Introduction to Isentropic Coordinates:

a new view of mean meridional & eddy circulations

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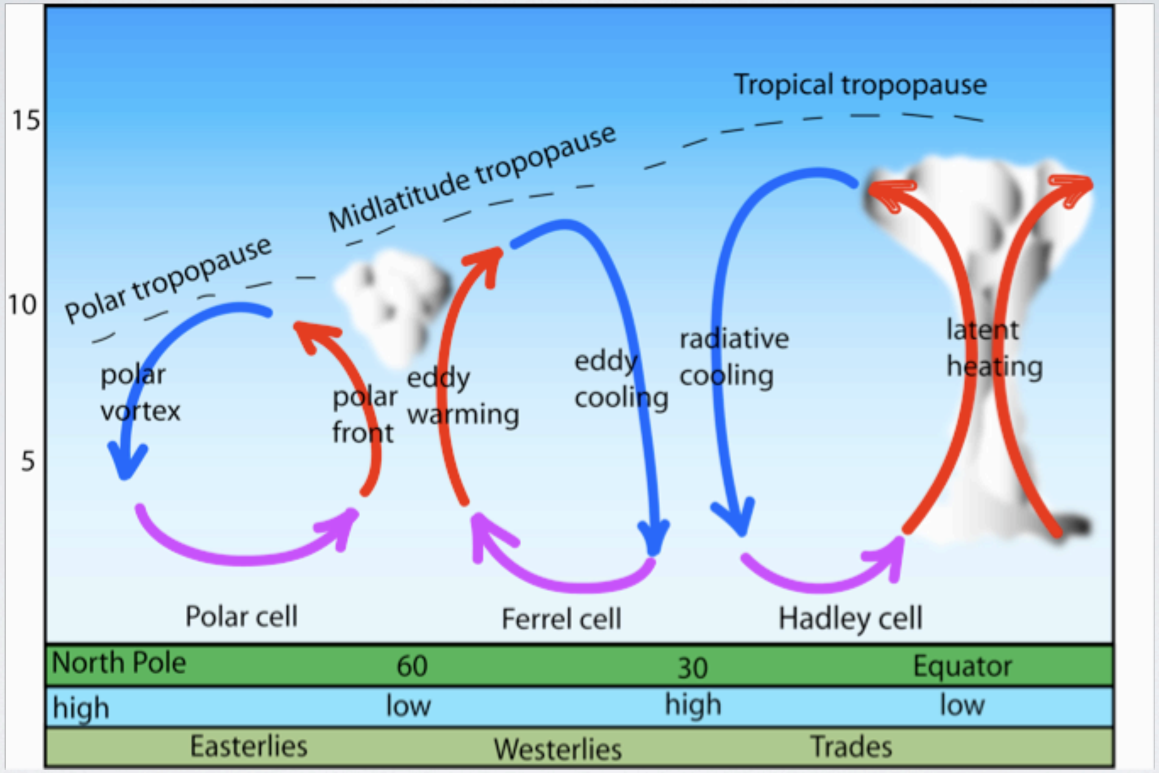
Mean Meridional Circulation

Mechanism for the meridional transports of energy and moisture in the atmosphere



What mathematical model do we use to represent these observed features?

- The heating processes at work in the atmosphere are very complicated, and motion-dependent. The heating is very closely related to moist processes, including cloud formation and precipitation, radiation, and diffusion.
- The response of the atmospheric circulation to the heating is complicated because of the existence of eddies and their interaction with the mean flow. These eddies are neither purely random, nor purely regular.

Introduction

The vertical stratification of the atmosphere can be represented using various coordinates such as:

- **physical height, z -coordinate: height above the Earth's surface**
- **pressure, p -coordinate: atmospheric pressure**
- **sigma, $\sigma = p/p_s$; where p is the air pressure and p_S is the surface-air pressure**
- **potential temperature, $\theta = T\left(\frac{p_0}{p}\right)^{\kappa}$; where T is the air temperature, p_0** = 100 kPa**, and** $\kappa = R/c_p$.

