

$$\Sigma = \begin{pmatrix} O & X \\ \bar{X} & T \end{pmatrix} + \left(1 + \frac{S}{v}\right)\langle\Sigma\rangle \tag{39}$$

We have to expand this matrix. In doing that we can take just one element of each representation, the other elements need to have the same mass since the SM symmetry is still preserved. Using a common normalization for all the fields we get

$$\Sigma = \begin{pmatrix} 2\frac{v+S}{\sqrt{30}} + \frac{O}{\sqrt{2}} & 0 & 0 & X & 0 \\ 0 & 2\frac{v+S}{\sqrt{30}} - \frac{O}{\sqrt{2}} & 0 & 0 & 0 \\ 0 & 0 & 2\frac{v+S}{\sqrt{30}} & 0 & 0 \\ \bar{X} & 0 & 0 & -3\frac{v+S}{\sqrt{30}} + \frac{T}{\sqrt{2}} & 0 \\ 0 & 0 & 0 & 0 & -3\frac{v+S}{\sqrt{30}} - \frac{T}{\sqrt{2}} \end{pmatrix} \tag{40}$$

We also do not need to expand all these together. It is enough to do separately for each of the fields O, T and S, while X and $\bar{X}$ have a common mass. We get

$$m_O^2 = \frac{\lambda}{6}v^2 \tag{41}$$

$$m_T^2 = \frac{2\lambda}{3}v^2 \tag{42}$$

$$m_{X\bar{X}}^2 = 0 \tag{43}$$

$$m_S^2 = 2\mu^2 \tag{44}$$

Few comments:

- to have a stable solution λ must be non-negative.
- for $\lambda = 0$ we have 23 massless fields. This is a consequence of the Nambu-Goldstone theorem: for this coupling the potential has more symmetry, SO(24). When this is broken by the fundamental of SO(24) (i.e. what we called the adjoint of SU(5)) to SO(23), we get

$$\frac{24 \times 23}{2} - \frac{23 \times 22}{2} = 23 \tag{45}$$

  massless particles. The symmetry is not what we decide, but what the potential tells us!
- for $\lambda > 0$, the usual Goldstones of the SU(5) breaking are X, $\bar{X}$, i.e $24 - 12 = 12$.

2.2 The Yukawa sector

On top of the usual fermion representation (ϵ_3 is the 3-D Levi-Civita tensor and $\epsilon_2 = i\tau_2$ is the corresponding 2-D one)

$$10_F = \begin{pmatrix} \epsilon_3 u^c & Q \\ -Q^T & \epsilon_2 e^c \end{pmatrix} \qquad 5_F^c = \begin{pmatrix} d^c \\ \epsilon_2 L \end{pmatrix} \tag{46}$$

we introduce the Higgs representation

$$5_H = \begin{pmatrix} T \\ H \end{pmatrix} \tag{47}$$

There is an easy way to find invariants in SU(5): fundamental representations (and their products) are the ones with indices up

$$F^{i_1 i_2 \ldots i_n} \tag{48}$$

and transform as

$$F^{i_1 i_2 \ldots i_n} \to U^{i_1}{}_{j_1} U^{i_2}{}_{j_2} \ldots U^{i_n}{}_{j_n} F^{j_1 j_2 \ldots j_n} \tag{49}$$

On the other way the antifundamentals

$$F_{i_1, i_2 \ldots i_n} \tag{50}$$

transform under SU(5) as

$$F_{i_1 i_2 \ldots i_n} \to F_{j_1 j_2 \ldots j_n} \left(U^\dagger\right)^{j_1}{}_{i_1} \left(U^\dagger\right)^{j_2}{}_{i_2} \ldots \left(U^\dagger\right)^{j_n}{}_{i_n} \tag{51}$$

There can be also mixed representations

$$F^{i_1 i_2 \ldots i_n}_{j_1, j_2 \ldots j_m} \tag{52}$$

that go like

$$F^{i_1 i_2 \ldots i_n}_{j_1, j_2 \ldots j_n} \to U^{i_1}{}_{k_1} U^{i_2}{}_{k_2} \ldots U^{i_n}{}_{j_n} F^{k_1 k_2 \ldots k_n}_{l_1, l_2 \ldots l_m} \left(U^\dagger\right)^{l_1}{}_{j_1} \left(U^\dagger\right)^{l_2}{}_{j_2} \ldots \left(U^\dagger\right)^{l_m}{}_{j_m} \tag{53}$$

Invariants are found as products of these fields so that upper indices match with lower ones (an implicit summation over two equal - one upper one lower - indices is assumed, as in general relativity), for example

$$M_{abc} N^{ab} K^c \tag{54}$$

On top of that one can use also the (5 index) Levi-Civita tensor

$$\epsilon_{i_1 i_2 \ldots i_n} \quad \text{or} \quad \epsilon^{i_1 i_2 \ldots i_n} \quad (\text{or mixed}) \tag{55}$$

Using this simple method, we find out that in our case of a single Higgs in the fundamental representation there are two SU(5) (and Lorentz) invariants for the renormalizable Yukawas (two fermions, one Higgs)

$$\mathcal{L}_Y = 5^c_F Y_5 10_F 5^*_H + \frac{1}{8} \epsilon_5 10_F Y_{10} 10_F 5_H \tag{56}$$

Y_5 and Y_{10} are matrices in generation space. The factor 1/8 is taken for convenience and is of course optional, since it just redefines the Yukawa matrix Y_{10}.

Here we are interested in the Yukawa terms with the light (SM) Higgs, so the first term in (56) can be rewritten as

$$\begin{aligned} 5^c_F Y_5 10_F 5^*_H &= \begin{pmatrix} d^c & -L\epsilon_2 \end{pmatrix} Y_5 \begin{pmatrix} \epsilon_3 u^c & Q \\ -Q^T & \epsilon_2 e^c \end{pmatrix} \begin{pmatrix} T^* \\ H^* \end{pmatrix} \\ &\to d^c Y_5 Q H^* + L Y_5 e^c H^* \end{aligned} \tag{57}$$

The two terms are essentially similar, except for the fact that the SU(2) doublet and singlet fields are interchanged. Rewriting the second term as

$$L Y_5 e^c H^* = e^c Y_5^T L H^* \tag{58}$$

it follows

$$Y_D = Y_E^T \tag{59}$$

i.e. the Yukawa (mass) matrix for down quarks is just the transpose of the Yukawa (mass) matrix of the charged leptons. This surprising result is just a consequence of SU(5) constraints. Unfortunately this simple relation is not satisfied in nature (it is not obvious to see it though, since these relations are valid at the GUT scale and one needs to run everything down by RGE to the low scale where these numbers are measured), but it is remarkable in any case.

The second term in (56) is a bit more tricky to calculate, since it contains the 5-index Levi-Civita tensor, but it is already clear from the structure that Y_{10} is symmetric. Let us see what it describes in the low energy theory.

$$\epsilon_5 10_F Y_{10} 10_F 5_H = \epsilon_{ijklm} \left(10_F\right)^{ij} Y_{10} \left(10_F\right)^{kl} \left(5_H\right)^m \tag{60}$$

Let us divide the indices

$$SU(5): \quad i, j, k, l, m = 1 \ldots 5 \tag{61}$$

into two groups as usual:

$$SU(3): \quad \alpha, \beta, \gamma = 1 \ldots 3 \qquad SU(2): \quad a, b = 1 \ldots 2 \tag{62}$$

We are interested in 5_H with a SU(2) index, and let us put the other possible SU(2) index into the first or second 10_F. (60) can be expanded then to

$$\begin{aligned}
(60) \quad &\rightarrow \quad 2\epsilon_{\alpha\beta\gamma ab}\left(10_F\right)^{\alpha\beta} Y_{10}\left(10_F\right)^{\gamma a}\left(5_H\right)^b \\
&+ \quad 2\epsilon_{\gamma a\alpha\beta b}\left(10_F\right)^{\gamma a} Y_{10}\left(10_F\right)^{\alpha\beta}\left(5_H\right)^b \\
&= \quad 2\epsilon_{\alpha\beta\gamma ab}\left(10_F\right)^{\alpha\beta}\left(Y_{10}+Y_{10}^T\right)\left(10_F\right)^{\gamma a}\left(5_H\right)^b
\end{aligned} \tag{63}$$

The factor of 2 comes from the two possibilities, $\left(10_F\right)^{\gamma a}$ and $\left(10_F\right)^{a\gamma}$. Obviously

$$\epsilon_{\alpha\beta\gamma ab} = \epsilon_{\alpha\beta\gamma}\epsilon_{ab} \tag{64}$$

so we get further

$$\begin{aligned}
&2\epsilon_{\alpha\beta\gamma}\epsilon^{\alpha\beta\delta}u^c_\delta\left(Y_{10}+Y_{10}^T\right)Q^{\gamma a}\epsilon_{ab}H^b \\
= \quad &4u^c_\delta\left(Y_{10}+Y_{10}^T\right)Q^{\delta a}\epsilon_{ab}H^b
\end{aligned}$$

Finally we have (again in a compact notation)

$$\frac{1}{8}\epsilon_5 10_F Y_{10} 10_F 5_H = \frac{1}{2}u^c\left(Y_{10}+Y_{10}^T\right)QH \tag{65}$$

and thus the Yukawa (mass) matrix for the up quarks is symmetric:

$$Y_U = Y_U^T \tag{66}$$

Let's summarize the relevant lesson we learned for the SM Yukawa couplings: the charged lepton mass matrix is proportional to the down quark mass matrix at the GUT scale, and the neutrinos are massless. How do we cure these shortcomings?

