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Workshop and School on Topological Aspects of Condensed Matter Physics

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THE BOTT CLOCK AND TOPOLOGICAL INSULATORS BOTT PERIODICITY: TOPOLOGY AND HAMILTONIANS

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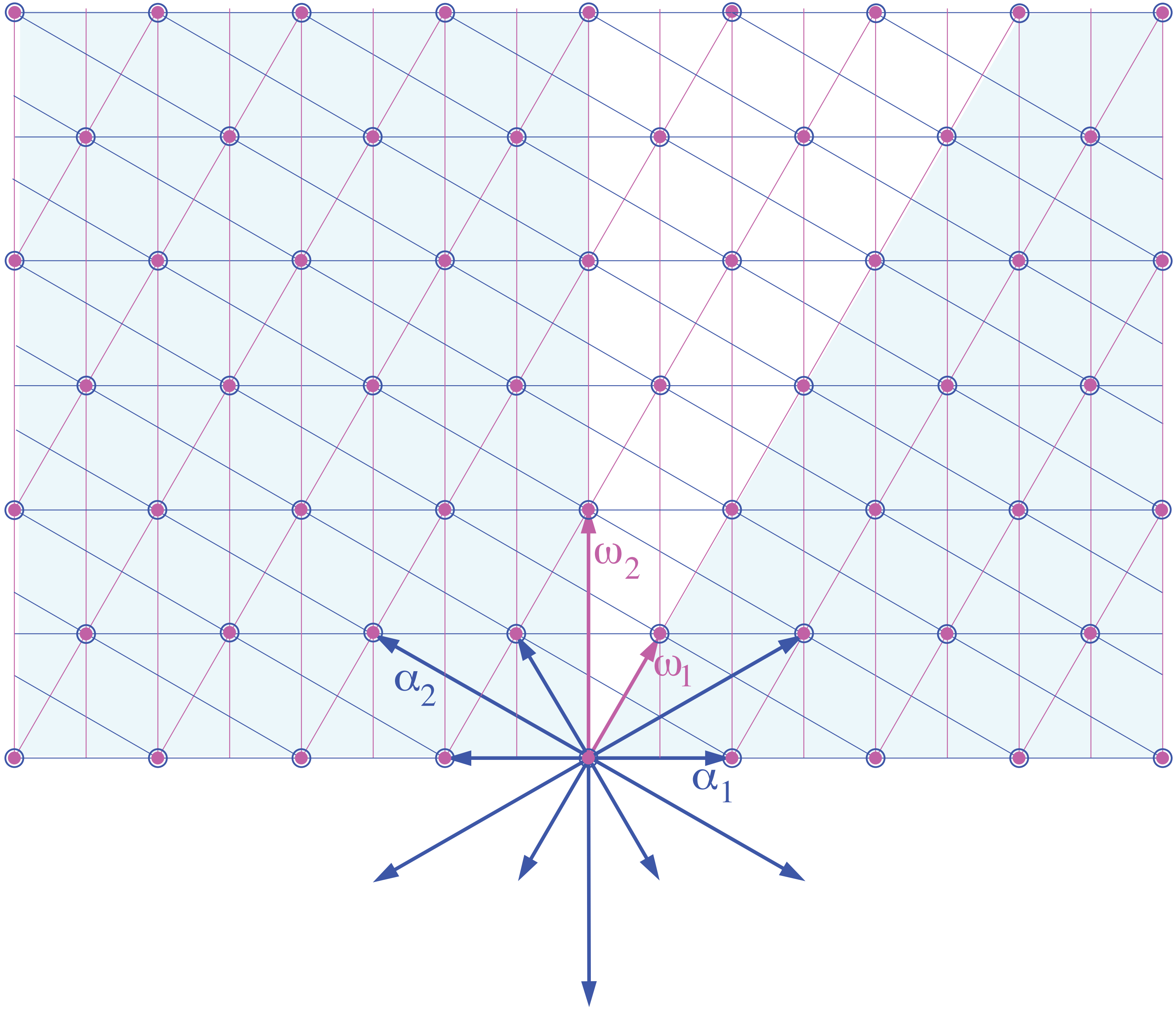
ω_2
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α_2
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Bott Periodicity: Topology and Hamiltonians

Michael Stone

ICTP Trieste, June 2011

- Work done with Ching-Kai Chiu and Abhishek Roy

Stone, Chiu, Roy, J. Phys. A 44, 045001 (2011)

The pattern we wish to understand...