Review of Potential Temperature Properties

✦ Physical Interpretation

θ **is the temperature of a material element would have if it were** *adiabatically** **expanded (for** $p > p_0$**) or compressed (for** $p < p_0$**) to the reference pressure** p_0**.**

✦ Materially conserved for adiabatic flow

$$\frac{\mathbf{D}\theta}{\mathbf{Dt}} = \mathbf{0}$$

Considering air to be an ideal gas, so that obeys the ideal gas law $p = \alpha RT$**, then the first law of thermodynamics can be written:**

$$c_p \frac{DT}{Dt} - \alpha \frac{Dp}{Dt} = Q \qquad \textbf{(2)}$$

*without heating

$$\frac{c_p}{T}\frac{DT}{Dt} - \frac{\alpha}{T}\frac{Dp}{Dt} = \frac{Q}{T} \qquad \textbf{(2)}$$

using the logarithm's property $\frac{1}{f}\frac{df}{dx} = \frac{d(\ln f)}{dx}$**, (2) can be written as**

$$c_p \frac{D}{Dt}(\ln T - \kappa \ln p) = \frac{Q}{T} \qquad \textbf{(3)}$$

and we can write (3) as

$$c_p \frac{D}{Dt}\left[\ln T\left(\frac{p_0}{p}\right)^{\kappa}\right] = \frac{Q}{T} \qquad \textbf{(4)}$$

Written in terms of potential temperature, (4) becomes

$$c_p \frac{D\theta}{Dt} = \frac{Q}{T}$$

When $Q = 0$**, the flow is termed adiabatic and** $D\theta/DT = 0$**. Thus the potential temperature is** *materially conserved** **for adiabatic flow.**

$$^{*}\ \frac{D}{Dt}() = \frac{\partial}{\partial t}() + \vec{V}\cdot\nabla()$$

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Transformation of the Quasi-static Primitive Equation to Isentropic Coordinate

Using the longitude λ, the latitude ϕ, and the physical height z as the independent spatial coordinates, the quasi-static primitive equations for inviscid, adiabatic flow are:

$$\frac{Du}{Dt} - \left(2\Omega \sin\phi + \frac{u \tan\phi}{a}\right) v + \frac{1}{\rho}\left(\frac{\partial p}{a\cos\phi \partial\lambda}\right)_z = 0 \qquad \textbf{(1)}$$

$$\frac{Dv}{Dt} + \left(2\Omega \sin\phi + \frac{u \tan\phi}{a}\right) u + \frac{1}{\rho}\left(\frac{\partial p}{a \partial\phi}\right)_z = 0 \qquad \textbf{(2)}$$

$$g + \frac{1}{\rho}\frac{\partial p}{\partial z} = 0 \qquad \textbf{(3)}$$

$$\frac{1}{\rho}\frac{D\rho}{Dt} + \left(\frac{\partial u}{a\cos\phi \partial\lambda}\right)_z + \left(\frac{(v\cos\phi)}{a\cos\phi \partial\phi)}\right)_z + \frac{\partial w}{\partial z} = 0 \qquad \textbf{(4)}$$

$$c_p \frac{DT}{Dt} - \frac{1}{\rho}\frac{Dp}{Dt} = 0 \qquad (5)$$

$$p = \rho R T \qquad (6)$$

with the material derivative:

$$\frac{D}{Dt} = \left(\frac{\partial}{\partial t}\right)_z + u\left(\frac{\partial}{a\cos\phi\partial\lambda}\right)_z + v\left(\frac{\partial}{a\partial\phi}\right)_z + w\frac{\partial}{\partial z} \qquad (7)$$

Transformation to $(\lambda, \phi, \theta, t)$ coordinates:

1.Material Derivative (7)

$$\frac{D}{Dt} = \left(\frac{\partial}{\partial t}\right)_z + u\left(\frac{\partial}{a\cos\phi\partial\lambda}\right)_z + v\left(\frac{\partial}{a\partial\phi}\right)_z + w\frac{\partial}{\partial z}$$

$$\left(\frac{\partial}{\partial t}\right)_z = \left(\frac{\partial}{\partial t}\right)_\theta + \left(\frac{\partial\theta}{\partial t}\right)_z \frac{\partial}{\partial\theta}$$

$$\left(\frac{\partial}{a\cos\phi\partial\lambda}\right)_z = \left(\frac{\partial}{a\cos\phi\partial\lambda}\right)_\theta + \left(\frac{\partial\theta}{a\cos\phi\partial\lambda}\right)_z \frac{\partial}{\partial\theta}$$

