The Abdus Salam
International Centre for Theoretical Physics

United Nations
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2253-13

Workshop on Synergies between Field Theory and Exact Computational Methods in Strongly Correlated Quantum Matter

24 - 29 July 2011

Entanglement spectrum and boundary theories with projected entangled-pair states

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Entanglement spectrum and boundary theories with projected entangled-pair states

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OUTLINE

- Some motivations

- Entanglement concepts & tools for studying many-body systems

- Entanglement spectra of Heisenberg ladder

  D.P., PRL **105**, 077202 (2010)

- Boundary Hamiltonians for Heisenberg ladders and with PEPS - "Holographic Principle"

  - 2D AKLT
  - 2D transverse field Ising
  - Kitaev code (LRE)

COLLABORATORS

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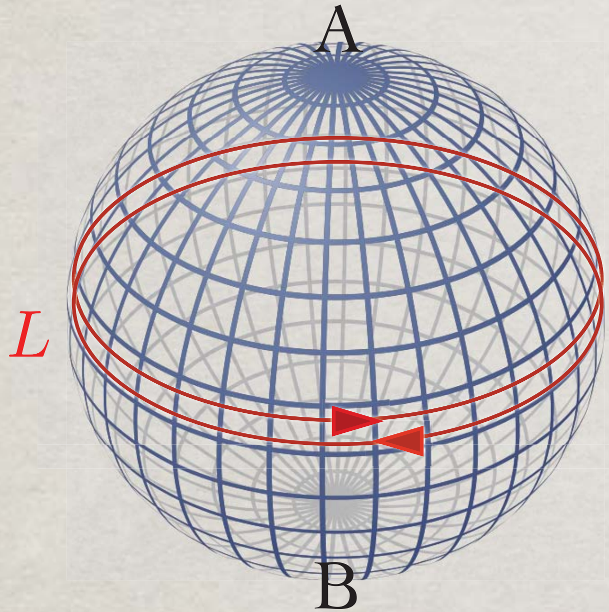
* Norbert Schuch
(Cal. Tech.)

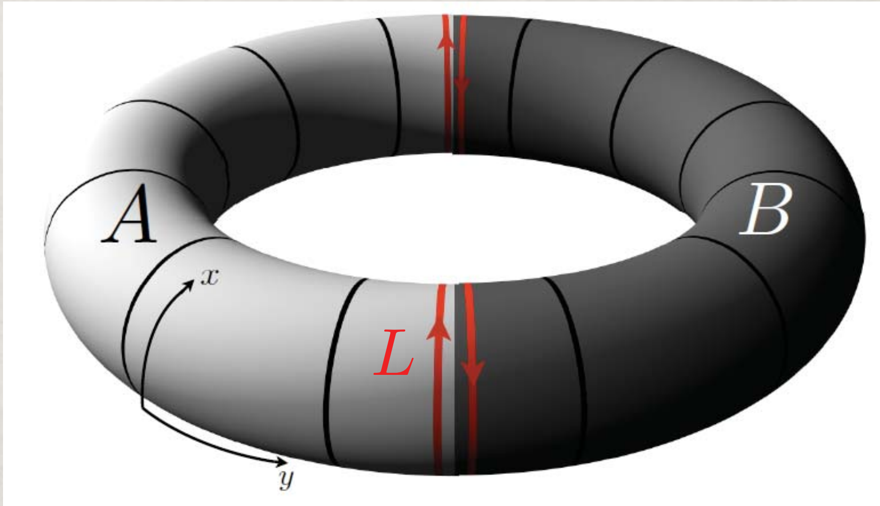
* Frank Vertraete
(Univ. Vienna)

Phys. Rev. B 83, 245134 (2011)

Boundaries in Condensed Matter Systems



Li & Haldane
PRL 2008



Lauchli et al., 2009

Edge states in (topological) FQH systems

Also topological insulators, etc...

Entanglement Concepts:

The basis of Quantum Information Science

Historically associated to seminal work:

Einstein A, Podolsky B, Rosen N (1935). "Can Quantum-Mechanical Description of Physical Reality Be Considered Complete?". *Phys. Rev.* **47** (10): 777–780.

Schrödinger E (1935). "Discussion of probability relations between separated systems". *Mathematical Proceedings of the Cambridge Philosophical Society* **31** (04): 555–563.

J. S. Bell (1964). "On the Einstein- Poldolsky-Rosen paradox"

Entangled states as a new “non-classical” ressource:

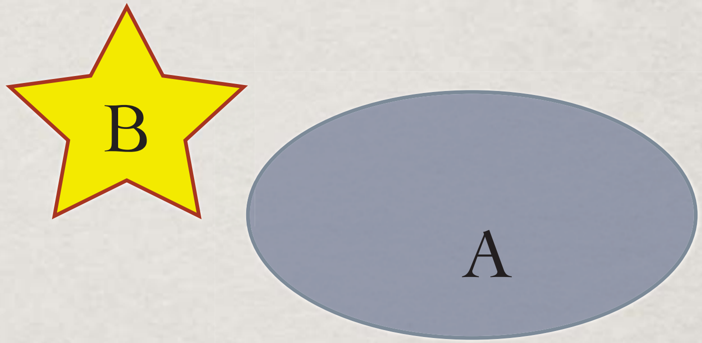
- Quantum information
- Quantum computing
- Quantum Cryptography

Powerful tools to understand correlated systems ?

Entanglement Concept

Hilbert space:

$$\mathcal{E}_A \otimes \mathcal{E}_B$$



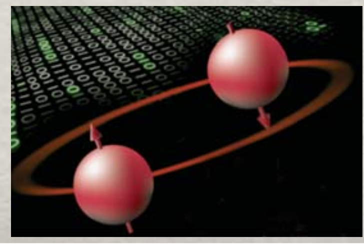
$$|\Psi\rangle_A \otimes |\Phi\rangle_B$$ separable state

More general case: **entangled state**

$$|\Psi\rangle_{AB} = \sum_{i,j} c_{ij}\, |i\rangle_A \otimes |j>_B$$

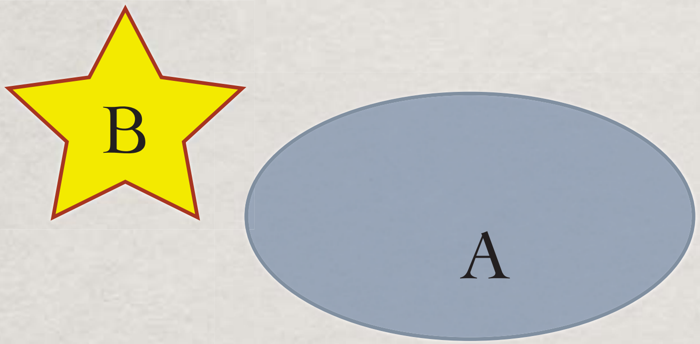
Exemple: $\frac{1}{\sqrt{2}}(|\uparrow\rangle_A \otimes |\downarrow\rangle_B - |\downarrow\rangle_A \otimes |\uparrow\rangle_B)$ singlet state !