The first part, a correct description of the charged lepton and down quark masses, is relatively easy. One has essentially two choices in SU(5): either add a new Higgs representation, in this case for example a $45_{\alpha\beta}{}^{\gamma}$, which contains also the standard model Higgs doublet, or allow non-renormalizable operators using the same minimal field content. To show the point we will consider now the second option. Let us add to (56) the following terms

$$
\begin{aligned}
\mathcal{L}_Y &= 5^c_F Y_5^{(1)} 10_F \left(\frac{\Sigma}{\Lambda} 5_H\right)^* + 5^c_F Y_5^{(2)} \left(\frac{\Sigma}{\Lambda} 10_F\right) 5^*_H \\
&+ \frac{1}{8} \epsilon_5 10_F Y_{10}^{(1)} 10_F \left(\frac{\Sigma}{\Lambda} 5_H\right) + \frac{1}{8} \epsilon_5 10_F Y_{10}^{(2)} \left(\frac{\Sigma}{\Lambda} 10_F\right) 5_H
\end{aligned} \tag{67}
$$

where Λ is a UV cutoff.

Defining the SM Yukawa couplings through (3) we arrive at

$$
\begin{aligned}
Y_U &= \frac{1}{2}\left(Y_{10} + Y_{10}^T\right) - \frac{3}{2}\frac{v}{\sqrt{30}\Lambda}\left(Y_{10}^{(1)} + Y_{10}^{(1)T}\right) - \frac{1}{4}\frac{v}{\sqrt{30}\Lambda}\left(Y_{10}^{(2)} - 2Y_{10}^{(2)T}\right) \\
Y_D &= Y_5 - 3\frac{v}{\sqrt{30}\Lambda} Y_5^{(1)} + 2\frac{v}{\sqrt{30}\Lambda} Y_5^{(2)} \\
Y_E^T &= Y_5 - 3\frac{v}{\sqrt{30}\Lambda} Y_5^{(1)} - 3\frac{v}{\sqrt{30}\Lambda} Y_5^{(2)}
\end{aligned} \tag{68}
$$

We have now enough freedom to fit the charged lepton and down quark masses. Of course, at the expense of predictiveness. Remember that all these relations are valid at the GUT scale.

2.3 The gauge boson mass

We will write the adjont's vev in a shorthand notation as

$$
\langle\Sigma\rangle = \frac{v}{\sqrt{30}} \begin{pmatrix} 2_{3\times3} & 0_{3\times2} \\ 0_{2\times3} & -3_{2\times2} \end{pmatrix} \to \frac{v}{\sqrt{30}} \begin{pmatrix} 2 & 0 \\ 0 & -3 \end{pmatrix} \tag{69}
$$

with

$$
A_\mu = A_\mu^a T^a \tag{70}
$$

summed over all 24 generators of SU(5). Gauge boson's mass can be calculated as in any Higgs mechanism through the kinetic terms of the Higgs in question, i.e. through the covariant derivative

$$
D_\mu \Sigma = \partial_\mu \Sigma + i\frac{g}{2}\left[A_\mu, \Sigma\right] \tag{71}
$$

That this is the right combination for the covariant derivative can be seen from

$$\Sigma \to U\Sigma U^\dagger \quad , \quad A_\mu \to U A_\mu U^\dagger + \frac{2}{ig} U \partial_\mu U^\dagger \tag{72}$$

The Hermitian conjugate of (69) is

$$(D^\mu \Sigma)^\dagger = \partial^\mu \Sigma + i\frac{g}{2}\left[A^\mu, \Sigma\right] \tag{73}$$

where we took into account that the matrices A_μ and Σ are Hermitian.

We are interested into the masses of the $X - \bar{X}$, all other vanishing, so in the compact notation introduced before X_μ ($\bar{X}_\mu$) is a 3×2 (2×3) matrix:

$$A_\mu = \begin{pmatrix} 0 & X_\mu \\ \bar{X}_\mu & 0 \end{pmatrix} \tag{74}$$

The commutator becomes

$$[A_\mu, \Sigma] = -\frac{5v}{\sqrt{30}} \begin{pmatrix} 0 & X_\mu \\ \bar{X}_\mu & 0 \end{pmatrix} \tag{75}$$

and the mass term in the Lagrangian is just

$$\begin{aligned} \mathcal{L} = Tr\left[(D^\mu\Sigma)^\dagger D_\mu \Sigma\right] \quad &\to \quad -\frac{g^2}{4}\left(\frac{25v^2}{30}\right) Tr \begin{pmatrix} 0 & X_\mu \\ \bar{X}_\mu & 0 \end{pmatrix} \begin{pmatrix} 0 & X^\mu \\ \bar{X}^\mu & 0 \end{pmatrix} \\ &\to \quad \frac{5g^2v^2}{12} X_\mu \bar{X}^\mu \end{aligned} \tag{76}$$

so that the mass is

$$M_X^2 = \frac{5}{12} g^2 v^2 \tag{77}$$

We will call it also M_{GUT}, i.e. the scale at which the three SM gauge couplings get unified.

2.4 The violation of baryon and lepton numbers

Baryon and lepton number conservation are peculiar to the SM: it is simply impossible in the SM to write down a baryon and/or lepton number violating term at tree level. We say that baryon and lepton numbers are an accidental symmetries of the SM, they do not need to be imposed, but they follow from the field content and the requirement of gauge and Lorentz invariance. Thus, apart from anomalies (that give however a far too small contribution, proportional to $\exp(-4\pi/\alpha_2) \approx 10^{-150}$) baryon and lepton numbers remain conserved, and thus loops cannot generate a nonzero nucleon decay rate or neutrino mass. Of course higher dimensional operators can violate baryon and lepton numbers, but the SM itself cannot tell at which scale this has to happen. In short, the SM is not a theory of baryon and lepton number violation.

In GUTs different SM representations lie in same multiplets so baryon (and lepton) number is not conserved, not even at tree level, so there is nothing that prevents protons from decaying. Since in the limit of the GUT scale to infinity proton must become stable, it is clear that the decay lifetime must be proportional to some positive power of M_{GUT}. To get it a bit more precisely, remember that the heavy GUT gauge bosons have mass M_{GUT}, and that their interaction violates B and L. So a B and L violating amplitude between four fermions gets a contribution from the exchange of such a gauge boson. The amplitude is (similar as the W exchange in muon decay, where the amplitude goes as $1/M_W^2$)

$$A(qq \to \bar{q}\bar{l}) \approx \frac{1}{M_{GUT}^2} \tag{78}$$

and thus the decay rate

$$\Gamma(p = qqq \to q\bar{q}\bar{l} = \pi^0 e^+) \approx \frac{m_p^5}{M_{GUT}^4} \tag{79}$$

One can thus estimate that the experimental lifetime $\tau_p = 1/\Gamma_p$ of 10^{34} yrs or so constrains

$$M_{GUT} \gtrsim 10^{16}\,\mathrm{GeV} \tag{80}$$

which, as we will see, is relatively close (in logarithmic scale) to the unification point.

Now it is time to derive these operators more precisely in the Georgi-Glashow model. The main contribution comes from gauge interaction, i.e. from the kinetic term of the fermions. Using the usual transformation rule

$$\hat{T}^a 5^c = -T^{aT} 5^c \tag{81}$$

where T^a on the right are the SU(5) generators in fundamental representation (the Gell-Mann matrices), while 10_F and 5^c_F are written in the matrix form (46), we get

$$\begin{aligned} i\overline{5^c_F}\gamma^\mu D_\mu 5^c_F &\to i\left(\,\overline{d^c} \quad \overline{\epsilon_2 L}\,\right)\gamma^\mu \frac{g}{2}(-i)\begin{pmatrix} 0 & X_\mu \\ \bar{X}_\mu & 0 \end{pmatrix}^T \begin{pmatrix} d^c \\ \epsilon_2 L \end{pmatrix} \\ &= \frac{g}{2} Tr\left(\overline{d^c}\gamma^\mu \bar{X}^T_\mu \epsilon_2 L + \overline{\epsilon_2 L}\gamma^\mu X^T_\mu d^c\right) \\ &= \frac{g}{2}\left[\left(\overline{d^c}\right)^\beta \gamma^\mu \left(\bar{X}_\mu\right)^b{}_\beta \epsilon_{ba} L^a + \left(\bar{L}\right)_a \epsilon^{ab}\gamma^\mu (X_\mu)^\beta{}_b (d^c)_\beta\right] \end{aligned} \tag{82}$$