$$\left(\frac{\partial}{a\partial\phi}\right)_z = \left(\frac{\partial}{a\partial\phi}\right)_\theta + \left(\frac{\partial\theta}{a\partial\phi}\right)_z \frac{\partial}{\partial\theta}$$

$$\frac{\partial}{\partial z} = \frac{\partial\theta}{\partial z}\frac{\partial}{\partial\theta}$$

$$\boxed{\frac{D}{Dt} = \left(\frac{\partial}{\partial t}\right)_\theta + u\left(\frac{\partial}{a\cos\phi\partial\lambda}\right)_\theta + v\left(\frac{\partial}{a\partial\phi}\right)_\theta + \dot{\theta}\frac{\partial}{\partial\theta}}$$

2.Continuity Equation (4)

$$\frac{1}{\rho}\frac{D\rho}{Dt}+\left(\frac{\partial u}{a\cos\phi\partial\lambda}\right)_z+\left(\frac{(v\cos\phi)}{a\cos\phi\partial\phi)}\right)_z+\frac{\partial w}{\partial z}=0$$

$$\left(\frac{\partial u}{a\cos\phi\partial\lambda}\right)_z=\left(\frac{\partial u}{a\cos\phi\partial\lambda}\right)_\theta+\left(\frac{\partial\theta}{a\cos\phi\partial\lambda}\right)_z\frac{\partial u}{\partial\theta}$$

$$\left(\frac{\partial(v\cos\phi)}{a\partial\phi}\right)_z=\left(\frac{\partial(v\cos\phi)}{a\partial\phi}\right)_\theta+\left(\frac{\partial\theta}{a\partial\phi}\right)_z\frac{\partial v}{\partial\theta}$$

To transform $\partial w/\partial z$ **we first note that**

$$\frac{\partial\dot\theta}{\partial\theta}=\frac{\partial}{\partial\theta}\left(\frac{D\theta}{Dt}\right)=\frac{\partial z}{\partial\theta}\frac{\partial}{\partial z}\left\{\left(\frac{\partial\theta}{\partial t}\right)_z+u\left(\frac{\partial\theta}{a\cos\phi\partial\lambda}\right)_z+v\left(\frac{\partial\theta}{a\partial\phi}\right)_z+w\frac{\partial\theta}{\partial z}\right\}$$

$$\frac{\partial w}{\partial z}=\frac{\partial\dot\theta}{\partial\theta}-\frac{\partial z}{\partial\theta}\frac{D}{Dt}\left(\frac{\partial\theta}{\partial z}\right)-\frac{\partial u}{\partial\theta}\left(\frac{\partial\theta}{a\cos\phi\partial\lambda}\right)_z-\frac{\partial v}{\partial\theta}\left(\frac{\partial\theta}{a\partial\phi}\right)_z$$

To transform $D \ln \rho / Dt$ we first note that hydrostatic equation (3)

$$\rho = -\frac{1}{g}\frac{\partial p}{\partial z} = -\frac{1}{g}\frac{\partial p}{\partial \theta}\frac{\partial \theta}{\partial z} = \sigma \left|\frac{\partial \theta}{\partial z}\right|$$

where the pseudo-density, σ is defined by

$$\sigma = \frac{1}{g}\left|\frac{\partial p}{\partial \theta}\right|$$

$$\frac{D \ln \rho}{Dt} = \frac{D \ln \sigma}{Dt} + \frac{D \ln |\partial \theta / \partial z|}{Dt}$$

$$\frac{1}{\sigma}\frac{D\sigma}{Dt} + \left(\frac{\partial u}{a \cos \phi \partial \lambda}\right)_\theta + \left(\frac{\partial (v \cos \phi)}{a \cos \phi \partial \phi}\right)_\theta + \frac{\partial \dot{\theta}}{\partial \theta} = 0$$