Maximally entangled state
(equiv. to Bell state for qubits)



Reduced density matrix

P. Dirac (1930)



$|\Psi\rangle \in \mathcal{E}_A \otimes \mathcal{E}_B \qquad \rho = |\Psi\rangle\langle\Psi|$ projector

Definition: $\rho_A = \sum_j \langle j|_B \left(|\Psi\rangle\langle\Psi|\right) |j\rangle_B = \mathrm{Tr}_B\, \rho$

Example of entangled state: $\frac{1}{\sqrt{2}}(|\uparrow\rangle_A \otimes |\downarrow\rangle_B - |\downarrow\rangle_A \otimes |\uparrow\rangle_B)$

$$\rho_A = (1/2)(|\uparrow\rangle_A\langle\uparrow|_A + |\downarrow\rangle_A\langle\downarrow|_A)$$

In general: $\rho_A = \sum_i \lambda_i^2 \, |i\rangle_A\langle i|_A$ "mixed" ensemble

Entanglement Entropy

Kitaev & Preskill, 2006

Levin & Wen, 2006

A quantitative **measure** of entanglement

Reduced density matrix: $\rho_A = \mathrm{Tr}_B |\Psi\rangle\langle\Psi|$

$$S_{\mathrm{entanglement}} = -\mathrm{Tr}\{\rho_A \ln \rho_A\}$$ (Von Neumann)

$S_{\mathrm{entanglement}} \propto L^{d-1}$ “area” law

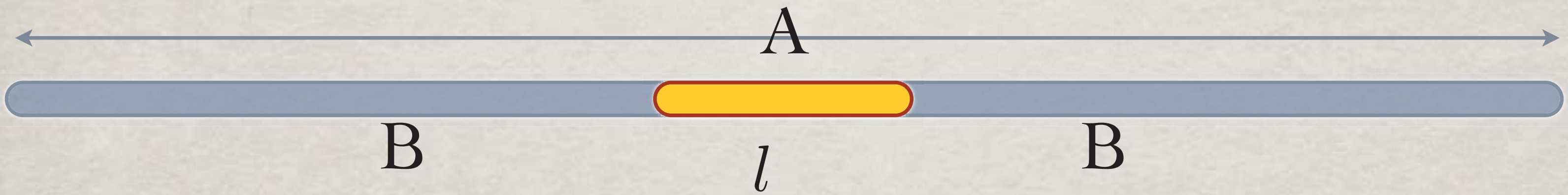
d=2: $\propto L$ (perimeter)

d=1 or critical: ?

Example: segment entanglement

Special Issue: Entanglement Entropy in Extended Quantum Systems, J. Phys. A **42**, N° 50, 500301-504012 (2009); Guest Editors: P. Calabrese, J. Cardy and B. Doyon.

$L \gg 1$



1D critical (spin) chain

$$S_{\mathrm{VN}} \propto c \ln l$$

central charge (universality class)

Rewrite ρ_A as thermal density matrix

$$\rho(T) = \frac{1}{Z}\exp(-\beta H) = \sum_\alpha \exp(-\beta e_\alpha)|\alpha\rangle\langle\alpha|$$

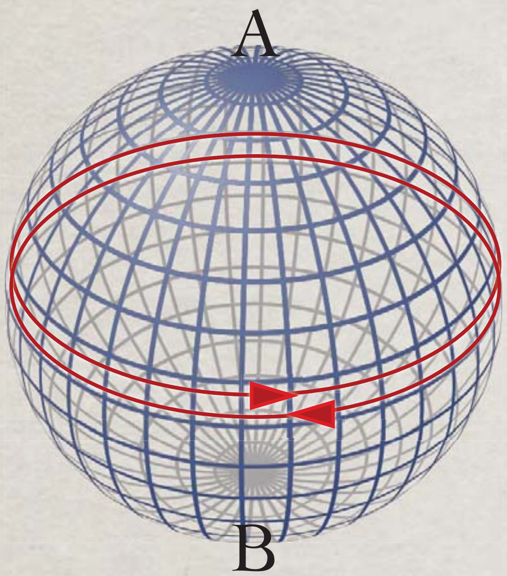
$\beta = 1/T$ inverse temperature

$$\rho_A = \sum_i \lambda_i^2 \, |i\rangle_A \langle i|_A$$

rewrite the weights as: $\lambda_i = \exp(-\xi_i/2)$

$$\boxed{\text{Entanglement spectrum : } \{\xi_i\}}$$

$$\rho_A = \exp(-\hat{\xi})$$



Li & Haldane, 2008

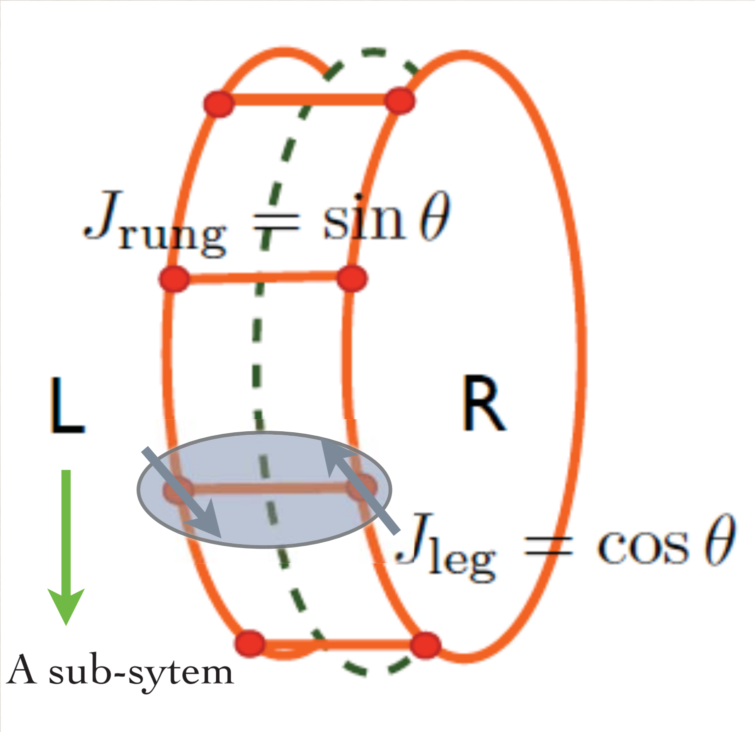
"Haldane" Conjecture:

Precise correspondence between the entanglement spectrum of a FQH system (with LRE) partitioned into two sub-systems linked by some "edge" and the true edge spectrum