For the two index antisymmetric 10 the transformation rule is

$$\hat{T}^a 10 = T^a 10 - 10^T T^{aT} \tag{83}$$

Due to antisymmetry of 10 this gives

$$\frac{1}{2} Tr\left(\overline{10}\hat{T}^a 10\right) = Tr\left(\overline{10} T^a 10\right) \tag{84}$$

i.e. it is enough to transform just the first index in 10. We continue thus

$$\begin{aligned} &\frac{i}{2} Tr\left[\overline{10_F}\gamma^\mu D_\mu 10_F\right] \to \\ &iTr\left[\begin{pmatrix} \overline{\epsilon_3 u^c} & -\bar{Q}^T \\ \bar{Q} & \overline{\epsilon_2 e^c} \end{pmatrix}\gamma^\mu \frac{g}{2} i \begin{pmatrix} 0 & X_\mu \\ \bar{X}_\mu & 0 \end{pmatrix}\begin{pmatrix} \epsilon_3 u^c & Q \\ -Q^T & \epsilon_2 e^c \end{pmatrix}\right] \\ = \;& \frac{g}{2} Tr\left[\overline{\epsilon_3 u^c}\gamma^\mu X_\mu Q^T + \bar{Q}^T \gamma^\mu \bar{X}_\mu \epsilon_3 u^c - \bar{Q}\gamma^\mu X_\mu \epsilon_2 e^c - \overline{\epsilon_2 e^c}\gamma^\mu \bar{X}_\mu Q\right] \\ = \;& \frac{g}{2}\left(\epsilon_{\alpha\beta\delta}\left(\overline{u^c}\right)^\alpha \gamma^\mu Q^{\delta b} + \left(\bar{Q}\right)_{a\beta}\epsilon^{ab}\gamma^\mu e^c\right)(X_\mu)^\beta{}_b \\ - \;& \frac{g}{2}\left(\epsilon^{\alpha\beta\delta}\left(\bar{Q}\right)_{b\alpha}\gamma^\mu (u^c)_\delta + \overline{e^c}\gamma^\mu \epsilon_{ba} Q^{\beta a}\right)\left(\bar{X}_\mu\right)^b{}_\beta \end{aligned} \tag{85}$$

To this we have to add the gauge boson mass term (76):

$$\frac{M_X^2}{2} Tr\left(\bar{X}_\mu X^\mu\right) = \frac{M_X^2}{2}\left(\bar{X}_\mu\right)^b{}_\beta (X_\mu)^\beta{}_b \tag{86}$$

These heavy fields we want to integrate out to get the effective 4-fermion (dimension 6) interaction. We thus sum up (82), (85) and (86), take the derivative first over $\left(\bar{X}_\mu\right)^b{}_\beta$ to get

$$(X_\mu)^\beta{}_b = \frac{g}{M_X^2}\left[\epsilon^{\alpha\beta\delta}\left(\bar{Q}\right)_{b\alpha}\gamma_\mu\left(u^c\right)_\delta + \overline{e^c}\gamma_\mu\epsilon_{ba}Q^{\beta a} + \left(\overline{d^c}\right)^\beta\gamma_\mu\epsilon_{ab}L^a\right] \tag{87}$$

then over $(X_\mu)^\beta{}_b$ to obtain

$$\left(\bar{X}_\mu\right)^b{}_\beta = \frac{-g}{M_X^2}\left[\epsilon_{\alpha\beta\delta}\left(\overline{u^c}\right)^\alpha\gamma_\mu Q^{\delta b} + \left(\bar{Q}\right)_{a\beta}\epsilon^{ab}\gamma_\mu e^c + \left(\bar{L}\right)_a\epsilon^{ab}\gamma_\mu\left(d^c\right)_\beta\right] \tag{88}$$

The original Lagrangian thus becomes (only the B and L violating terms)

$$\mathcal{L}_{d=6} = \frac{g^2}{2M_X^2}\epsilon_{\alpha\beta\delta}\left(\overline{u^c}\right)^\alpha\gamma_\mu Q^{\delta b}\left(\overline{e^c}\gamma^\mu\epsilon_{ba}Q^{\beta a} + \left(\overline{d^c}\right)^\beta\gamma^\mu\epsilon_{ab}L^a\right) + h.c. \tag{89}$$

2.5 Magnetic monopoles $\leftrightarrow$ charge quantization

One of the great mysteries of electrodynamics is the quantization of the electric charge. Since the non-Abelian generators have quantized eigenvalues, this then mean that in the SM what is quantized is the hypercharge. This connection obviously reminds us of a possible explanation. If the SM gauge group derives from a non-Abelian common group, all the diagonal generators of it have quantized eigenvalues and thus the hypercharge and electric charge that follows from them are quantized as well.

One can get to a similar conclusion in an apparently completely different way. Imagine that there is a magnetic monopole with magnetic charge q_M. Dirac showed that due to quantum mechanical arguments (the wave-function must be single-valued) all the electric charges q_E must be quantized as

$$q_E q_M = n2\pi \qquad n \in \mathcal{Z} \tag{90}$$

But it turns out that a non-Abelian gauge group that gets broken into a final one with at least one Abelian factor actually has as a classical solution to the equation of motion a magnetic monopole. So the two explanations are connected.

GUT magnetic monopoles are heavy, order M_{GUT}, or even a bit more. Their magnetic field is quantized, and their eventual presence from the sky has been searched at the Gran Sasso National Laboratories. The experiment MACRO has put the best limit on their abundance in the universe [7].

Due to time constraints I will unfortunately not pursue this very fascinating subject in the following.

2.6 The doublet-triplet splitting

Before going into description of particular realistic models, I would like to mention another peculiar characteristic of grand unified theories. The SM Higgs doublet, when embedded in a GUT representation, typically has as partners color triplets which mediate proton decay. For example, in Georgi-Glashow SU(5) the Higgs stays in the fundamental, and the triplet partner T has Yukawa couplings that can be derived from (56) and look like

$$\mathcal{L}_Y(T) \quad = \quad T^* \left(LY_5Q - d^cY_5u^c\right) - \left(\frac{1}{2}QY_{10}Q + u^cY_{10}e^c\right)T \tag{91}$$

This triplet is heavy and its exchange leads to the Fermi interaction

$$\mathcal{L}_{d=6} = \frac{1}{2M_T^2}\left(QY_{10}Q\right)\left(QY_5^TL\right) + \frac{1}{M_T^2}\left(d^cY_5u^c\right)\left(u^cY_{10}e^c\right) \tag{92}$$

Since nucleons are made of first generation quarks, the corresponding Yukawas are typically of that order, so that

$$\frac{y_uy_d}{M_T^2} \lesssim \frac{1}{M_{GUT}^2} \tag{93}$$

and so

$$M_T \gtrsim 10^{12}\text{GeV} \tag{94}$$

On the other side, our Higgs doubet, the SU(5) partner of this heavy singlet, has a (negative) mass term of order $-M_Z^2$, i.e. practically massless. The two requirement, a heavy color triplet and a light weak doublet from the same multiplet, are difficult to achieve in a natural way. This is called the doublet-triplet splitting problem. Although formally one can satisfy these constraints, a light doublet and a heavy tripelt from the same multiplet, this will need fine-tuning of the model parameters, unless one works in complicated and non-minimal set-ups. Let us see this a bit more precisely. The interaction between the Higgs responsible for SU(5) breaking (the 24_H) and the Higgs responsible for the electro-weak breaking (the 5_H) looks like

$$5_H^*\left(a24_H + b\right)5_H \tag{95}$$

Once the heavy Higgs gets a vev (10) the masses of the two parts of the fundamenal multiplets split as

$$\left(2a\frac{v}{\sqrt{30}}+b\right)|T|^2+\left(-3a\frac{v}{\sqrt{30}}+b\right)|H|^2=M_T^2|T|^2+M_H^2|H|^2 \tag{96}$$

From here it follows that

$$b=M_H^2+3a\frac{v}{\sqrt{30}}=\frac{3}{5}M_T^2+\mathcal{O}(M_H^2) \tag{97}$$

Due to (35) we need a fine-tuning of the parameters of the Lagrangian of order $\mathcal{O}(M_H^2/M_T^2)\approx 10^{-20}$. This is called the doublet-triplet splitting problem.

There are various ways of solving this problem, but unfortunately no minimal model has such solutions. So the solution of it, at least at the state of the art, needs non-minimal generalizations, and supersymmetry on top of that to stabilize it. Such solutions typically do not have particular predictions, i.e. they cannot be experimentally differentiated at low-energies from the minimal, fine-tuned, models. The issue, although probably important, seems thus at the moment a bit philosophical. For this reason I will not pursue the subject any longer.

3 Two realistic SU(5) models

As we have just seen, the Georgi-Glashow model is ruled out, because it predicts wrong gauge couplings at the scale M_Z (another way of saying is that the 3 gauge couplings of the SM do not unify). On top of that, this model suffers from the same problem as the SM: it predicts massless neutrinos. It is actually even worse than the SM: there we could at least phenomenologically write down an effective Weinberg operator

$$\mathcal{L}_{Weinberg\,SM}=y_{ij}^{SM}\frac{(L_iH)(HL_j)}{M}+h.c. \tag{98}$$

With properly chosen values of y^{SM}/M we could fit the experimental numbers, since M can be essentially anything. This is not allowed anymore in SU(5). Although we can write down a similar effective operator

$$\mathcal{L}_{Weinberg\,SU(5)}=y_{ij}^{SU(5)}\frac{(5^c_{Fi}5_H)(5_H5^c_{Fj})}{M}+h.c. \tag{99}$$

the cutoff M cannot be lower than M_{GUT} if we want this theory to make sense at the unification scale. Due to proton decay constraints $M_{GUT} \gtrsim 10^{16}$ GeV the resulting neutrino masses turn out too small ($y \lesssim 1$ because of perturbativity assumption).