3.Pressure Gradient Force $(1/\rho)\nabla_z p$

$$\frac{1}{\rho}\left(\frac{\partial p}{\partial \lambda}\right)_z = \frac{1}{\rho}\left(\frac{\partial p}{\partial \lambda}\right)_\theta + \frac{1}{\rho}\left(\frac{\partial \theta}{\partial \lambda}\right)_z \left(\frac{\partial p}{\partial \theta}\right)$$

$$= \frac{1}{\rho}\left(\frac{\partial p}{\partial \lambda}\right)_\theta - \frac{1}{\rho}\left(\frac{\partial p}{\partial z}\right)\left(\frac{\partial z}{\partial \lambda}\right)_\theta$$

$$= \frac{1}{\rho}\left(\frac{\partial p}{\partial \lambda}\right)_\theta + \left(\frac{\partial (gz)}{\partial \lambda}\right)_\theta$$

using definition of potential temperature we can write

$$\underbrace{\nabla_\theta \ln \theta}_{=0} = \nabla_\theta \ln T - \frac{R}{C_p}\nabla_\theta \ln p$$

using the ideal gas law $p = \rho R T$ **we can write**

$$\frac{1}{\rho}\left(\frac{\partial p}{\partial \lambda}\right)_\theta = C_p \left(\frac{\partial T}{\partial \lambda}\right)_\theta$$

$$\frac{1}{\rho}\left(\frac{\partial p}{\partial \lambda}\right)_z = \left(\frac{\partial (C_p T + gz)}{\partial \lambda}\right)_\theta = \left(\frac{\partial M}{\partial \lambda}\right)_\theta$$

where $M = C_p T + gz = C_p T + \Phi$ **is the Montgomery potential.**

$$\frac{1}{\rho}\nabla_z p = \nabla_\theta M$$

4.Hydrostatic equation (3)

$$g + \frac{1}{\rho}\frac{\partial p}{\partial z} = 0$$

$$\frac{\partial M}{\partial \theta} = \frac{\partial}{\partial \theta}(C_p T + gz)$$

$$= C_p \frac{\partial T}{\partial \theta} + g\frac{\partial z}{\partial \theta}$$

$$= C_p \left(\frac{p}{p_0}\right)^{\kappa}$$

$$\frac{\partial M}{\partial \theta} = \Pi$$

where $\Pi = C_p\,(p/p_0)^{\kappa}$ **is the Exner function.**

The quasi-static primitive equations in isentropic coordinates

$$\frac{Du}{Dt} - \left(2\Omega \sin\phi + \frac{u\tan\phi}{a}\right)v + \left(\frac{\partial M}{a\cos\phi\partial\lambda}\right)_\theta = 0$$

$$\frac{Dv}{Dt} + \left(2\Omega \sin\phi + \frac{u\tan\phi}{a}\right)u + \left(\frac{\partial M}{a\partial\phi}\right)_\theta = 0$$

$$\frac{\partial M}{\partial\theta} = \Pi$$

$$\frac{1}{\sigma}\frac{D\sigma}{Dt} + \left(\frac{\partial u}{a\cos\phi\partial\lambda}\right)_\theta + \left(\frac{\partial(v\cos\phi)}{a\cos\phi\partial\phi}\right)_\theta + \frac{\partial\dot{\theta}}{\partial\theta} = 0$$

where σ is the pseudo-density, M the Montgomery potential, Π the Exner function, and

$$\frac{D}{Dt} = \left(\frac{\partial}{\partial t}\right)_\theta + u\left(\frac{\partial}{a\cos\phi\partial\lambda}\right)_\theta + v\left(\frac{\partial}{a\partial\phi}\right)_\theta + \dot{\theta}\frac{\partial}{\partial\theta}$$

Zonal Mean

$$[A] = \frac{1}{2\pi}\int_0^{2\pi} A(\lambda)d\lambda$$

Departure from the Zonal Mean or

$$A^* = A - [A]$$

Statistics of interest

$$[A^*] = 0$$

$$\frac{1}{2\pi}\int_0^{2\pi}\left(\frac{\partial A}{\partial \lambda}\right)d\lambda = 0$$

$$\underbrace{[vA]}_{\text{total flux}} = \underbrace{[A][v]}_{\text{symmetric circulation}} + \underbrace{[A^*v^*]}_{\text{flux due to eddies}}$$

Eddies

alternating trains of low and high pressure systems moving in circular motions in the westerly flow

push the warm air from the subtropics poleward and cool air from high latitudes equatorward; the net effect is a reduction of equator-to-pole temperature gradient

transport also momentum and the eddy momentum flux influences the zonal mean temperature through the thermal wind balance that dominates the middle latitudes.