Class	T	C	P	d=0	d=1	d=2	d=3	d=4	d=5	d=6	d=7
AIII	0	0	1	0	Z	0	Z	0	Z	0	Z
A	0	0	0	Z	0	Z	0	Z	0	Z	0
D	0	+1	0	Z_2	Z_2	Z	0	0	0	Z	0
DIII	-1	+1	1	0	Z_2	Z_2	Z	0	0	0	Z
AII	-1	0	0	Z	0	Z_2	Z_2	Z	0	0	0
CII	-1	-1	1	0	Z	0	Z_2	Z_2	Z	0	0
C	0	-1	0	0	0	Z	0	Z_2	Z_2	Z	0
CI	+1	-1	1	0	0	0	Z	0	Z_2	Z_2	Z
AI	+1	0	0	Z	0	0	0	Z	0	Z_2	Z_2
BDI	+1	+1	1	Z_2	Z	0	0	0	Z	0	Z_2

Qi, Hughes, Zhang, PRB 78 195424 (2008)
Schnyder, Ryu, Furusaki, Ludwig, PRB 78 195125 (2008)
Kitaev, AIP conf Proc. 1134, 22 (2009)

- The patterns are usually understood via homotopy:

$$\left.\begin{aligned}\pi_{n+2}(\mathrm{U}(N)) &= \pi_n(\mathrm{U}(N)) \\ \pi_{n+8}(\mathrm{O}(N)) &= \pi_n(\mathrm{O}(N))\end{aligned}\right\}\text{(Bott periodicity)}$$

- Take the point of view that topology is hard, but representation theory is familiar.

Can you see the pattern?

d_{pq}	q=0	1	2	3	4
p=1	2	2	2_2	4	8
2	4	4	4	4_2	8
3	4_2	8	8	8	8_2
4	8	8_2	16	16	16
5	8	16	16_2	32	32
6	8	16	32	32_2	64
7	8_2	16	32	64	64_2
8	16	16_2	32	64	128

	d=0	1	2	3	4
D	Z_2	Z_2	Z	0	0
DIII	0	Z_2	Z_2	Z	0
AII	Z	0	Z_2	Z_2	Z
CII	0	Z	0	Z_2	Z_2
C	0	0	Z	0	Z_2
CIII	0	0	0	Z	0
AI	Z	0	0	0	Z
BDI	Z_2	Z	0	0	0

The numbers are the dimensions of the irreducible representations of real orthogonal matrices J_i obeying

$$J_i J_j + J_j J_i = -2\eta_{ij}$$

$$\eta_{ij} = \mathrm{diag}(\underbrace{+1, \ldots, +1}_{p\,\text{entries}}, \underbrace{-1, \ldots, -1}_{q\,\text{entries}}).$$

- Why real?

Antilinear discrete symmetries

$$\mathcal{C}H\mathcal{C}^{-1} = -H, \quad \mathcal{C}^2 = \pm\mathbb{I}; \qquad \mathcal{T}H\mathcal{T}^{-1} = H, \quad \mathcal{T}^2 = \pm\mathbb{I}.$$

$$CH^*C^{-1} = -H, \quad C^*C = \pm\mathbb{I}, \qquad TH^*T^{-1} = -H, \quad T^*T = \pm\mathbb{I}.$$

- Problem is that, in a vector space over the complex numbers, complex conjugation is not a basis-independent concept.

- Make everything real

A real structure is an antilinear map:

$$\varphi(\lambda\mathbf{x}) = \lambda^*\varphi(\mathbf{x}) \qquad \varphi^2 = \mathrm{Id}$$

•Such a map decomposes a complex vector space into a real vector space of twice the dimension:

$$V = W \oplus_R iW$$

where:

$$W = \{\mathbf{x} \in V : \varphi(\mathbf{x}) = \mathbf{x}\}$$

•Let e_n be a basis for W, then

$$(u_n + iv_n)\mathbf{e}_n \mapsto u_n\mathbf{e}_n + v_n(i\mathbf{e}_n)$$

Recover the complex vector space from the real space by introducing a complex structure.