I will show now two examples of realistic models that can overcome these problems, the missing unification and the practically vanishing neutrino mass.

3.1 Minimal non-supersymmetric SU(5)

As we mentioned in the previous chapter, the idea is to include new degrees of freedom. For this purpose I will add to this model a fermionic adjoint [8], [9]. Under the SM it decomposes into

$$24_F = S(1,1,0) + T(1,3,0) + O(8,1,0) + X(3,2,-5/6) + \overline{X}(\bar{3},2,5/6) \quad (100)$$

Exercise: Derive (100). Hint: $24 \sim \bar{5} \times 5$.

The Higgs 24_H obviously decomposes in a similar way. We have thus the following possibility for light states (the gauge singlets do not contribute to the beta function, while the X_H, $\overline{X}_H$ get eaten by the longitudinal components of the SU(5) heavy gauge bosons via the Higgs mechanism):

$$\begin{aligned} \text{spin} = 0 : &\quad T_H(1,3,0)\,,\, O_H(8,1,1)\,,\, H^C(3,1,-1/3) \\ \text{spin} = 1/2 : &\quad T(1,3,0)\,,\, O(8,1,1)\,,\, X(3,2,-5/6)\,,\, \bar{X}(\bar{3},2,5/6) \end{aligned} \quad (101)$$

Although apparently a lot of freedom, there is not much choice for their masses, if we want unification. An important point is that in order to get lighter triplets and octets in 24_F, higher dimensional operators has to be used, and so the maximum mass for the leptoquark is $m_X \approx M_{GUT}^2/\Lambda$, where Λ is the cutoff of the SU(5) model, at least $100\,M_{GUT}$ or so, to make it perturbative. For this reason one can show that

$$m_T \approx 1\,\mathrm{TeV} \quad (102)$$

a neat prediction of the model. One can also show, that higher is the triplet mass, lower turns out to be the GUT scale, which means faster is the proton decay. So if we do not find it at the LHC, we should definitely find soon the proton decay, or discard the model.

Exercise: Derive (102) at 1-loop.

It is interesting that part of the spectrum is determined by the requirement of the SM being embedded in a GUT. And, even more exciting, the fermionic triplet lies in the range of the LHC.

We have now to solve the neutrino mass issue yet. We have two candidates for mediators of the see-saw mechanism, the fermionic singlet S (type I see-saw) and the fermionic triplet T (type III see-saw). They are coupled to the SM leptons as

$$\mathcal{L}_{Yuk} = y_T^i L_i T H + y_S^i L_i S H \tag{103}$$

to give the neutrino mass matrix ($M_{T,S}$ are the triplet and singlet masses)

$$m_\nu^{ij} = \frac{v^2}{2}\left(\frac{y_T^i y_T^j}{M_T} + \frac{y_S^i y_S^j}{M_S}\right) \tag{104}$$

with rank two, so the model predicts one massless neutrino.

The fermionic weak triplet $T = (T^+, T^0, T^-)$ decays through weak interactions mainly into a lepton and a gauge boson:

$$\begin{aligned} T^\pm &\rightarrow W^\pm \nu \quad \text{or} \quad Z^0 e^\pm & (105)\\ T^0 &\rightarrow W^\pm e^\mp \quad \text{or} \quad Z^0 \nu & (106) \end{aligned}$$

with a decay width estimate

$$\Gamma_T \approx |y_T|^2 m_T \tag{107}$$

The decay rate depends on the same Yukawa couplings that are responsible for the neutrino mass. LHC could thus give us information on the yet unmeasured parameters of the neutrino sector.

To summarize, the minimal non-supersymmetric SU(5) model predicts

- a weak fermionic triplet with mass $m_T \approx 1$ TeV;
- one neutrino massless;
- neutrino mass matrix a mixture of type I and type III see-saw;
- triplet decays constrained by neutrino masses and mixings.

3.2 Supersymmetric SU(5) version

In the MSSM the beta coefficients are $b_i = (-33/5, -1, 3)$. If we put all the superpartners at TeV, the three couplings unify in a single point at $\mu \approx 10^{16}$ GeV [15] [1]. To appreciate this fact one should remember that this unification fails badly in the nonsupersymmetric case (compare the two runnings on Fig. 1). So, if we have supersymmetric partners at M_Z or close to 1 TeV as required by naturalness (hierarchy problem), then we have the unification of gauge couplings for free. This is one of the (main) motivations for supersymmetry with low scale (TeV) superpartners (and of unification in supersymmetric theories). Let us now construct a supersymmetric SU(5) GUT.

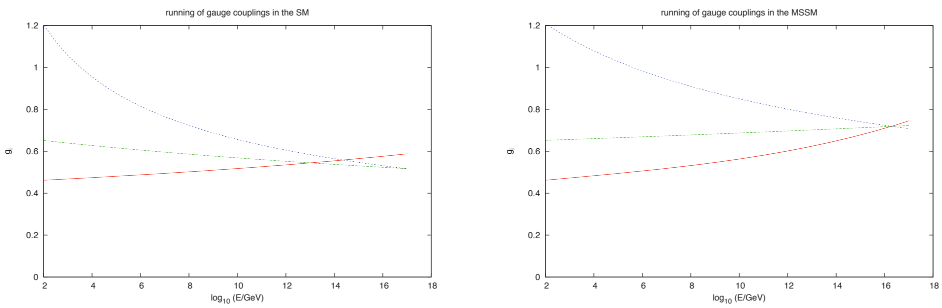

Figure 1: The running of the gauge couplings in SM (left) and low energy MSSM (right).

3.2.1 The Yukawa sector

The Yukawa structure does not change, except that we have now two Higgs fundamental representations, call them 5_H^u and 5_H^d. They are needed for two reasons: anomaly cancellation and nonzero Yukawa couplings for both up and down sectors. Both requirements are just GUT generalizations of the well known reason for two Higgs doublets in MSSM. So we get the Yukawa

[1] In order to get unification from low energy susy, the authors of [15] predicted $\sin^2\theta_W$ to be higher than known at that time and the top mass to be around 200 GeV instead of the ten times lighter believed at that time, both predictions confirmed by later experiments.

(56) with 5_H and 5^*_H replaced by 5^u_H and 5^d_H from the following superpotential (now all the fields are actually chiral superfield)

$$W_Y = 5^c_F Y_5 10_F 5^d_H + \epsilon_5 10_F Y_{10} 10_F 5^u_H \tag{108}$$

The subscript F denotes matter (fermionic) multiplets. Regarding group theory, supersymmetry does not change the conclusions of symmetric Y_U and equality of $Y_D = Y_E^T$. Although running from M_{MSSM} to M_{GUT} changes with respect to the SM case, it does not get substantially closer to these relations. In practice we can make such model realistic as in the ordinary - non-supersymmetric case: either by adding a new Higgs superfield, for example a 45_H (and a $\overline{45}_H$), or allowing non-renormalizable terms. Let's stick for definiteness to the second possibility in the following.

3.2.2 The Higgs sector

Now what about the Higgs potential? The 24_H is now a complex field. Its superpotential is given at the renormalizable level as

$$W_H = \frac{\mu}{2} Tr 24_H^2 + \frac{\lambda}{3} Tr 24_H^3 \tag{109}$$

As in the non-supersymmetric case will consider SM-like vacua, in which

$$\langle 24_H \rangle = \frac{v}{\sqrt{30}} diag(2, 2, 2, -3, -3) \tag{110}$$

with now $v = \mu/\lambda$.

Exercise: Show that other (degenerate) vacua are possible.

It is easy to show that in such renormalizable superpotential all the SM decomposed fields are at the same scale M_{GUT}. This is not necessarily true anymore if one includes also higher, non-renormalizable terms in the superpotential (109). Since we are forced to use non-renormalzible terms to cure bad mass relations in the Yukawa sector, we should allow for such possibility. Let's add thus

$$\delta W_H = \frac{c_1}{4\Lambda} Tr 24_H^4 + \frac{c_2}{4\Lambda} \left(Tr 24_H^2 \right)^2 \tag{111}$$

where we leave for the moment the cut-off scale Λ free. It is now straightforward to find out in the limit $\lambda \to 0$ the following relation

$$m_3 = 4m_8 \approx c\frac{v^2}{\Lambda} \tag{112}$$

Exercise: Calculate the Higgs spectrum in the general case and verify (112) in the $\lambda \to 0$ limit. Explain why the X and $\bar{X}$ are massless if only the superpotential is considered.