The Zonal Mean Equations in Isentropic Coordinates

Transform the quasi-static primitive equations in isentropic coordinates

$$\frac{Du}{Dt} - \left(2\Omega \sin\phi + \frac{u \tan\phi}{a}\right) v + \left(\frac{\partial M}{a \cos\phi \partial\lambda}\right)_\theta = 0$$

$$\frac{Dv}{Dt} + \left(2\Omega \sin\phi + \frac{u \tan\phi}{a}\right) u + \left(\frac{\partial M}{a \partial\phi}\right)_\theta = 0$$

$$\frac{\partial M}{\partial\theta} = \Pi$$

$$\frac{D\sigma}{Dt} + \sigma \left\{ \left(\frac{\partial u}{a \cos\phi \partial\lambda}\right)_\theta + \left(\frac{\partial(v \cos\phi)}{a \cos\phi \partial\phi}\right)_\theta + \frac{\partial\dot{\theta}}{\partial\theta} \right\} = 0$$

in flux form

$$\frac{\partial}{\partial t}(\sigma u) + \frac{1}{a\cos\phi}\frac{\partial}{\partial\lambda}(\sigma uu) + \frac{1}{a\cos^2\phi}\frac{\partial}{\partial\phi}(\sigma uv\cos^2\phi) - 2\Omega\sigma v\sin\phi$$

$$= -\frac{\sigma}{a\cos\phi}\left(\frac{\partial M}{\partial\lambda}\right) - \frac{\partial}{\partial\theta}(\sigma u\dot{\theta})$$

$$\sigma\left(\frac{\partial M}{\partial\lambda}\right) = -\frac{1}{g}\frac{\partial p}{\partial\theta}\frac{\partial M}{\partial\lambda} = -\frac{\partial}{\partial\theta}\left(p\frac{\partial M}{\partial\lambda}\right) - \frac{p}{g}\left(\frac{\partial p}{\partial\lambda}\right)\frac{d\Pi}{dp}$$

$$\frac{\partial}{\partial t}(\sigma u) + \frac{1}{a\cos\phi}\frac{\partial}{\partial\lambda}(\sigma uu) + \frac{1}{a\cos^2\phi}\frac{\partial}{\partial\phi}(\sigma uv\cos^2\phi) - 2\Omega\sigma v\sin\phi$$

$$= -\frac{1}{a\cos\phi}\left\{\frac{1}{g}\frac{\partial}{\partial\theta}\left(p\frac{\partial M}{\partial\lambda}\right) + \frac{\partial\Gamma(p)}{\partial\lambda}\right\} - \frac{\partial}{\partial\theta}(\sigma u\dot{\theta})$$

$$\frac{\partial\sigma}{\partial t} + \frac{\partial(\sigma u)}{a\cos\phi\partial\lambda} + \frac{\partial(\sigma v\cos\phi)}{a\cos\phi\partial\phi} + \frac{\partial(\sigma\dot{\theta})}{\partial\theta} = 0$$

$$\frac{\partial}{\partial t}[\sigma u] + \frac{1}{a\cos^2\phi}\frac{\partial}{\partial\phi}[\sigma uv\cos^2\phi] - 2\Omega[\sigma v]\sin\phi + \frac{\partial}{\partial\theta}[\sigma u\dot{\theta}] = -\frac{1}{ga\cos\phi}\frac{\partial}{\partial\theta}\left[p\frac{\partial M}{\partial\lambda}\right]$$

$$\frac{\partial[\sigma]}{\partial t} + \frac{\partial([\sigma v]\cos\phi)}{a\cos\phi\partial\phi} + \frac{\partial[\sigma\dot{\theta}]}{\partial\theta} = 0$$