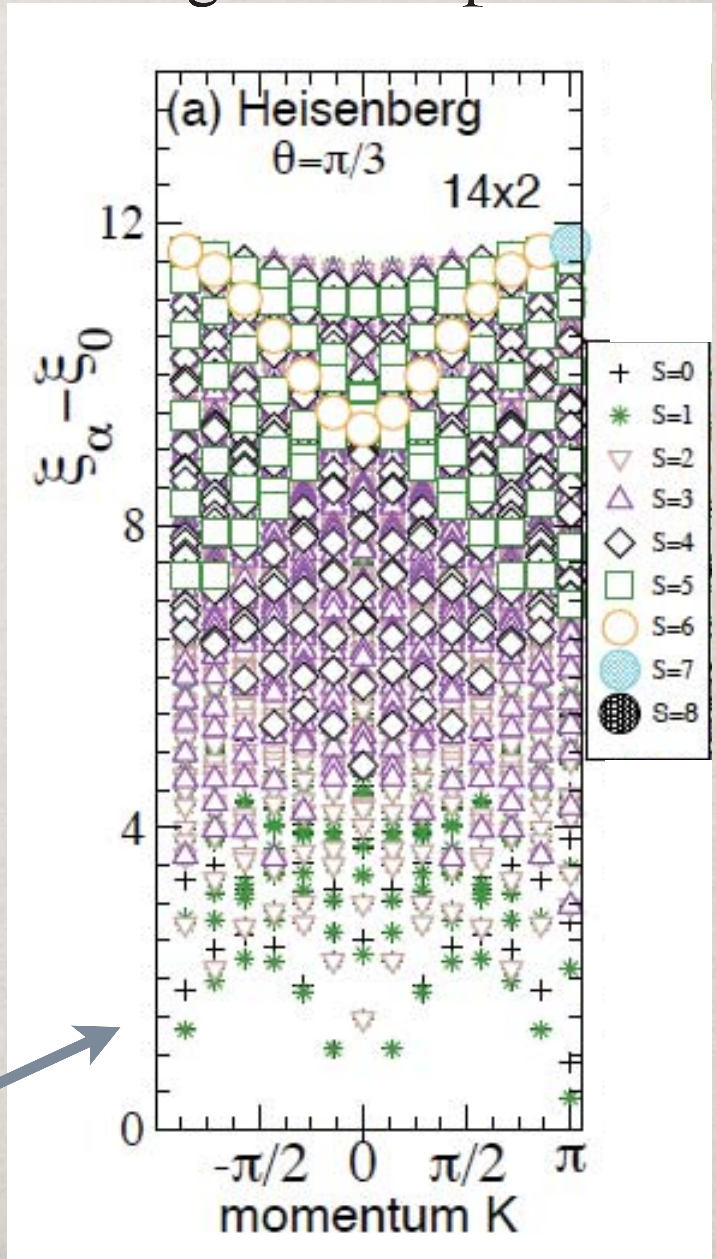
Questions:

- Is the ES always connected to the edges/boundary ?
- How does it reflect bulk properties ?

A simple example:
the 2-leg antiferromagnetic
spin "ladder"



Entanglement spectrum



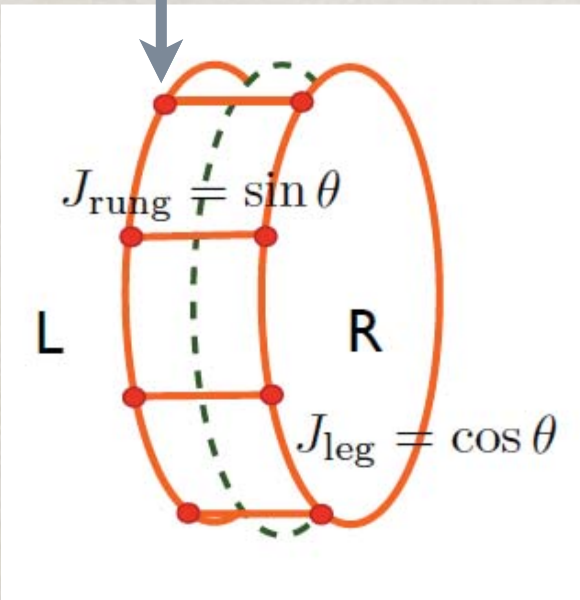
c=1 CFT

D.P., PRL **105**, 077202 (2010)

A precise characterization of the "boundary hamiltonien" is in fact possible !

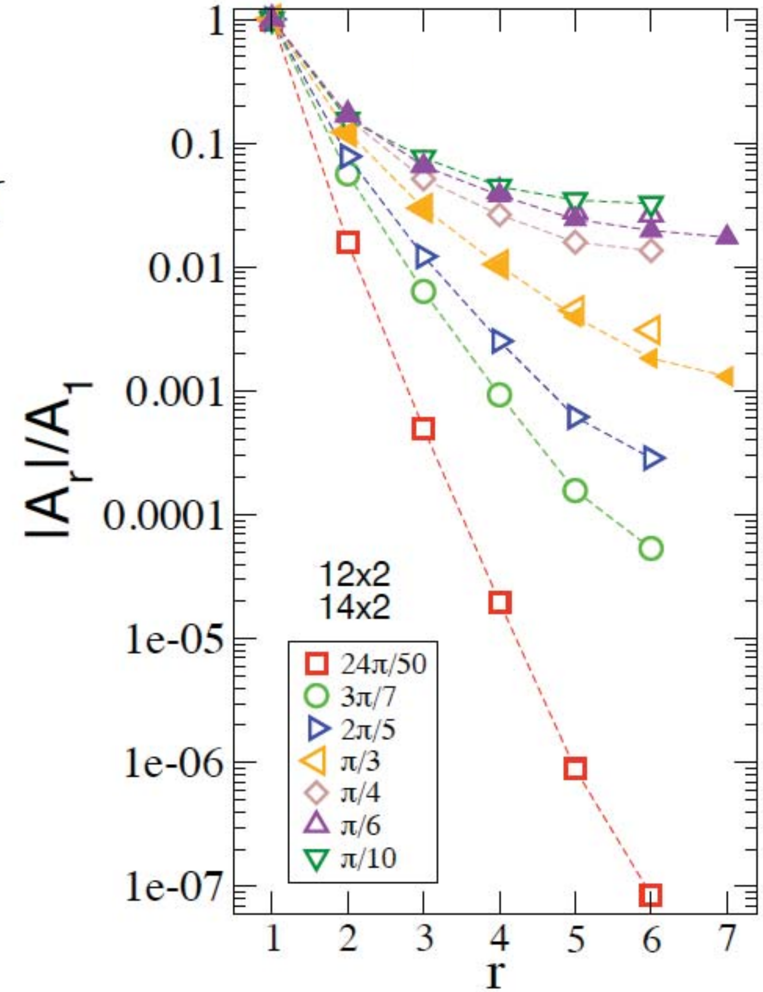
$$\rho_A = \exp\left(-H_b\right)$$

$$H_b = A_0 N_v + \sum_{r,k} A_r\, \mathbf{S}_k \cdot \mathbf{S}_{k+r} + R\hat{X}$$



2-leg ladder

Exponential decay !



Cirac, D.P., Schuch, Verstraete, PRB (2011)