3.2.3 Running in the non-renormalizable case

What we just found out is very important, because we have now new states below the GUT scale v. For this reason we have to redo the renormalization group analysis for the gauge couplings:

$$\begin{aligned}
2\pi\left(\alpha_1^{-1}(M_Z) - \alpha_U^{-1}\right) &= -\frac{5}{2}\log\frac{M_{SUSY}}{M_Z} + \frac{33}{5}\log\frac{M_{GUT}}{M_Z} + \frac{2}{5}\log\frac{M_{GUT}}{m_T} \\
2\pi\left(\alpha_2^{-1}(M_Z) - \alpha_U^{-1}\right) &= -\frac{25}{6}\log\frac{M_{SUSY}}{M_Z} + \log\frac{M_{GUT}}{M_Z} + 2\log\frac{M_{GUT}}{m_3} \\
2\pi\left(\alpha_3^{-1}(M_Z) - \alpha_U^{-1}\right) &= -4\log\frac{M_{SUSY}}{M_Z} - 3\log\frac{m_8}{M_Z} + \log\frac{M_{GUT}}{m_T}
\end{aligned} \tag{113}$$

Taking two linear combinations we can get rid of the experimentally unknown gauge coupling at the unification scale

$$\begin{aligned}
2\pi\left(3\alpha_2^{-1} - 2\alpha_3^{-1} - \alpha_1^{-1}\right) &= -2\log\frac{M_{SUSY}}{M_Z} + \frac{12}{5}\log\frac{m_T}{M_Z} + 6\log\frac{m_8}{m_3} \\
2\pi\left(5\alpha_1^{-1} - 3\alpha_2^{-1} - 2\alpha_3^{-1}\right) &= 8\log\frac{M_{SUSY}}{M_Z} + 12\log\frac{\sqrt{m_3 m_8}M_{GUT}^2}{M_Z^3}
\end{aligned} \tag{114}$$

Denoting with m_T^0 and M_{GUT}^0 the values for the renormalizable case in which $m_3 = m_8 = M_{GUT}^0$ we get first

$$m_T = m_T^0\left(\frac{m_3}{m_8}\right)^{5/2} \tag{115}$$

$$M_{GUT} = M_{GUT}^0\left(\frac{M_{GUT}^0}{\sqrt{m_3 m_8}}\right)^{1/2} \tag{116}$$

and then, since $m_3 = 4m_8 \approx M_{GUT}^2/\Lambda$

$$m_T = 32 m_T^0 \tag{117}$$
$$M_{GUT} \approx \left[\left(M_{GUT}^0\right)^3 \Lambda\right]^{1/4} \tag{118}$$

i.e. the GUT scale and the color triplet mass get increased. This is very important, and we will use it in the next section when considering proton decay.

3.2.4 Dimension 5 proton decay

In supersymmetry we have on top of the usual (dimension 6) heavy gauge boson (and gaugino) mediated proton decay modes also another, potentially much more dangerous decay mode coming from the exchange of the heavy color triplet Higgs from $5_H^{u,d}$. Using (46) and the corresponding

$$5_H^d = \begin{pmatrix} T^c \\ H_d \end{pmatrix} \qquad 5_H^u = \begin{pmatrix} T \\ H_u \end{pmatrix} \tag{119}$$

one can easily derive in the renormalizable case (108) the coupling of these triplets to the SM chiral fermions

$$W_Y(T) = T^c \left(L Y_5 Q - d^c Y_5 \epsilon_3 u^c\right) + \left(\frac{1}{2} Q Y_{10} Q - u^c Y_{10} e^c\right) T \tag{120}$$

These triplets are heavy, with mass term

$$-M_T T^c T \tag{121}$$

so they can be integrated out by solving the equations of motion, getting

$$W_{d=5} = \frac{1}{2M_T} \epsilon_3 \left(Q Y_{10} Q\right) \left(Q Y_5^T L\right) + \frac{1}{M_T} \left(d^c Y_5 u^c\right) \left(u^c Y_{10} e^c\right) \tag{122}$$

By itself this does not yet produce a proton decay 4-fermion operator at tree level, but only a baryon and lepton number violating term among two fermions and two sfermions, for example between two quarks, and a slepton and a squark. This is a dimension 5 operator. It can be however closed in

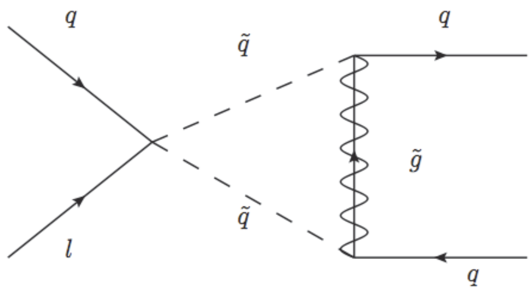

Figure 2: The $d = 5$ proton decay operator closed by a gluino exchange loop.

a loop by for example exchange of a gaugino (gluino or wino) or Higgsino, giving rise to the usual 4-fermion interaction. An example of such diagrams is given on fig. 2. For a complete analysis of such processes and formulae involved see for example [10, 11].

Assuming the sfermion masses are bigger than the gluino one, this gives rise to an operator of the form (schematically)

$$\left(\frac{Y_{10}Y_5}{M_T}\right)\left(\frac{\alpha_3}{4\pi}\right)\left(\frac{m_{\tilde{g}}}{m_{\tilde{q}}^2}\right) qqql \tag{123}$$

What is however peculiar here, is that such an operator is suppressed only by one inverse power of the heavy triplet mass, instead of the two powers of the heavy gauge boson mass in the usual $d = 6$ operator. In principle this could give rise to a huge enhancement of the decay rate [12, 13]. There are however several reasons that make the proton lifetime long enough though [14]:

- the proton is made from first generation quarks, so at least some of the Yukawas involved are typically small.
- due to Yukawa higher dimensional operators needed to cure the wrong mass relations, the corresponding Yukawas appearing in the $d = 5$ operator do not need to be connected to the fermion masses and can thus conspire to cancel the dangerous decay modes.
- a similar uncertainty is present in the squark sector: the mixing angles need not be related to the fermion ones, even if one takes into account the stringent constraints from flavor violating transitions

- most important, due to non-renormalizable operators in the Higgs sector the color octet and weak triplets can be, as shown in the previous section, lighter than the GUT scale, and thus can change the running. It is thus easy to accommodate a higher GUT scale and color triplet mass, thus suppressing further (123).

Due to all these uncertainties, proton decay is not yet a problem in the minimal supersymmetric SU(5). What is however a problem, or better, a shortcoming of this model, is the description of neutrinos. The non-supersymmetric model described previously was an exception: due to its minimality and simplicity the model was predictive, but this is no longer true in the supersymmetric version. In general cases the first (and probably last) non-trivial GUT model of neutrino mass is SO(10), which we will consider now.

4 SO(10) grand unification

SO(10) models are richer than SU(5) and there are more choices for the possible representations that can embed the SM. We will insist as so far to have the gauge symmetry as our only guidance and not include any more global continuous or discrete symmetries. This is not the only possible choice and much work has been done considering for example family (horizontal) and other symmetries on top of the gauge one.

We will go through SO(10) describing a specific supersymmetric model that has been first proposed 30 years ago [16, 17] but has been studied in detail only in the last decade.

4.1 Representations

There are two types of representations in SO(10) (and SO(N) in general): tensorial and spinorial. The first type is a bit what we were using in SU(5), although now there are no differences between upper and lower indices. For example the combination

$$M_{ijk}N_{ij}P_k \tag{124}$$

where repeated indices automatically run from 1 to 10 (N in SO(N)), is an SO(10) invariant. All this follows from the transformation rule of a fundamental index:

$$M_i' = O_{ij} M_j \tag{125}$$

which is easily generalized for more indeces:

$$R'_{i_1 \dots i_N} = O_{i_1 j_1} \dots O_{i_N j_N} R_{j_1 \dots j_N} \tag{126}$$

where the transformation matrix

$$O_{ij} = \exp\left(i\alpha_{kl} T_{kl}\right)_{ij} \tag{127}$$

T_{kl} are the anti-symmetric generators (10×10 matrices, 45 of them independent) of SO(10), that satisfy the commutation relations

$$[T_{ij}, T_{kl}] = i\left(\delta_{ik} T_{jl} + \delta_{jl} T_{ik} - \delta_{il} T_{jk} - \delta_{jk} T_{il}\right) \tag{128}$$

All this is completely analogous to the well known SO(3) case of ordinary rotations in 3D space.

The different tensorial representations have one or more fundamental indices, and usually some extra constraint on them, for example symmetry, antisymmetry, tracelessness, and, as we will see later, (anti)self-duality. We will consider obviously only the lower dimensional ones, although in SO(10) very few representations are really low dimensional. The building block among tensorial representations is the fundamental 1-index 10_i. With two indices we can construct either an antisymmetric $45_{ij} = -45_{ji}$ or a symmetric $54_{ij} = 54_{ji}$ combinations

$$10 \times 10 = 45 + 54 + 1 \tag{129}$$

We may use in the following also the 3 indices completely antisymmetric 120 ($= 10 \times 9 \times 8/3!$), a 4 indices completely antisymmetric 210 ($= 10 \times 9 \times 8 \times 7/4!$) and 5-indices completely antisymmetric with an extra self (or anti-self) duality relation

$$126_{ijklm} = \pm \frac{i}{5} \epsilon_{ijklmnopqr} 126_{nopqr} \tag{130}$$

In fact, $126 = (1/2) 10 \times 9 \times 8 \times 7 \times 6/5!$.