Define a "mass-weighted zonal mean" $\hat{A} \equiv \frac{[\sigma A]}{[\sigma]}$

$$\frac{\partial}{\partial t}[u] + \frac{\hat{v}}{a\cos\phi}\frac{\partial}{\partial\phi}([u]\cos\phi) - 2\Omega\hat{v}\sin\phi + \hat{\dot{\theta}}\frac{\partial[u]}{\partial\theta} = -\frac{1}{[\sigma]}\frac{\partial}{\partial t}[\sigma^* u^*]$$

$$-\frac{1}{[\sigma]a\cos^2\phi}\frac{\partial}{\partial\phi}([(\sigma v)^* u^*]cos^2\phi)+$$

$$\frac{1}{ga\cos\phi}\frac{\partial}{\partial\theta}\left[p^*\frac{\partial M^*}{\partial\lambda}\right] - \frac{1}{[\sigma]}\frac{\partial}{\partial\theta}[(\sigma\dot{\theta})^* u^*]$$

$$F_\phi = -a\cos\phi[(\sigma v)^* u^*] \qquad F_\theta = \frac{1}{g}\left[p^*\frac{\partial M^*}{\partial\lambda}\right] - a\cos\phi[(\sigma\dot{\theta})^* u^*]$$

$$\frac{\partial}{\partial t}[u] + \frac{\hat{v}}{a\cos\phi}\frac{\partial}{\partial \phi}[u\cos\phi] - 2\Omega\hat{v}\sin\phi + \hat{\dot{\theta}}\frac{\partial [u]}{\partial \theta} =$$

$$-\frac{1}{[\sigma]}\frac{\partial}{\partial t}[\sigma^* u^*] - \frac{1}{[\sigma]a\cos\phi}\nabla\cdot\mathbf{F}$$

$$\frac{\partial[\sigma]}{\partial t} + \frac{\partial([\sigma]\hat{v}\cos\phi)}{a\cos\phi\partial\phi} + \frac{\partial([\sigma]\hat{\dot{\theta}})}{\partial\theta} = 0$$

where $\nabla\cdot\mathbf{F}$ **denotes the divergence of the Eliassen-Palm flux,** $\mathbf{F}$**.**

$\nabla\cdot\mathbf{F}$

Mean Meridional Circulation

$$\frac{\partial([\sigma]\hat{v}\cos\phi)}{a\cos\phi\partial\phi} = -\frac{\partial([\sigma]\hat{\dot{\theta}})}{\partial\theta}$$

- **heating induces a meridional circulation, i.e., $[\dot{\theta}] \neq 0$implies that $[v] \neq 0$.**

- **meridional mass convergence (divergence) requires an increase (decrease) of the upward mass flux with height in regions of cooling.**

Mass stream function:

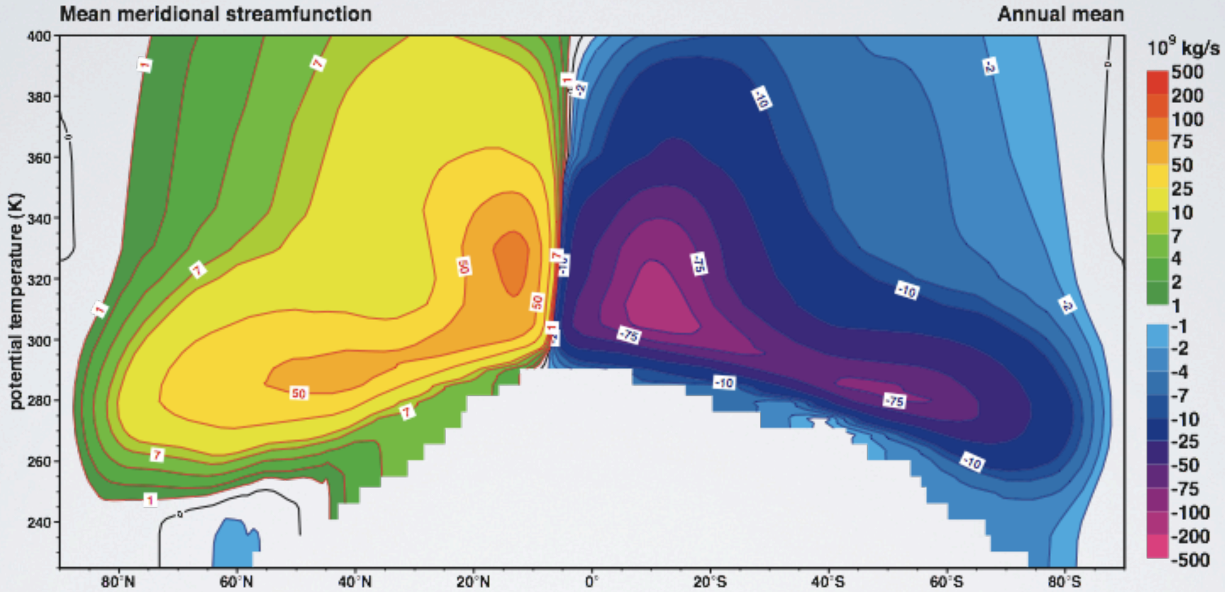
$$\Psi_\theta = -2\pi\cos\phi\int_{\theta_s}^{\theta}[\sigma]\hat{v}d\theta = -2\pi\cos\phi\int_{\theta_s}^{\theta}[\sigma v]d\theta$$