Effective temperature

$$\rho_A = \frac{1}{z_\theta} \exp\left(-\beta_\theta \hat{h}\right)$$

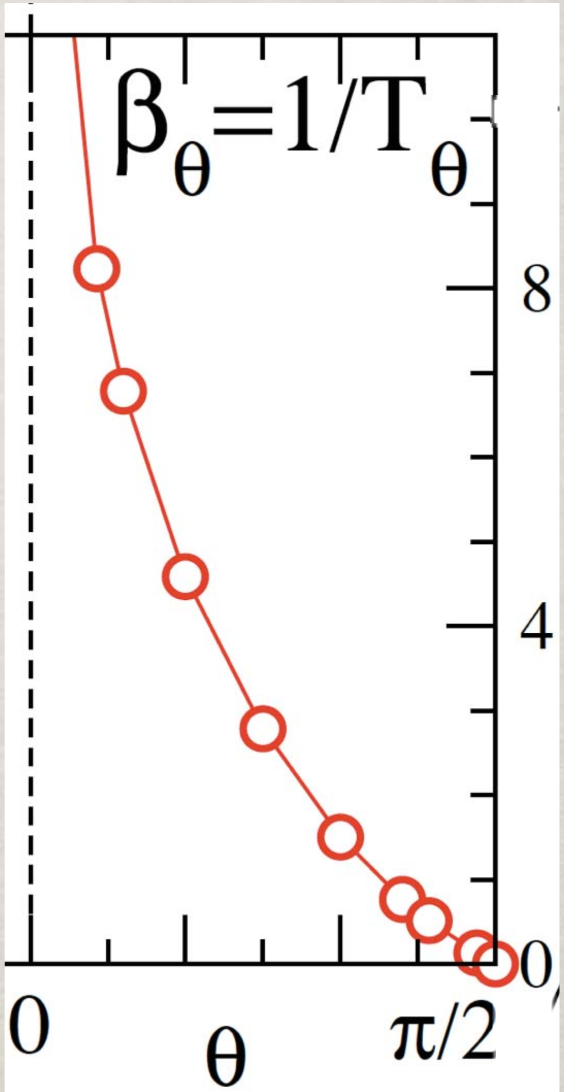
H_b

Heisenberg ladder

$$J_{\text{rung}} = \sin\theta$$
$$J_{\text{leg}} = \cos\theta$$

$$J_{\text{leg}} = 0 \;\rightarrow\; T_\theta = \infty$$

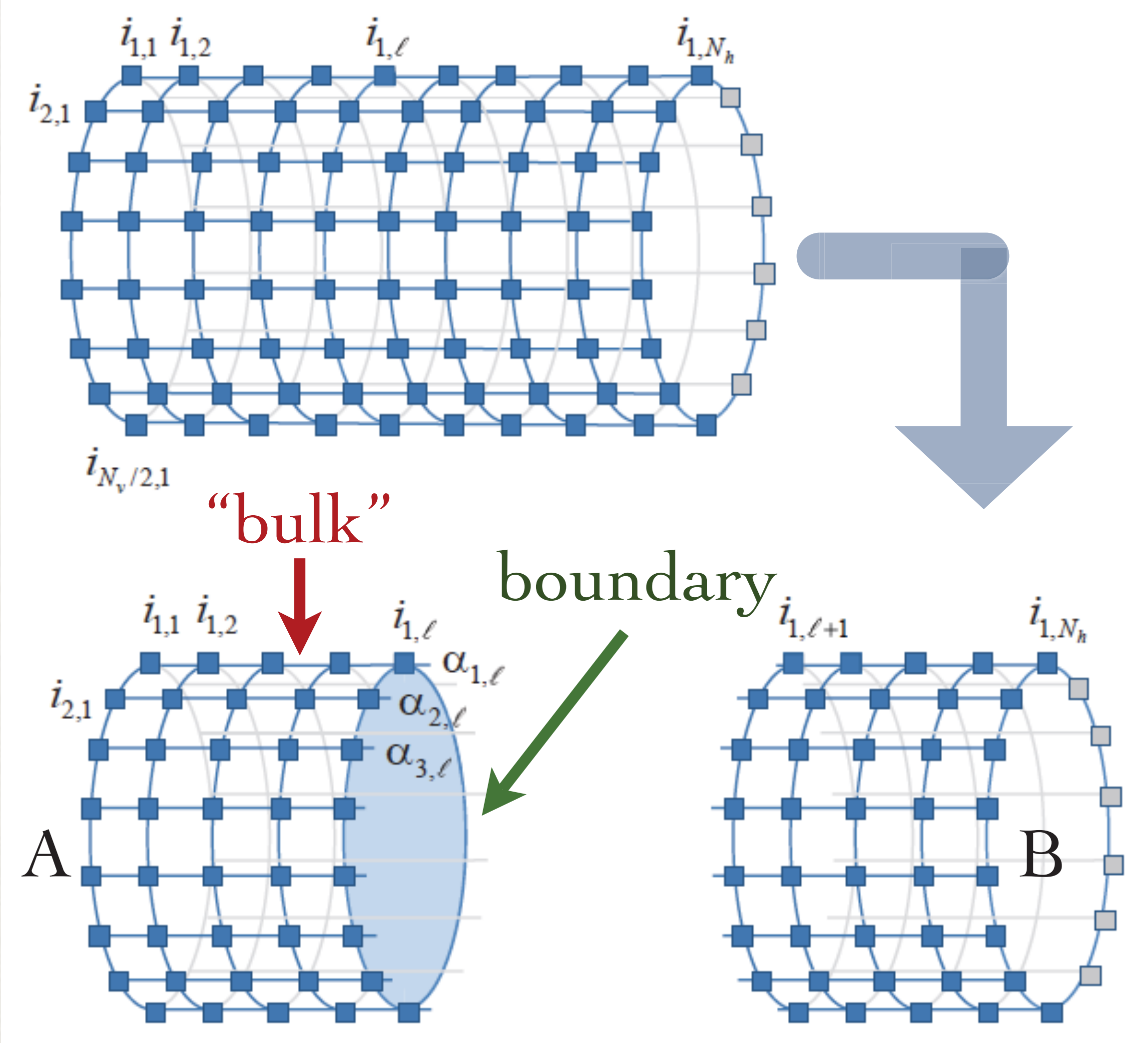
$$J_{\text{rung}} = 0 \;\rightarrow\; T_\theta = 0$$



D.P., PRL **105**, 077202 (2010)

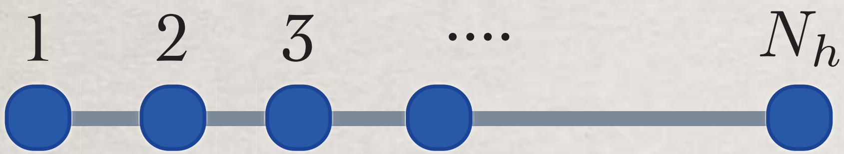
Extend to long cylinders with Nh legs ?

$N_h \to \infty$?



Tensor Network approaches

I. Cirac
F. Verstraete
G. Vidal



$$|\Psi\rangle = \sum_I c_I |i_1, i_2, ..., i_{N_h}\rangle \qquad i_k = -S, -S+1, \dots, S-1, S$$

Matrix Product States (1D) : $M^i_{\alpha_1,\alpha_2}$ $D \times D$ matrix

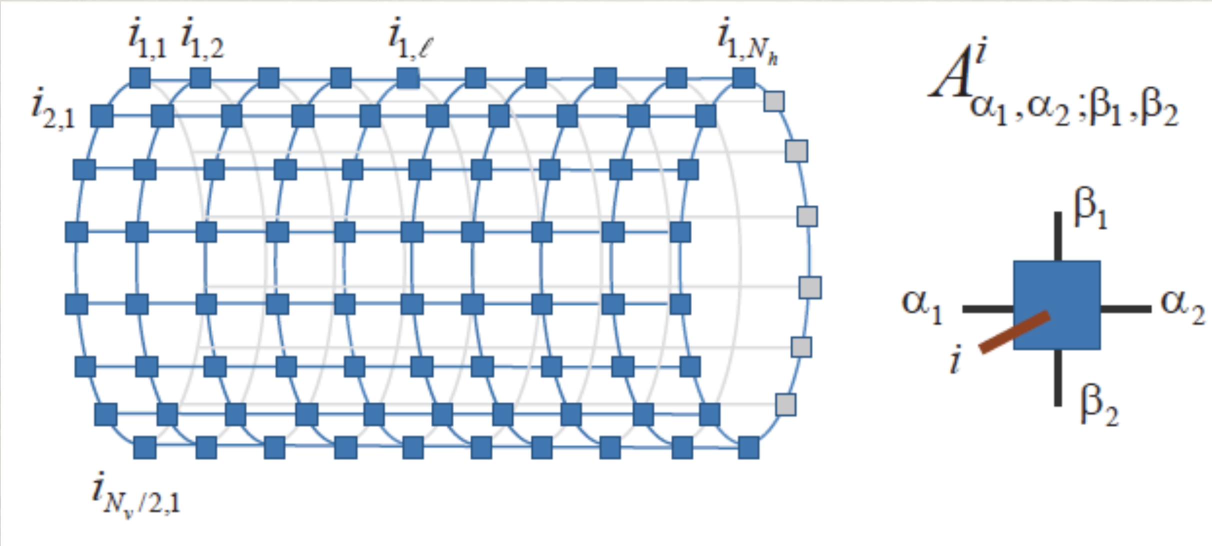
$$c_I = \sum_\alpha L^{i_1}_{\alpha_1} M^{i_2}_{\alpha_1\alpha_2} \dots M^{i_{N_h-1}}_{\alpha_{N_h-2}\alpha_{N_h-1}} R^{i_{N_h}}_{\alpha_{N_h-1}}$$