The spinorial representations are a bit more tricky. They follow from a different type of generators that satisfy (128). You obtain it by first generating the 10 (N) different $2^5 = 32$-dimensional (in a general SO(N) the power is $N/2$ for N even and $(N-1)/2$ for N odd) Γ matrices that satisfy the anticommutation relation

$$\{\Gamma_i, \Gamma_j\} = 2\delta_{ij} \tag{131}$$

Then the 45 matrices 32×32

$$\Sigma_{ij} = \frac{1}{4i}\left[\Gamma_i, \Gamma_j\right] \tag{132}$$

satisfy the SO(10) commutation relations (128). The explicit form of the Γ and thus Σ matrices can be found in [18, 19].

One can think that the spinorial representation is 32-dimensional, but actually this is reducible. One can in fact define the analogue of γ_5 in Minkowski spacetime as

$$\Gamma_{FIVE} = i\Gamma_2\Gamma_4 \ldots \Gamma_{10} \tag{133}$$

and project the left 16 and right $\overline{16}$ states as

$$16 = \frac{1}{2}\left(1 + \Gamma_5\right)32 \qquad \overline{16} = \frac{1}{2}\left(1 - \Gamma_5\right)32 \tag{134}$$

Now we have enough knowledge to see better into the useful representations of SO(10).

4.2 Our choice of representations

First, the matter fields of one generation live in a single 16 dimensional (spinorial) representation of SO(10). It is great that all SM fermions are unified, and the 16^{th} element is a singlet, the right-handed neutrino:

$$16_F = (Q, u^c, d^c, L, \nu^c, e^c) \tag{135}$$

This obviously calls for the see-saw mechanism [20, 21]. Also, it is not strange that different Yukawas will be connected now. So one can derive in SO(10) various constraints among SM Yukawa couplings (quarks and leptons, neutrino included).

Second, only three types of Yukawas are possible, i.e. only 10, 120 and 126 dimensional Higgses of SO(10) can couple to spinorial bilinears

$$16 \times 16 = 10 + 120 + 126 \tag{136}$$

We will keep just of them, 10 and 126 only, with the SM Higgs doublets (remember that in MSSM there must be two Higgs doublets) living in both 10 and 126 (i.e. linear combinations of doublets there) [22, 23]. Schematically

$$W_{Yukawa\,SO(10)} = 16_F(Y_{10}10_H + Y_{126}126_H)16_F \tag{137}$$

SO(10) constraints the Yukawa matrices in generation space Y_{10} and Y_{126} to be symmetric (Y_{120} turns out to be antisymmetric).

Third, SO(10) has rank 5, the SM rank 4. So to break the rank one needs to give a vev to the SM singlet in 126 (another, non-minimal option is to add a new Higgs in a 16 dimensional representation). But since we are in supersymmetry, another superfield, the $\overline{126}$, must be introduced to cancel the nonzero D-terms (or, better, to allow the rank breaking). Notice that here the situation is different from the introduction of the second Higgs doublet in MSSM: SO(10) is anomaly free by construction, no matter what representation one chooses.

This same (126) vev is the one that gives mass to the right-handed neutrino. Notice that this means that its mass matrix has the same Yukawa Y_{126} that is used for other fermion masses, a powerful consequence of SO(10) gauge invariance.

Finally, the renormalizable Higgs sector needed to break SO(10) into SM can be constructed with 54 and 45 or 210 only, on top of the above-mentioned $126 - \overline{126}$ pair. Since only the second choice allow in supersymmetry weak doublet mixing in 10 and 126, we will stick to this choice.

Finally, an adjoint 45 dimensional vector multiplet will describe the gauge part of SO(10).

To summarize: we will work with

$$3 \times 16_F, 10_H, 126_H, \overline{126}_H, 210_H, 45_V \tag{138}$$

The index F means that our light fermions are living there, H that sooner or later some of the fields are getting a nonzero vev, and V refers to the vector superfield.

4.3 The Pati-Salam subgroup

Here it is perhaps time to introduce the Pati-Salam (PS) subgroup of SO(10). It is a left-right symmetric model with 4 colors, i.e. the product group $\mathrm{SU}(2)_L \times \mathrm{SU}(2)_R \times \mathrm{SU}(4)_C$. The matter fields under it are

$$16 = (2,1,4) + (1,2,\bar{4}) \tag{139}$$

where the left and right handed doublets are in

$$(2,1,4) = \begin{pmatrix} u_1 & u_2 & u_3 & \nu \\ d_1 & d_2 & d_3 & e \end{pmatrix} \quad , \quad (1,2,\bar{4}) = \begin{pmatrix} u_1^c & u_2^c & u_3^c & \nu^c \\ d_1^c & d_2^c & d_3^c & e^c \end{pmatrix} \tag{140}$$

Notice that leptons are just the 4^{th} color.

The 10 and 126 dimensional Higgses get decomposed under the PS subgroup (not the SM anymore!) as (for this and most other group theory results the reader should consult the famous review of Slansky [24])

$$\begin{aligned}
10 &= (2,2,1) + (1,1,6) && (141)\\
126 &= (2,2,15) + (3,1,\overline{10}) + (1,3,10) + (1,1,6) && (142)\\
\overline{126} &= (2,2,15) + (1,3,\overline{10}) + (3,1,10) + (1,1,6) && (143)\\
210 &= (1,1,1) + (2,2,6) + (3,1,15) + (1,3,15) && \\
&+ (2,2,10) + (2,2,\overline{10}) + (1,1,15) && (144)\\
45 &= (3,1,1) + (1,3,1) + (2,2,6) + (1,1,15) && (145)
\end{aligned}$$

I derived the above in the following way. Remember that the PS theory is locally equivalent to SO(4)×SO(6), since locally SO(4)$\sim$SU(2)$_L$×SU(2)$_R$ and SO(6)$\sim$SU(4)$_C$.

10_i has one index of SO(10), i, which runs from 1 to 10. The elements in 10 with index i from 1 to 4 represent a 4 of SO(4), i.e. a (2,2,1) under Pati-Salam. The remaining elements 10_i with $i = 5, \ldots 10$ are a 6 of SO(6), thus a (1,1,6) of PS.

On the other side 126 is a 5-index completely antisymmetric matrix with a self-dual relation that modes out half of the degrees of freedom. We can just repeat the previous case of 10, but now with 5 indeces. For example, taking all 5 indices running from 5 to 10 and antisymmetrizing them we get just a 6 of SO(6) (in 6 dimensions a 1-form is dual to a 5-form, i.e. in d-dimensions an object with p completely antisymmetric indices has the same number of components as an object with d-p completely antisymmetric indices), i.e. a (1,1,6) of PS. We continue then with 4 indices of SO(6) and one index of SO(4) to get a (2,2,15) of PS, etc.

Exercise: Derive the decompositions in (141)-(145).

One last thing will be useful in future: the electric charge can be written with the following symmetric combination of SO(10) generators:

$$Q_{em} = T_{3L} + T_{3R} + \frac{B-L}{2} \tag{146}$$

Here $T_{3L,R}$ are the eigenvalues of the third generator in $\mathrm{SU}(2)_{L,R}$, in fundamental representation for example from the usual Pauli matrix $\tau_3/2$:

$$T_3 = \frac{1}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \tag{147}$$

while $B-L$ is proportional to the last, 15^{th} generator of $\mathrm{SU}(4)_C$, in fundamental representation for example by

$$\frac{B-L}{2} = \frac{1}{3}\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -3 \end{pmatrix} \tag{148}$$

4.4 The Higgs sector

We have now most of the ingredients needed to describe the SO(10) breaking into the SM. First of all, which of the HIggs involved contain SM singlets? After a look to their Pati-Salam decomposition we find these candidates in $\Phi(210)$, $\Sigma(126)$ and $\overline{\Sigma}(\overline{126})$. We denote their vevs as

$$\begin{aligned} p &= \langle\Phi(1,1,1)\rangle \quad , \quad a = \langle\Phi(1,1,15)\rangle \quad , \quad \omega = \langle\Phi(1,3,15)\rangle \\ \sigma &= \langle\Sigma(1,3,10)\rangle \quad , \quad \bar{\sigma} = \langle\overline{\Sigma}_H(1,3,10)\rangle \end{aligned} \tag{149}$$

The most general renormalizable SO(10) invariant superpotential with fields Φ, Σ and $\overline{\Sigma}$ can be written ([25])

$$\begin{aligned} W_{Higgs} &= \frac{m_\Phi}{4!}\Phi_{ijkl}\Phi_{ijkl} + \frac{m_\Sigma}{5!}\Sigma_{ijklm}\overline{\Sigma}_{ijklm} \qquad (150) \\ &+ \frac{\lambda}{4!}\Phi_{ijkl}\Phi_{klmn}\Phi_{mnij} + \frac{\eta}{4!}\Phi_{ijkl}\Sigma_{ijmno}\overline{\Sigma}_{klmno} \end{aligned}$$

The next step is to rewrite this same superpotential (150) in terms of SM singlets (149). To do that we need to localize the SM, i.e. find out in which components of representations Φ, Σ and $\overline{\Sigma}$ they live (i.e. to calculate the Clebsch-Gordan coefficients).

The fundamental representation of SO(10) satisfies (128), and is given by

$$(T_{ij})_{kl} = -i\left(\delta_{ik}\delta_{jl} - \delta_{il}\delta_{jk}\right) \tag{151}$$

The Cartan subalgebra of SO(10) (the maximal Abelian subgroup of SO(10)) is 5-dimensional, and is composed of

$$T_{12}, T_{34}, T_{56}, T_{78}, T_{90} \tag{152}$$

The SM generators and $B-L$ are linear combinations of these Cartan generators. Let's see it.