which satisfies the equations:

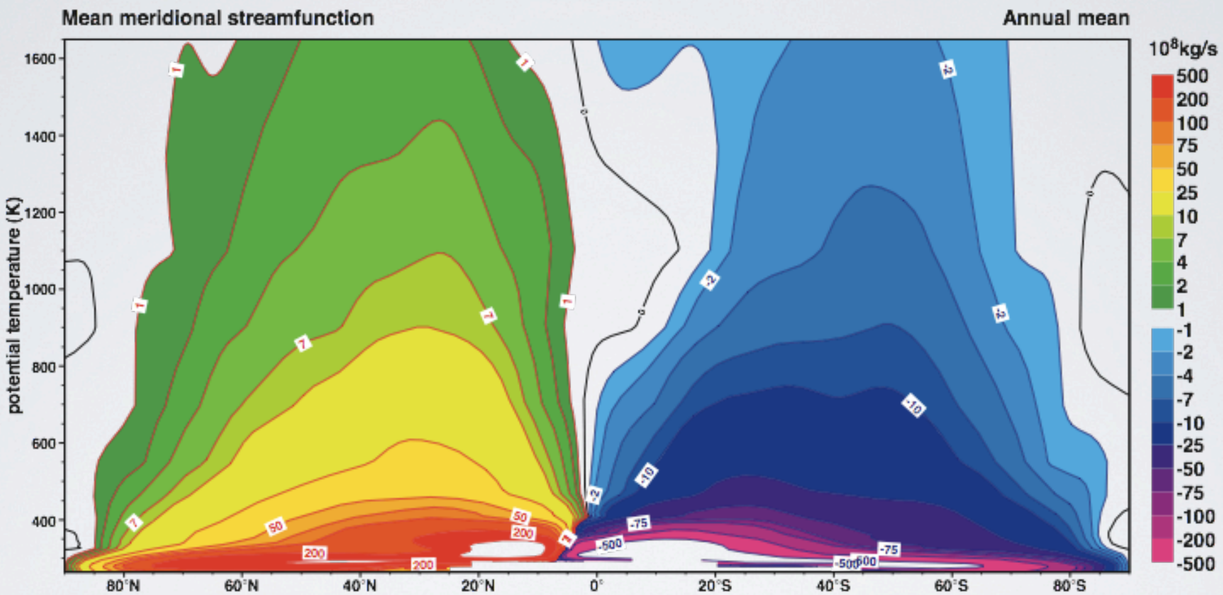
$$\frac{\partial\Psi_\theta}{\partial\theta} = -2\pi a\cos\phi[\sigma]\hat{v}$$

$$\frac{\partial\Psi_\theta}{\partial\phi} = 2\pi a^2\cos\phi[\sigma]\hat{\dot{\theta}}$$