$$= tr\{L^{i_1} M^{i_2} \dots M^{i_{N_h-1}} R^{i_{N_h}}\}$$

Equivalent to DMRG !!

D ~ m parameter controling the DMRG truncation

Tensor Network for d=2 (and higher): Projected Entangled Paired States (PEPS)



“contract” product of tensors

$$c_I = \sum_{\Lambda} L^{I_1}_{\Lambda_1} B^{I_2}_{\Lambda_1,\Lambda_2} \dots B^{I_{N_h-1}}_{\Lambda_{N_h-2},\Lambda_{N_h-1}} R^{I_{N_h}}_{\Lambda_{N_h-1}}$$

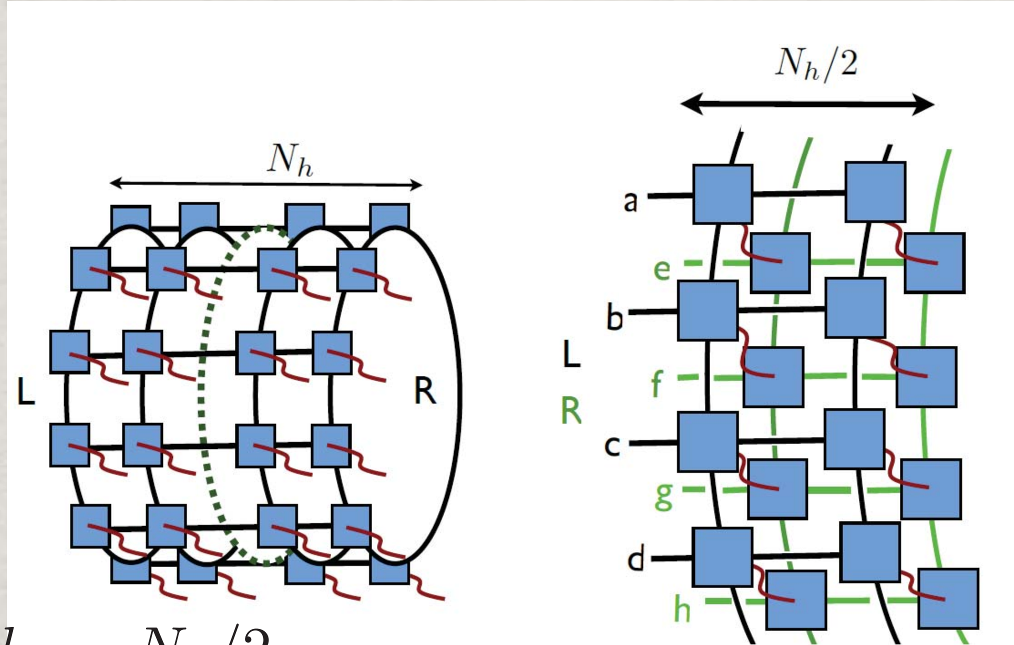
$$B^{I_n}_{\Lambda_{n-1},\Lambda_n} = \mathrm{tr}\left[\prod_{k=1}^{N_v} \hat{A}^{i_{k,n}}_{\alpha_{k,n-1},\alpha_{k,n}}\right]$$

$$\Lambda_n = (\alpha_{1,n}, \alpha_{2,n}, \dots, \alpha_{N_v,n})$$

$$I_n = (i_{1,n}, i_{2,n}, \dots, i_{N_v,n})$$

Holographic framework

ρ_A



$l_A = N_h/2$

σ_b^2

“leaves” on the boundary

Basic formula: $\rho_A = U \sigma_b^2 U^\dagger$

isometry: maps 2D onto 1D

$$\sigma_b^2 = \exp(-H_b)$$

Consequence: expect area law !

Boundary theories: main message

Can we only describe gapped systems ?

To what extend H_b is a local Hamiltonian ?

* gapped systems (AKLT):
  H_b is short-range

* approaching a critical point (deformed AKLT or Ising PEPS):
  H_b becomes long-range

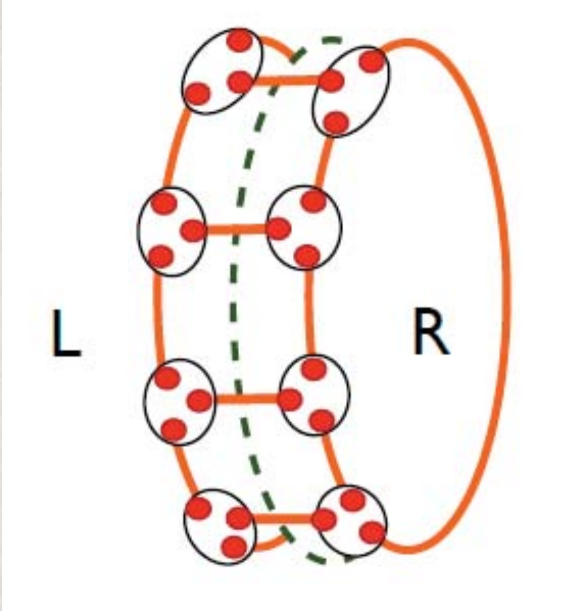
* for topological GS (toric code):
  LR entanglement => H_b non-local