Remember that indices from 1 to 4 mean the left-right SU(2)$_L\times$SU(2)$_R$ subgroup, while those from 5 to 10 (this last denoted for simplicity just by 0) live in SU(4)$_C$. Let's consider the first case. It boils down to the known way of writing SU(2)$\times$SU(2) generators $T^{1,2,3}_{L,R}$ from the SO(4) generators T_{ij} ($i<j$ and running from 1 to 4).

Take one index as particular, and define (a,b,c now run from 1 to 3):

$$T_{a4} = K_a \qquad T_{ab} = \epsilon_{abc}J_c \tag{153}$$

Since T_{ij} satisfy (128), the new generators satisfy

$$[K_a, K_b] = i\epsilon_{abc}J_c \quad [J_a, K_b] = i\epsilon_{abc}K_c \quad [J_a, J_b] = i\epsilon_{abc}J_c \tag{154}$$

Just one step more and define

$$T^a_L = \frac{1}{2}\left(J_a + K_a\right) \qquad T^a_R = \frac{1}{2}\left(J_a - K_a\right) \tag{155}$$

with the SU(2)$_L\times$SU(2)$_R$ commutation relations

$$\left[T^a_{L,R}, T^b_{L,R}\right] = i\epsilon_{abc}T^c_{L,R} \qquad \left[T^a_{L,R}, T^b_{R,L}\right] = 0 \tag{156}$$

In components one has in terms of the original generators

$$
\begin{aligned}
T_L^1 &= \frac{1}{2}\left(T_{23}+T_{14}\right) \qquad T_R^1 = \frac{1}{2}\left(T_{23}-T_{14}\right) \\
T_L^2 &= \frac{1}{2}\left(T_{31}+T_{24}\right) \qquad T_R^2 = \frac{1}{2}\left(T_{31}-T_{24}\right) \\
T_L^3 &= \frac{1}{2}\left(T_{12}+T_{34}\right) \qquad T_R^3 = \frac{1}{2}\left(T_{12}-T_{34}\right)
\end{aligned}
\tag{157}
$$

Similarly we find the SU$(4)_C$ generators explicitly from the SO(6) ones:

$$
\begin{aligned}
T_{3L} &\propto T_{12}+T_{34} \\
T_{3R} &\propto T_{12}-T_{34} \\
B-L &\propto T_{56}+T_{78}+T_{90} \\
T_{3C} &\propto T_{56}-T_{78} \\
T_{8C} &\propto T_{56}+T_{78}-2T_{90}
\end{aligned}
\tag{158}
$$

Exercise: Find out the proportionality factors in (158).

Let's come back to our original problem, i.e. finding where the SM singlets live. Imagine the generator T_{12}. It acts on a one index object as

$$
\left(\hat{T}_{12}X\right)_k = \left(T_{12}\right)_{kl} X_l = -i\left(\delta_{1k}X_2 - \delta_{2k}X_1\right) \tag{159}
$$

Since for $X_{k_1k_2\ldots k_N}$ the transformation rule is as usual

$$
\left(\hat{T}_{ij}X\right)_{k_1k_2\ldots k_N} = \left(T_{ij}\right)_{k_1l} X_{lk_2\ldots k_N} + \left(T_{ij}\right)_{k_2l} X_{k_1l\ldots k_N} + \ldots + \left(T_{ij}\right)_{k_Nl} X_{k_1k_2\ldots l} \tag{160}
$$

we can immediately find out that

$$
T_{2i-1,2i}X_{2k-1,2k} = 0 \tag{161}
$$

for any $i,k = 1,5$. This is a necessary constraint for a SM singlet, but not sufficient. For example a 15 of SU$(4)_C$ has 3 such objects with the eigenvalues of all Cartan generators (U(1) quantum numbers) zero. The fundamental of SU(4) get decomposed into SU(3)×U(1) as

$$
4 = (3,-1/3) + (1,1) \tag{162}
$$

so that the adjoint becomes

$$
\begin{aligned}
15 &= (4 \times \bar{4}) - 1 = [(\bar{3}, 1/3) + (1, -1)] \times [(3, -1/3) + (1, 1)] - 1 \\
&= (8, 0) + (1, 0) + (3, -4/3) + (\bar{3}, 4/3)
\end{aligned} \tag{163}
$$

In the octet of SU(3) there are two elements with zero T_3 and T_8, i.e. the neutral pion and eta. These should not be counted.

So which are the SM singlet states in 210? The easiest one is the PS singlet (all 4 SO(4) indices) which is

$$
p = \langle \Phi_{1234} \rangle \tag{164}
$$

All the possible permutations are also possible (remember that Φ is completely antisymmetric in its 4 indices).

Next comes the $(1, 1, 15)$, which is made from all 4 SO(6) indices:

$$
a = \langle \Phi_{5678} \rangle = \langle \Phi_{5690} \rangle = \langle \Phi_{7890} \rangle \tag{165}
$$

Finally we have the mixed $(1, 3, 15)$, so that

$$
\omega = \langle \Phi_{1256} \rangle = \langle \Phi_{1278} \rangle = \langle \Phi_{1290} \rangle = \langle \Phi_{3456} \rangle = \langle \Phi_{3478} \rangle = \langle \Phi_{3490} \rangle \tag{166}
$$

We will not go through the whole derivation for the Σ, which is a bit more complicated, but one can have a look for example to [26]. Plugging this directions into the superpotential we get

$$
\begin{aligned}
W_{Higgs} &= m_\Phi \left(p^2 + 3a^2 + 6\omega^2 \right) + 2\lambda \left(a^3 + 3p\omega^2 + 6a\omega^2 \right) \\
&+ m_\Sigma \bar{\sigma}\sigma + \eta \bar{\sigma}\sigma \left(p + 3a - 6\omega \right)
\end{aligned} \tag{167}
$$

The minimization of this superpotential leads to non-zero values of the vevs p, a, ω and $\sigma = \bar{\sigma}$. This last equality follows from D-terms.

Exercise: Analize the possible minima of (167). Check the results in [26].

4.5 The Yukawa sector

From (137) and the above decomposition it is easy to get the SM Yukawas. For example

$$\begin{aligned} 16_F 10_H 16_F &\rightarrow (2,1,4)_F(2,2,1)_H(1,2,\overline{4})_F \\ 16_F 126_H 16_F &\rightarrow (2,1,4)_F(2,2,15)_H(1,2,\overline{4})_F + (1,2,\overline{4})_F(1,3,10)_H(1,2,\overline{4})_F \\ &+ (2,1,4)_F(3,1,\overline{10})_H(2,1,4)_F + \ldots \end{aligned} \quad (168)$$

The SM doublets live in $(2,2,1)_H$ and $(2,2,15)_H$, the SM singlet that break the rank of SO(10) is in $(1,3,10)_H$, while the $SU(2)_W$ triplet Higgs that gives rise to a type II see-saw is in $(3,1,\overline{10})$. Remember again that now the decomposition is under Pati-Salam, not the SM!

It is now relatively simple to guess the SM fermion masses for down quarks (D), up quarks (U), charged leptons (E) and neutrinos (N), valid for any number of generations:

$$M_D = v_{10}^d Y_{10} + v_{126}^d Y_{126} \quad (169)$$

$$M_U = v_{10}^u Y_{10} + v_{126}^u Y_{126} \quad (170)$$

$$M_E = v_{10}^d Y_{10} - 3 v_{126}^d Y_{126} \quad (171)$$

$$M_N = -M_{\nu_D} M_{\nu_R}^{-1} M_{\nu_D} + M_{\nu_L} \quad (172)$$

where we defined the Dirac (ν_D), left Majorana (ν_L) and right Majorana (ν_R) neutrino masses as

$$M_{\nu_D} = v_{10}^u Y_{10} - 3 v_{126}^u Y_{126} \quad (173)$$

$$M_{\nu_L} = v_L Y_{126} \quad (174)$$

$$M_{\nu_R} = v_R Y_{126} \quad (175)$$

and the vevs are

$$v_{10}^{u,d} = \langle (2,2,1)_H^{u,d} \rangle \quad , \quad v_{126}^{u,d} = \langle (2,2,15)_H^{u,d} \rangle \quad (176)$$

$$v_R = \langle (1,3,10)_H \rangle \quad , \quad v_L = \langle (3,1,\overline{10})_H \rangle \quad (177)$$

The only thing that has to be still understood is the factor of -3 in front of Y_{126} in M_E and M_{ν_D}. It is due to the vev of the (traceless) adjoint 15 of SU(4)$_C$ in (2,2,15)$_H$:

$$\langle 15_C \rangle \propto diag(1,1,1,-3) \tag{178}$$

and thus give an extra factor -3 to leptons with respect to quarks.

Remember also that every left-right bidoublet (2,2) is (as any chiral superfield spin 0 component) complex in supersymmetry, so there are two possible vevs, which we denoted with indices u and d.

Finally, we have still to specify how SO(10) gets broken to the SM, i.e. the Higgs sector. It turns out that on top of the fields I have mentioned so far (the matter 16_F and the Higgses 10_H and 126_H) we need two other representations, the 5 indices antisymmetric and anti-self-adjoint $\overline{126}_H$ and the 4 indices antisymmetric 210.