A complex structure on a real vector space is a linear map

$$J : V \to V, \quad J^2 = -\mathbb{I}$$

$$\varphi \to \begin{pmatrix} 1 & & & & \\ & -1 & & & \\ & & 1 & & \\ & & & -1 & \\ & & & & \ddots \end{pmatrix} \qquad J \to \begin{pmatrix} 0 & -1 & & & \\ 1 & 0 & & & \\ & & 0 & -1 & \\ & & 1 & 0 & \\ & & & & \ddots \end{pmatrix}$$

- Now antilinear φ becomes linear with $\varphi J = -J\varphi$

- The subgroup of $\mathrm{O}(2N)$ that commutes with J

  is: $\mathrm{O}(2N) \cap \mathrm{Sp}(2N, \mathbb{R}) = \mathrm{U}(N)$

$$a + ib \to a \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + b \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$

$$= \begin{pmatrix} a & -b \\ b & a \end{pmatrix}$$

On a complex vector space, an antilinear map

$$\chi(\lambda \mathbf{x}) = \lambda^* \chi(\mathbf{x}), \quad \chi^2 = -\mathbb{I}$$

is a quaternionic structure.

- The subgroup of $\mathrm{U}(2N)$ that commutes with χ is: $\mathrm{U}(2N) \cap \mathrm{Sp}(2N, \mathbb{C}) = \mathrm{Sp}(N) \equiv \mathrm{U}(N, \mathbb{H})$

$$\mathbf{i} \mapsto \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}, \quad \mathbf{j} \mapsto \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \quad \mathbf{k} \mapsto \begin{pmatrix} 0 & -i \\ -i & 0 \end{pmatrix}.$$

(these are 4-by-4 real, skew symmetric, matrices)

- Rename $J \to J_1$ and $\chi \to J_2$
- Now $J_1^2 = J_2^2 = -\mathbb{I}$
- Antilinearity is $J_1 J_2 + J_2 J_1 = 0$
- In other words:

$$\boxed{J_i J_j + J_j J_i = -2\delta_{ij}}$$

Consider more anticommuting "complex structure" matrices:

$$J_i J_j + J_j J_i = -2\delta_{ij}$$

and the subgroups of $\mathrm{O}(16r)$ that commute with them.

We find:

$$\ldots \mathrm{O}(16r) \supset \mathrm{U}(8r) \supset \mathrm{Sp}(4r) \supset \mathrm{Sp}(2r) \times \mathrm{Sp}(2r)$$

$$\supset \mathrm{Sp}(2r) \supset \mathrm{U}(2r) \supset \mathrm{O}(2r) \supset \mathrm{O}(r) \times \mathrm{O}(r) \supset \mathrm{O}(r) \ldots$$

Sequence repeats with period 8: Bott periodicity!

The topologist's view:

Cartan label	R_q	G/H
D	R_1	$\mathrm{O}(16r)\times\mathrm{O}(16r)/\mathrm{O}(16r)\simeq\mathrm{O}(16r)$
DIII	R_2	$\mathrm{O}(16r)/\mathrm{U}(8r)$
AII	R_3	$\mathrm{U}(8r)/\mathrm{Sp}(4r)$
CII	R_4	$\{\mathrm{Sp}(4r)/\mathrm{Sp}(2r)\times\mathrm{Sp}(2r)\}\times\mathbb{Z}$
C	R_5	$\mathrm{Sp}(2r)\times\mathrm{Sp}(2r)/\mathrm{Sp}(2r)\simeq\mathrm{Sp}(2r)$
CI	R_6	$\mathrm{Sp}(2r)/\mathrm{U}(2r)$
AI	R_7	$\mathrm{U}(2r)/\mathrm{O}(2r)$
BDI	R_0	$\{\mathrm{O}(2r)/\mathrm{O}(r)\times\mathrm{O}(r)\}\times\mathbb{Z}$