Mean Meridional Mass Stream Function, Tropospheric Perspective



Mean Meridional Mass Stream Function, Stratospheric Perspective



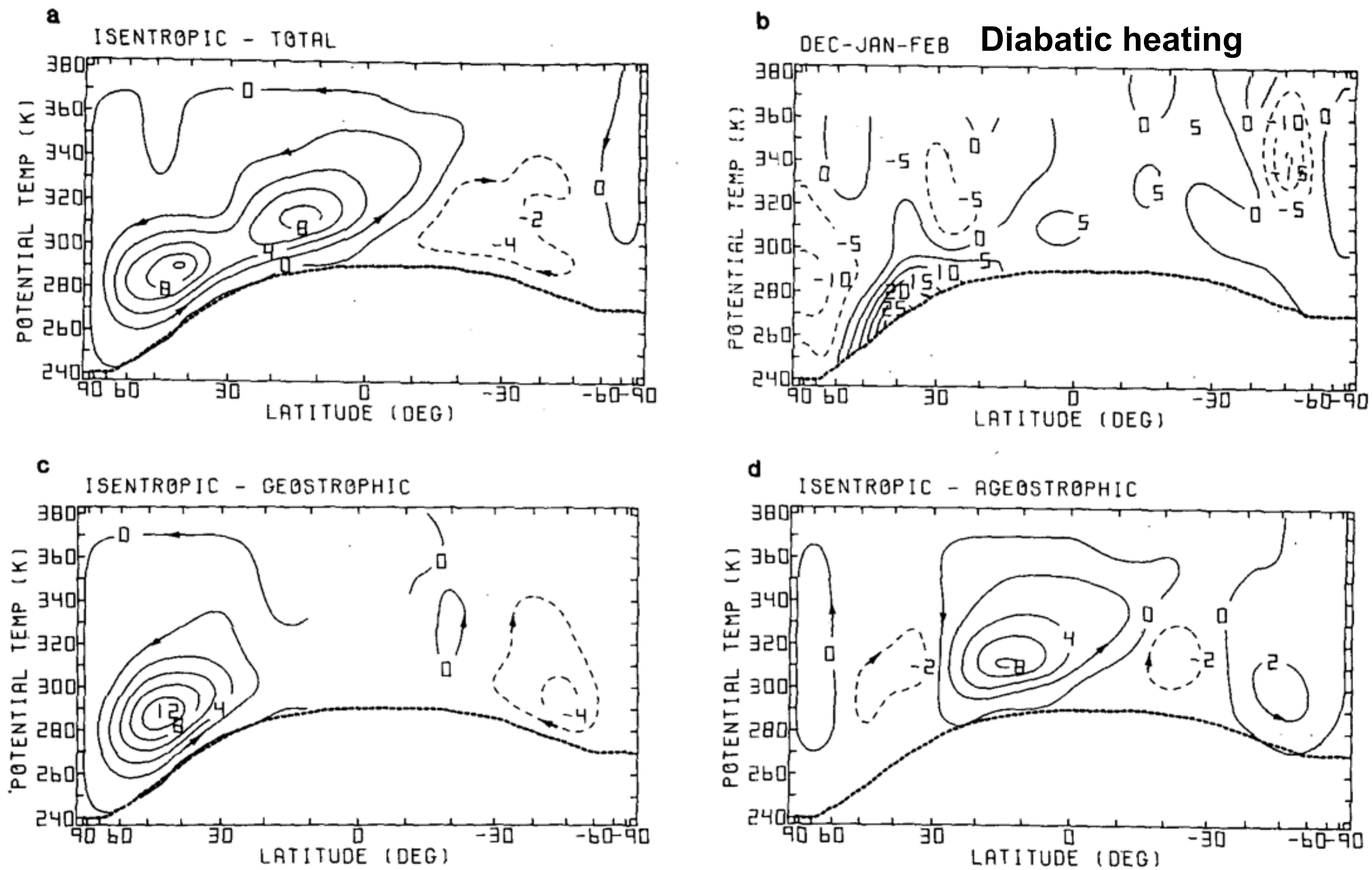


FIG. 7. Mass streamfunction for the (a) isentropic total, (c) geostrophic and (d) ageostrophic mean meridional circulations (10^{10} kg s^{-1}) and (b) meridional cross section of mass-weighted time- and zonally averaged diabatic heating (10^{-1} K day^{-1}) for December 1978–February 1979. Arrows indicate the direction of the circulation.

$$F_\phi = -a\cos\phi[(\sigma v)^* u^*]$$ **barotropic wave**

$$F_\theta = \frac{1}{g}\left[p^* \frac{\partial M^*}{\partial \lambda}\right] - a\cos\phi[(\sigma\dot{\theta})^* u^*]$$

baroclinic wave

$$\frac{1}{g}\left[p^* \frac{\partial M^*}{\partial \lambda}\right]$$ **pressure torque exerted by the fluid above an isentrope on that below.**