Just to taste the predictiveness of this model, consider the case of 2 generations (let us talk about the heaviest two, the second and the third generation of the SM) and limit ourselves to the real case. We can always go into the basis in which Y_{10} for example is diagonal:

$$v_{10}^d Y_{10} = \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix} \quad , \quad v_{126}^d Y_{126} = \begin{pmatrix} c & d \\ d & e \end{pmatrix} \tag{179}$$

Then the number of free parameters in the charged fermion sector is 7:

$$a\,,\, b\,,\, c\,,\, d\,,\, e\,,\, v_{10}^u/v_{10}^d\,,\, v_{126}^u/v_{126}^d \tag{180}$$

They can be determined by fitting 7 experimental data:

$$m_s\,,\, m_b\,,\, m_c\,,\, m_t\,,\, m_\mu\,,\, m_\tau\,,\, V_{cb} \tag{181}$$

With one single new parameter,

$$v_R/v_L \tag{182}$$

we can now calculate two measurable quantities from the neutrino sector (we assume here a normal hierarchy in the neutrino sector)

$$m_3/m_2 = \sqrt{|\Delta m_{31}^2/\Delta m_{21}^2|} \quad , \quad \theta_{23} = \theta_{ATM} \tag{183}$$

so we have in total one prediction.

Exercise: Show that by increasing the number of generations one gets more predictions, assuming all parameters real.

Now let's see this fit in detail. Rewrite (169)-(171) as

$$\begin{aligned} M_U &= D + S && (184)\\ M_D &= r_1 D + r_2 S && (185)\\ M_E &= r_1 D - 3 r_2 S && (186) \end{aligned}$$

Since all the matrices involved are symmetric we can invert them, i.e. calculate $M_{U,D}$ in terms of S, D

$$\begin{pmatrix} D \\ S \end{pmatrix} = \frac{1}{r_2 - r_1} \begin{pmatrix} r_2 & -1 \\ -r_1 & 1 \end{pmatrix} \begin{pmatrix} M_U \\ M_D \end{pmatrix} \tag{187}$$

and plug the expressions in the last equation to get

$$M_E = \frac{4 r_1 r_2}{r_2 - r_1} M_U - \frac{3 r_2 + r_1}{r_2 - r_1} M_D \tag{188}$$

From it we find two useful equations taking its trace, or trace its square:

$$\begin{aligned} Tr M_E &= \frac{4 r_1 r_2}{r_2 - r_1} Tr M_U - \frac{3 r_2 + r_1}{r_2 - r_1} Tr M_D && (189)\\ Tr M_E^2 &= \left(\frac{4 r_1 r_2}{r_2 - r_1}\right)^2 Tr M_U^2 + \left(\frac{3 r_2 + r_1}{r_2 - r_1}\right)^2 Tr M_D^2 \\ &- 2 \left(\frac{4 r_1 r_2}{r_2 - r_1}\right) \left(\frac{3 r_2 + r_1}{r_2 - r_1}\right) Tr M_U M_D && (190) \end{aligned}$$

We know all the above traces, the last one being

$$Tr M_U M_D = (m_t m_b + m_c m_s) - (m_t - m_c)(m_b - m_s) V_{cb}^2 \tag{191}$$

where we used the fact that in the $M_U = M_U^d$ basis

$$M_D = V_{CKM}^T M_D^d V_{CKM} \tag{192}$$

Let's simplify the neutrino sector assuming that type II seesaw dominates. In this case

$$M_N = cS = \frac{c}{r_2 - r_1}\left(-r_1 M_U + M_D\right) \tag{193}$$

Two other invariant combinations can be found. The first one is

$$\frac{TrM_N^2}{(TrM_N)^2} = \frac{r_1^2 TrM_U^2 + TrM_D^2 - 2r_1 TrM_U M_D}{(-r_1 TrM_U + TrM_D)^2} \tag{194}$$

where the left-hand-side equals

$$\frac{1 + \left(\frac{m_3}{m_2}\right)^2}{\left(1 + \frac{m_3}{m_2}\right)^2} \tag{195}$$

must be compared to the experimental value (assuming a hierarchical neutrino spectrum [2])

$$\left(\frac{m_3}{m_2}\right)^2 = \left|\frac{\Delta m^2_{ATM}}{\Delta m^2_{SOL}}\right| = 32 \pm 2 \tag{196}$$

Here and in the following we are using for the neutrino fit the values from [27, 28].

The second useful invariant is

$$\begin{aligned} \frac{TrM_N M_E}{TrM_N} &= \frac{1}{-r_1 TrM_U + TrM_D} \\ &\times \left[\frac{-4r_1 r^2}{r_2 - r_1} TrM_U^2 - \frac{3r_2 + r_1}{r_2 - r_1} TrM_D^2 + \frac{r_1^2 + 7r_1 r_2}{r_2 - r_1} TrM_U M_D\right] \end{aligned} \tag{197}$$

where the left-hand-side is

$$\frac{\left(\frac{m_3}{m_2} m_\tau + m_\mu\right) - \left(\frac{m_3}{m_2} - 1\right)(m_\tau - m_\mu) V_{23}^2}{\frac{m_3}{m_2} + 1} \tag{198}$$

with the experimental value

$$V_{23}^2 = \sin^2\theta_{ATM} = 0.51 \pm 0.06 \tag{199}$$

[2] In the case of inverse hierarchy the two generation analysis is probably not a good approximation, since we would neglect a large mass.

We have thus 4 equations (189), (190), (194) and (197) for two unknowns, r_1 and r_2, clearly an overconstrained system. The idea is to consider a χ^2 analysis for all the observable involves (except the charged lepton masses, which are known too well). Remember that all these quantities must be evaluated at the GUT scale, which has been fortunately already done. We can use for example [29] for the masses, while the neutrino parameters and the value $V_{cb} \approx 0.04 \pm 0.001$ at M_Z do not change significantly, see for example [30] for a discussion on this point. Remember also that all the masses can have arbitrary sign, so there are all together $2^5 = 32$ possibilities, since one mass can be fixed.

Exercise: Find numerically the minimal χ^2 for the above case.

Instead of doing this numerical fit, let me mention an argument for why we may hope it will work. One of the main problems is to get a large atmospheric mixing angle and a small corresponding quark angle. Assuming as above that type II seesaw dominates, we have

$$M_N \propto S \propto M_D - M_E \tag{200}$$

i.e. explicitly in the basis $M_E = M_E^d$ (θ_D is the angle between M_D and M_E)

$$M_N \propto V(\theta_D) \begin{pmatrix} m_s & 0 \\ 0 & m_b \end{pmatrix} V^T(\theta_D) - \begin{pmatrix} m_\mu & 0 \\ 0 & m_\tau \end{pmatrix} \tag{201}$$

Small off-diagonal entries automatically gives small V_{cb}, so we see that the only way to get a large atmospheric angle is to cancel as much as possible the 33 entry, which we obtain if $m_b \approx m_\tau$. This, so called $b - \tau$ unification is however a well-known phenomenon occuring in MSSM. Although not exact, it is typically correct up to 20-30%. So a large atmospheric angle can be connected to $b - \tau$ unification, assuming type II seesaw dominates [31].

The realistic case of three generations and complex parameters is of course much more involved. Allowing an arbitrary Higgs sector several fits are possible and summarized for example in [32, 33, 34].

It is possible to fit all the data also in the minimal model with the Higgs superpotential described above, providing the gaugini and higgsini of MSSM lie at about 10-100 TeV, while the sfermions and the second Higgs are much heavier (10^{13} GeV or so), which does not spoil one-step unification (one version of the so-called split supersymmetry scenario). Such a model determines all the parameters, among others predicts all proton decay rates and a relatively large value of the yet unmeasured neutrino mixing angle θ_{13} (see [35] and references therein).

Another possibility (for those who do not like the split susy scenario and almost nothing to find at LHC) is to include another multiplet, the 120. This has been recently done successfully. For details and references see [36] and the latest [37].

4.6 Proton decay in susy SO(10)

One last word regarding proton decay. It is similar to the SU(5) case, the dimension 5 decay dominating the rate unless the sfermions are too heavy. There are though more color triplets mediating it. They live in 10_H, 126_H, $\overline{126}_H$ and 210_H. They mix, so that their mass matrix is certainly not diagonal. But only some elements are coupled to the SM fermions, and thus only some entries of the inverse mass matrix are important for the proton decay rate. It is thus at least in principle possible to arrange cancellations if the rate becomes dangerously large. For detailed studies see for example [38, 39].

5 Conclusion

There were many aspects of grand unification not considered in these lectures. Let me just mention the groups SU(6) and E_6, the SO(10) models with 16_H instead of 126_H, non-supersymmetric SO(10), etc. They would need more time, and each of these models has its advantages but also drawbacks. It is correct to say that at the moment there is no really satisfactory model of grand unification. What prevents to be such are the successful solution to the doublet-triplet splitting problem, the origin of supersymmetry breaking and a better understanding of the hierarchies in general. But these are problems present in any physics beyond the standard model as well in the standard model itself. What grand unified theories do is what any physical theory should do: connect different phenomena. And GUTs provide links between proton decay, fermion masses and gauge symmetries.

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