$$\pi_{n+m}(R_q)=\pi_n(R_{q+m})$$

- We will take an algebraic view

- Let $G_n = \{g \in \mathrm{O}(16r) : gJ_i = J_i g, i = 1, \ldots, n\}$

  $\mathfrak{g}_n = \mathrm{Lie}(G_n)$

- then $\mathfrak{g}_n = \mathfrak{h}_n \oplus \mathfrak{m}_n$

- where $J_{n+1}\mathfrak{h}_n J_{n+1}^{-1} = \mathfrak{h}_n, \quad J_{n+1}\mathfrak{m}_n J_{n+1}^{-1} = -\mathfrak{m}_n$

- we also have that

$$[\mathfrak{h}_n, \mathfrak{h}_n] \in \mathfrak{h}_n, \quad [\mathfrak{h}_n, \mathfrak{m}_n] \in \mathfrak{m}_n, \quad [\mathfrak{m}_n, \mathfrak{m}_n] \in \mathfrak{h}_n.$$

SYMMETRIC SPACE

- The Hamiltonians in the p-th Altland-Zirnbauer class are i times elements of $\mathfrak{m}_{p-2}$

Example: The class AI (Real symmetric matrices) p=7

$$G_5 = \mathrm{U}(2r), \quad G_6 = \mathrm{O}(2r)$$

$$\begin{pmatrix} 0 & a_{12} & a_{13} & \cdots \\ -a_{12} & 0 & a_{23} & \cdots \\ -a_{13} & -a_{23} & 0 & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix} + i \begin{pmatrix} b_{11} & b_{12} & b_{13} & \cdots \\ b_{12} & b_{22} & b_{23} & \cdots \\ b_{13} & b_{23} & b_{33} & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix}$$

$$\mathfrak{h}_5 = \mathfrak{o}(2r) \qquad\qquad \mathfrak{m}_5$$

- The Hamiltonians in the p-th Altland-Zirnbauer class are i times elements of $\mathfrak{m}_{p-2}$

- How can we construct useful matrices in $\mathfrak{m}_n$?

- Answer:

$$\Gamma_i = J_{n+1} J_{n+1+i} \in \mathfrak{m}_n \qquad i = 1, \ldots$$

- Use these matrices to construct representative hamiltonians

What do we know?

$$\Gamma_i = J_{n+1} J_{n+1+i} \in \mathfrak{m}_n \qquad i = 1, \ldots$$

$$\Gamma_i \Gamma_j + \Gamma_j \Gamma_i = -2\delta_{ij}$$

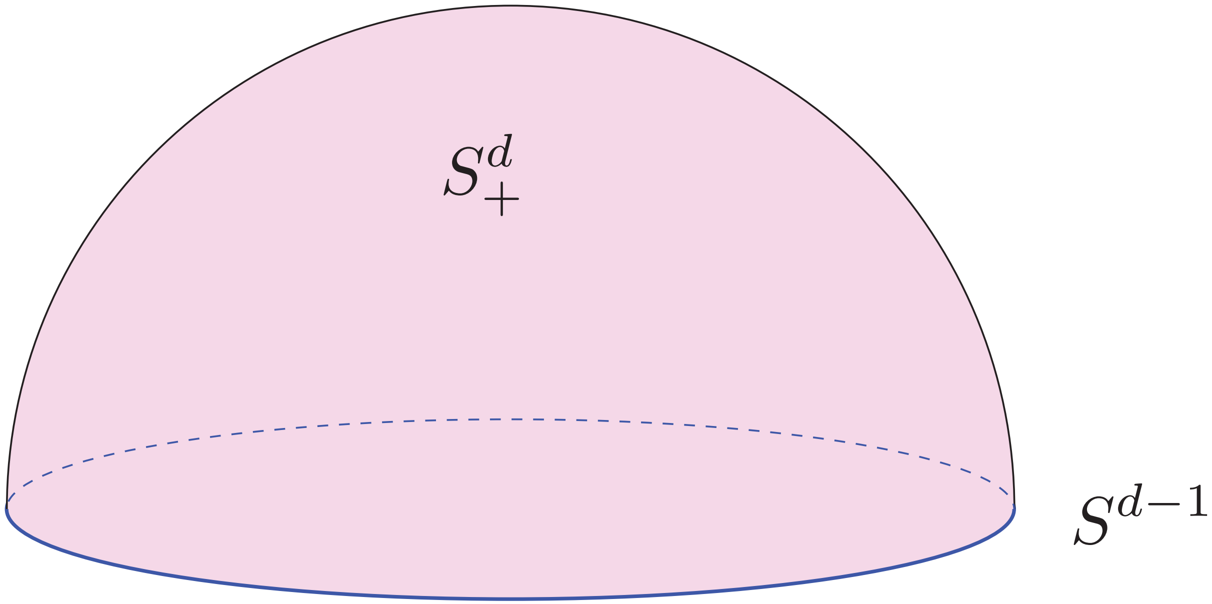
$$H(\mathbf{x}) = i \sum_{i=1}^{d} x_i \Gamma_i, \quad |\mathbf{x}|^2 = 1$$