Application to AKLT ladders

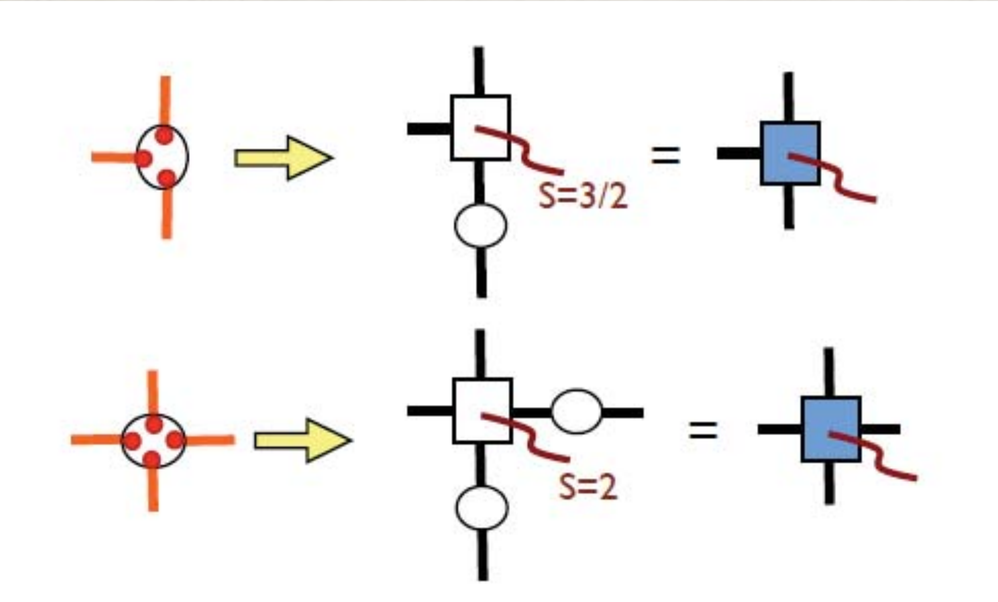
(Affleck-Lieb-Kennedy-Tasaki)

$$S_i = z_i/2$$

$$H_{AKLT} = \sum_{<ij>} P_{S_i+S_j}$$



$N_h = 2$

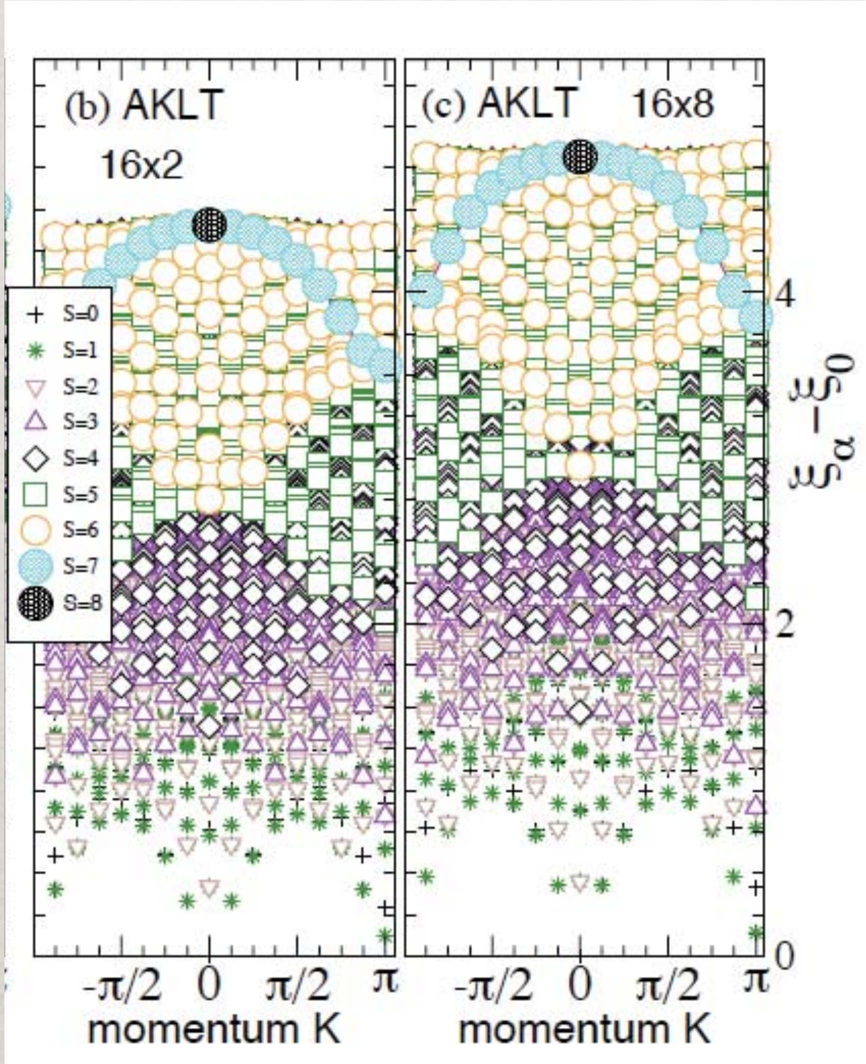


PEPS representation

D=2 !