$$H(\mathbf{x})^2 = \mathbb{I}$$

Topological or trivial?

- Know that $H(\mathbf{x})^2 = \mathbb{I}$
- Look at the $E = -1$ eigenspace for each $\mathbf{x} \in S^{d-1}$
- Is the bundle of eigenspaces over the sphere trivial or not?
- Take irreducible representation of the J_i

- Can we extend bundle from S^{d-1} to S^d_+ ?
- Why? -- A bundle over a contractable space is trivial!



- Need a Γ_{d+1}
- If we can make it with same size matrices: trivial
- If we need bigger matrices: nontrivial

Momentum-like dimensions

$$\mathcal{C}H(\mathbf{k})\mathcal{C}^{-1} = -H(-\mathbf{k}), \quad \mathcal{C}^2 = \pm\mathbb{I},$$

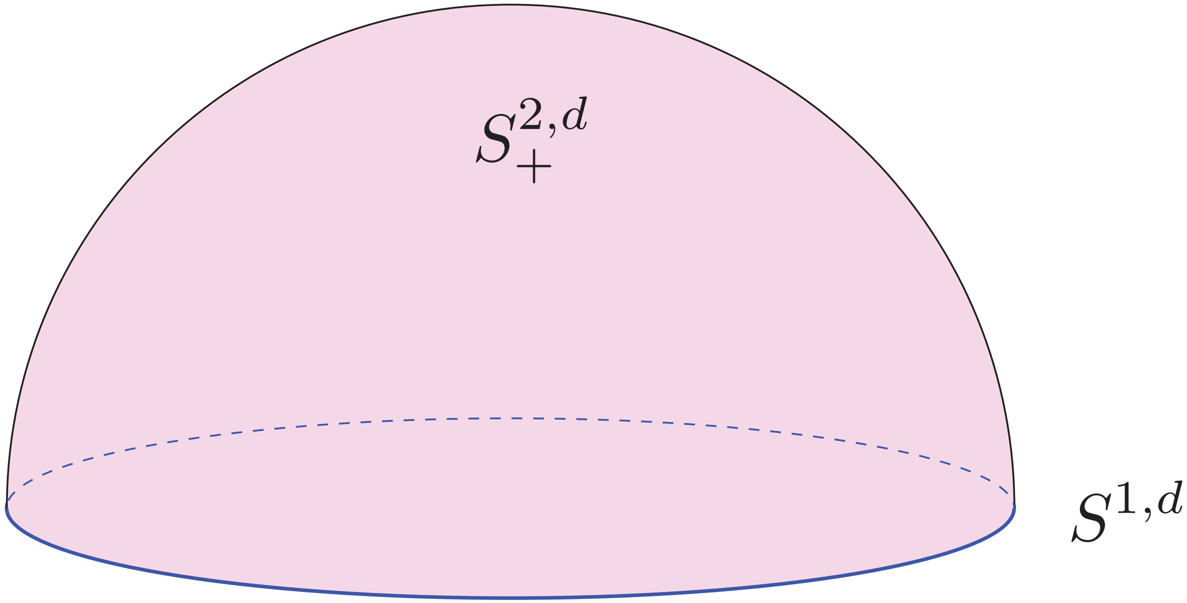
$$\mathcal{T}H(\mathbf{k})\mathcal{T}^{-1} = H(-\mathbf{k}), \quad \mathcal{T}^2 = \pm\mathbb{I}$$

- Introduce $\tilde{J}_i^2 = +\mathbb{I}$ and set $\tilde{\Gamma}_i = J_{n+1}\tilde{J}_{n+1+i}$
- Then $\tilde{\Gamma}_i^2 = +\mathbb{I}$, and can set

$$H(M, \mathbf{k}) = iM\Gamma_1 + \sum_{i=1}^{d} k_i\tilde{\Gamma}_i$$

$$M^2 + |\mathbf{k}|^2 = 1, \quad \Rightarrow (M, \mathbf{k}) \in S^{1,d}$$

- Can we extend bundle from $S^{1,d}$ to $S^{2,d}_{+}$?



- Need a Γ_2 . Not a $\tilde{\Gamma}_{d+1}$
- Extra mass term $\Rightarrow$ trivial

Now do you see the pattern?

d_{pq}	q=0	1	2	3	4		d=0	1	2	3	4
p=1	2	2	2_2	4	8	D	Z_2	Z_2	Z	0	0
2	4	4	4	4_2	8	DIII	0	Z_2	Z_2	Z	0
3	4_2	8	8	8	8_2	AII	Z	0	Z_2	Z_2	Z
4	8	8_2	16	16	16	CII	0	Z	0	Z_2	Z_2
5	8	16	16_2	32	32	C	0	0	Z	0	Z_2
6	8	16	32	32_2	64	CIII	0	0	0	Z	0
7	8_2	16	32	64	64_2	AI	Z	0	0	0	Z
8	16	16_2	32	64	128	BDI	Z_2	Z	0	0	0

- Complex representations

$$J_1 \to i, \qquad J_1 \to -i$$

- Inequivalent

- Real representations

$$J_1 \to \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \quad J_1 \to \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

$$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}^{-1} \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

Equivalent!

Can you see the pattern?

d_{pq}	q=0	1	2	3	4
p=1	2	2	2_2	4	8
2	4	4	4	4_2	8
3	4_2	8	8	8	8_2
4	8	8_2	16	16	16
5	8	16	16_2	32	32
6	8	16	32	32_2	64
7	8_2	16	32	64	64_2
8	16	16_2	32	64	128

	d=0	1	2	3	4
D	Z_2	Z_2	Z	0	0
DIII	0	Z_2	Z_2	Z	0
AII	Z	0	Z_2	Z_2	Z
CII	0	Z	0	Z_2	Z_2
C	0	0	Z	0	Z_2
CIII	0	0	0	Z	0
AI	Z	0	0	0	Z
BDI	Z_2	Z	0	0	0