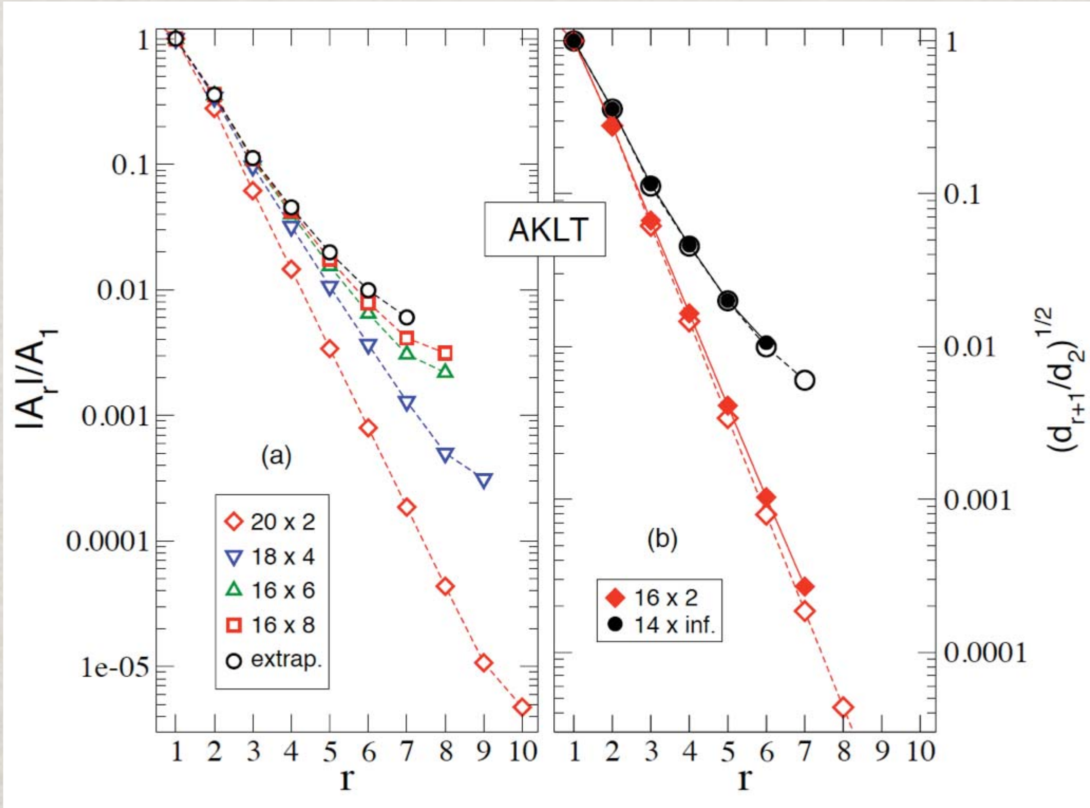
# of legs N_h from 2 to ∞

Entanglement spectra of AKL ladders/cylinders



c=1 CFT

Finite size Nh-leg AKLT ladders

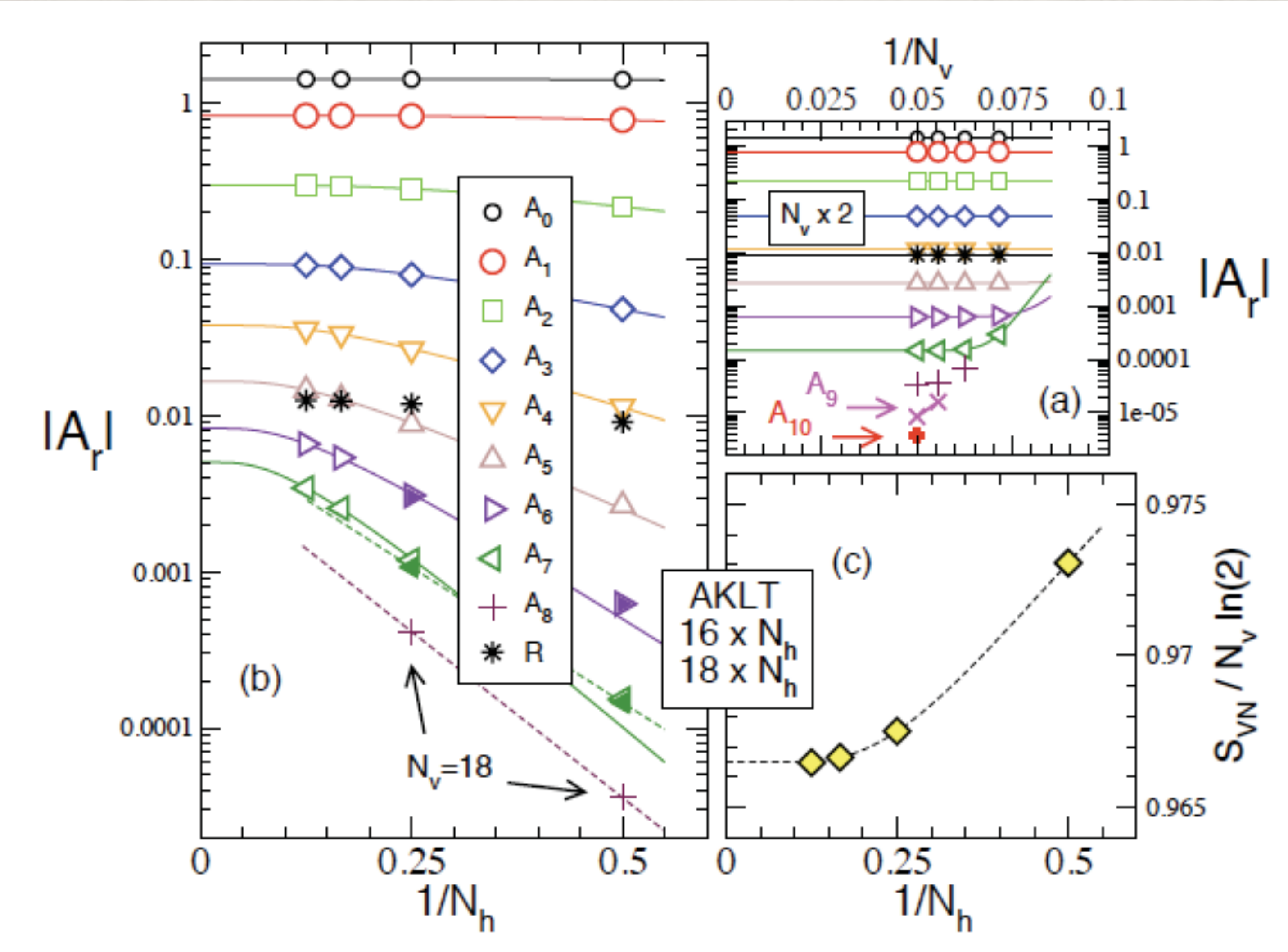


$$H_b = A_0 N_v + \sum_{r,k} A_r \mathbf{S}_k \cdot \mathbf{S}_{k+r} + R\hat{X}$$

$$H_b = \sum_n h_n$$

$$d_n = \mathrm{tr}(h_n^2)/2^{N_v}$$

Finite size scaling ("brute force" contractions)



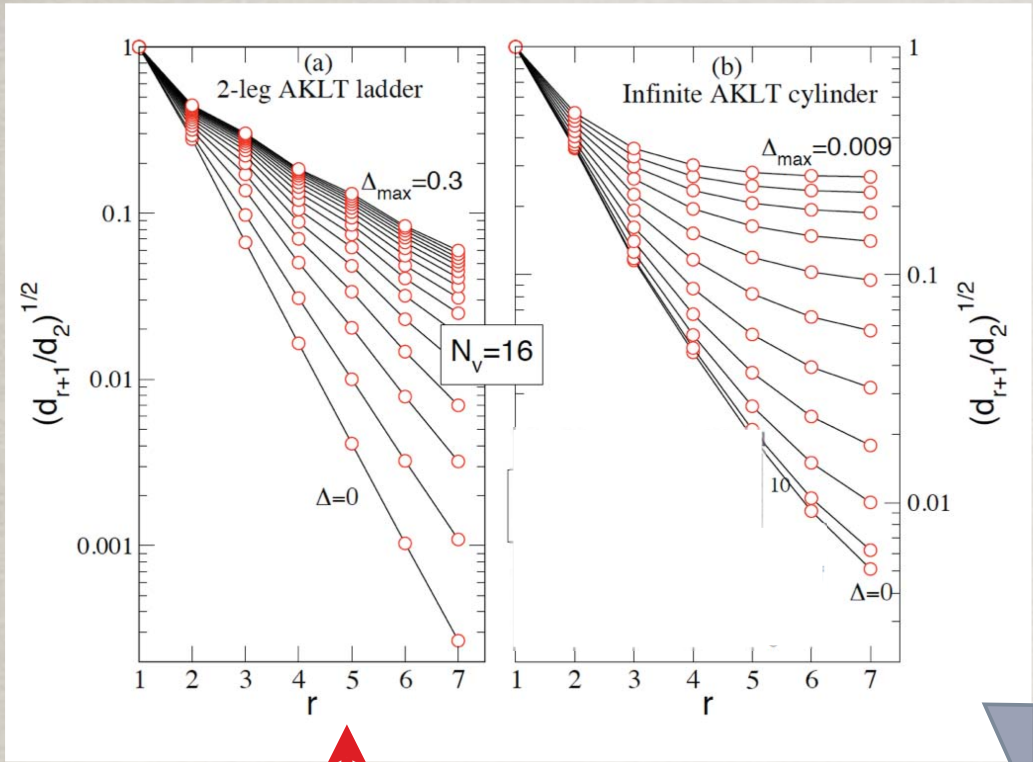
... and approximate ("cheaper") methods

Deformed AKLT model

$$A^m_{\alpha_1,\alpha_2,\alpha_3,\alpha_4} = \langle s_m|Q(\Delta)|\alpha_1,\alpha_2,\alpha_3,\alpha_4\rangle$$