A white lie

- The sizes of the Γ_i matrices are not those of the J_i
- Why? Because

$$K = J_1 J_2 J_3, \quad M = J_1 J_4 J_5, \quad P = J_1 J_6 J_7$$

mutually commute, obey

$$K^2 = M^2 = P^2 = \mathbb{I}$$

and commute with the Γ_i

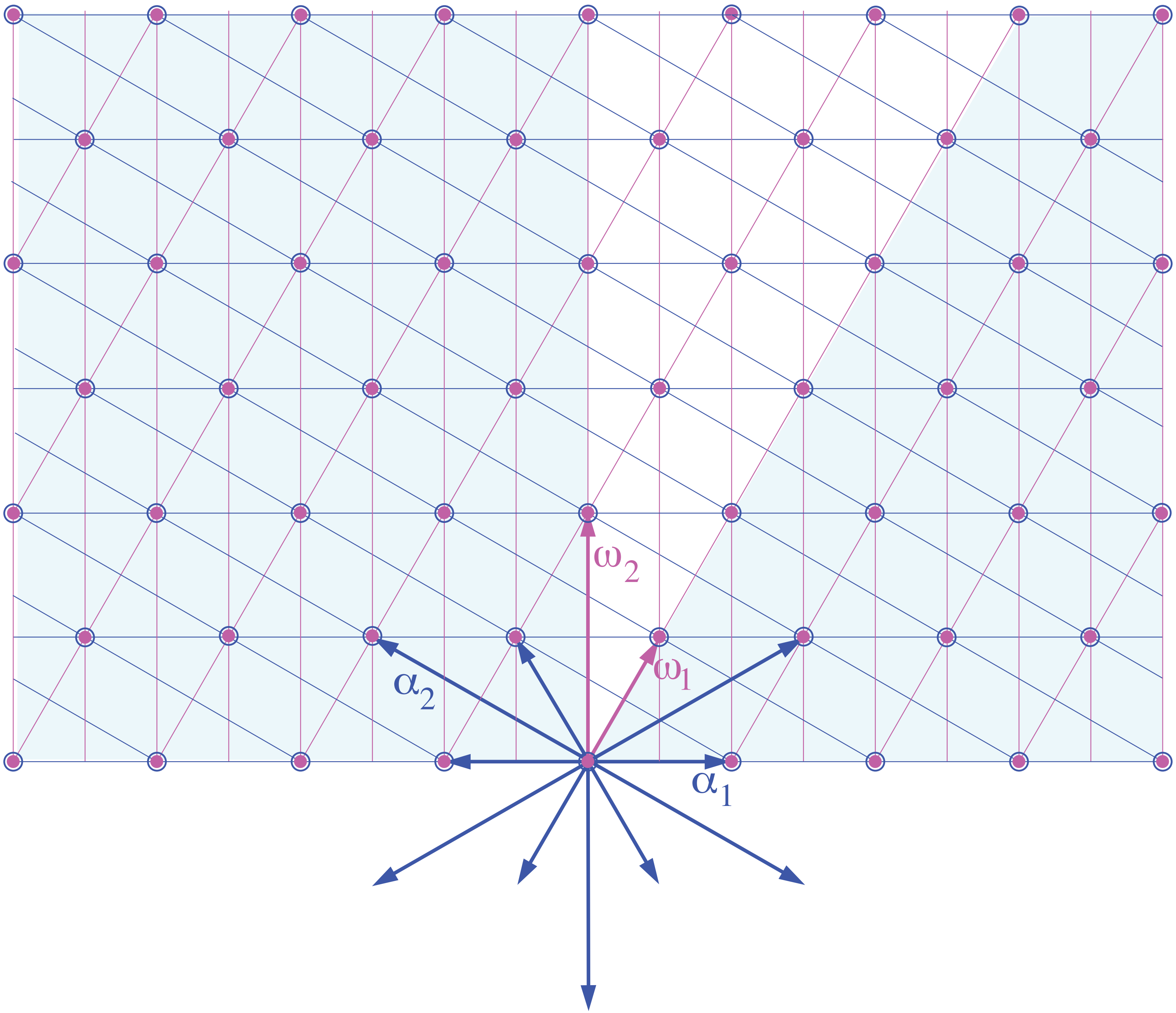
- Need to divide by 2,4 or 8

p	q=0	1	2	3	4			0d	1d	2d	3d	4d
p=1	2	2	2_2	4	8	/1	D	2	2	2_2	4	8
2	4	4	4	4_2	8		DIII	4	4	4	4_2	8
3	4_2	8	8	8	8_2	/2	AII	2_2	4	4	4	4_2
4	8	8_2	16	16	16		CII	4	4_2	8	8	8
5	8	16	16_2	32	32	/4	C	2	4	4_2	8	8
6	8	16	32	32_2	64		CI	2	4	8	8_2	16
7	8_2	16	32	64	64_2	/8	AI	1_2	2	4	8	8_2
8	16	16_2	32	64	128		BDI	2	2_2	4	8	16

Each symmetry class w/ or w/o an extra mass term is divided by the same number

Conclusions

- Can understand the periodic table from simple representation theory
- Do need to work with real representations (a bit harder than complex representation theory)
- No need to understand K-theory first (but it helps)



ω2
α2
ω1
α1