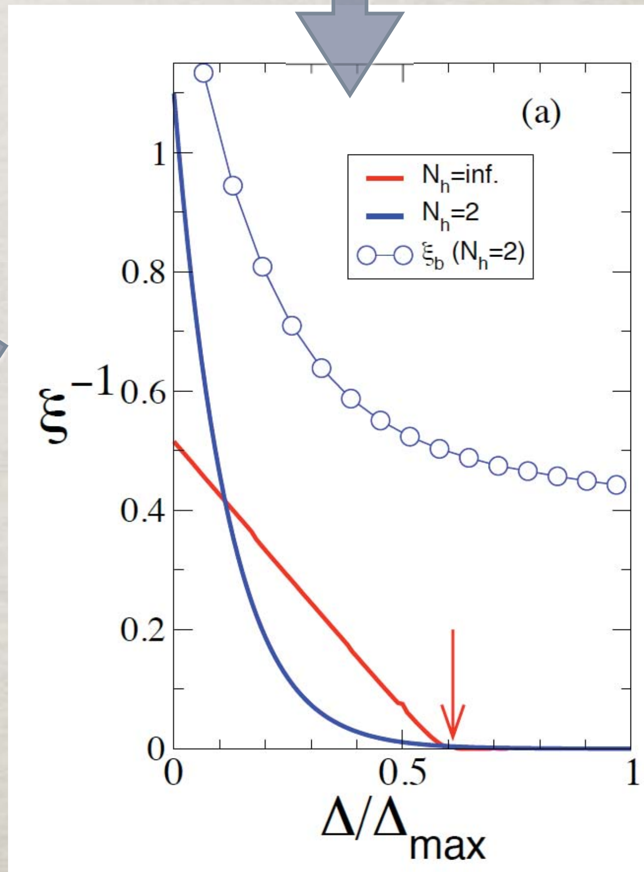
$$Q_n(\Delta) = e^{-8\Delta S^2_{z,n}}$$

breaks SU(2) down to U(1)



$$(d_{r+1}/d_2)^{1/2} \sim \exp(-r/\xi_b)$$

Critical point:

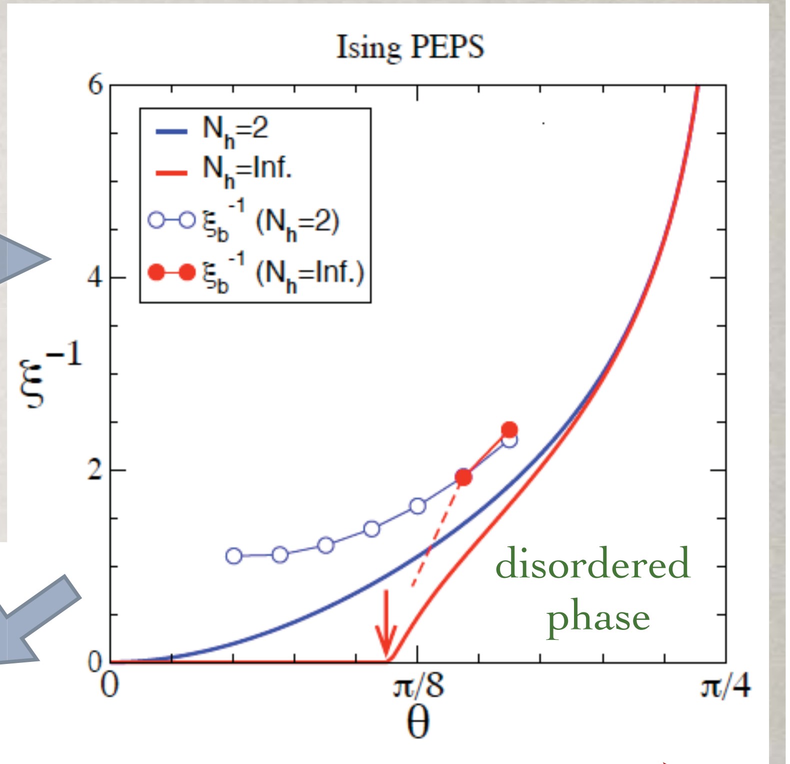


Quantum Ising PEPS

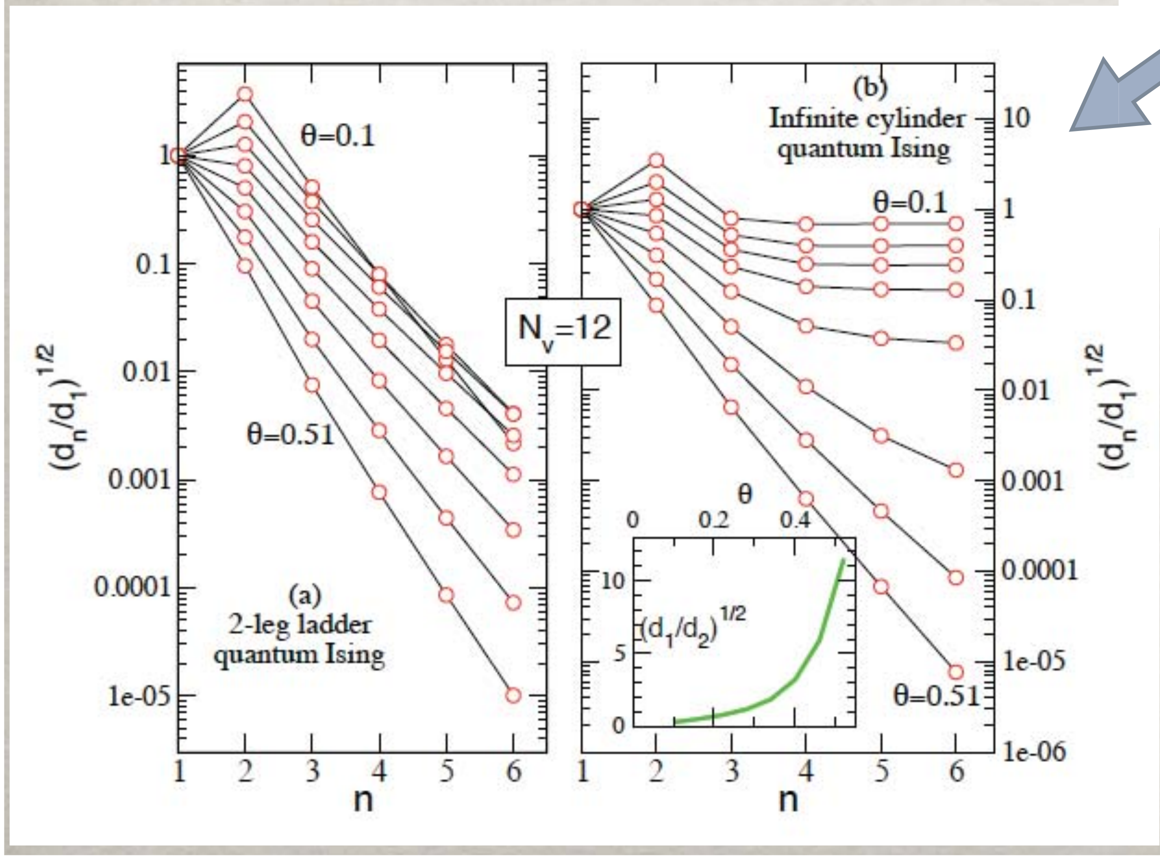
F. Verstraete, M. M. Wolf, D. Perez-Garcia, and J. I. Cirac, Phys. Rev. Lett. **96**, 220601 (2006).

$$A^{m}_{\alpha_1,\alpha_2,\alpha_3,\alpha_4} = a_m(\alpha_1)a_m(\alpha_2)a_m(\alpha_3)a_m(\alpha_4)$$

$$a_0(-1/2) = a_1(1/2) = \cos\theta$$
$$a_1(-1/2) = a_0(1/2) = \sin\theta$$



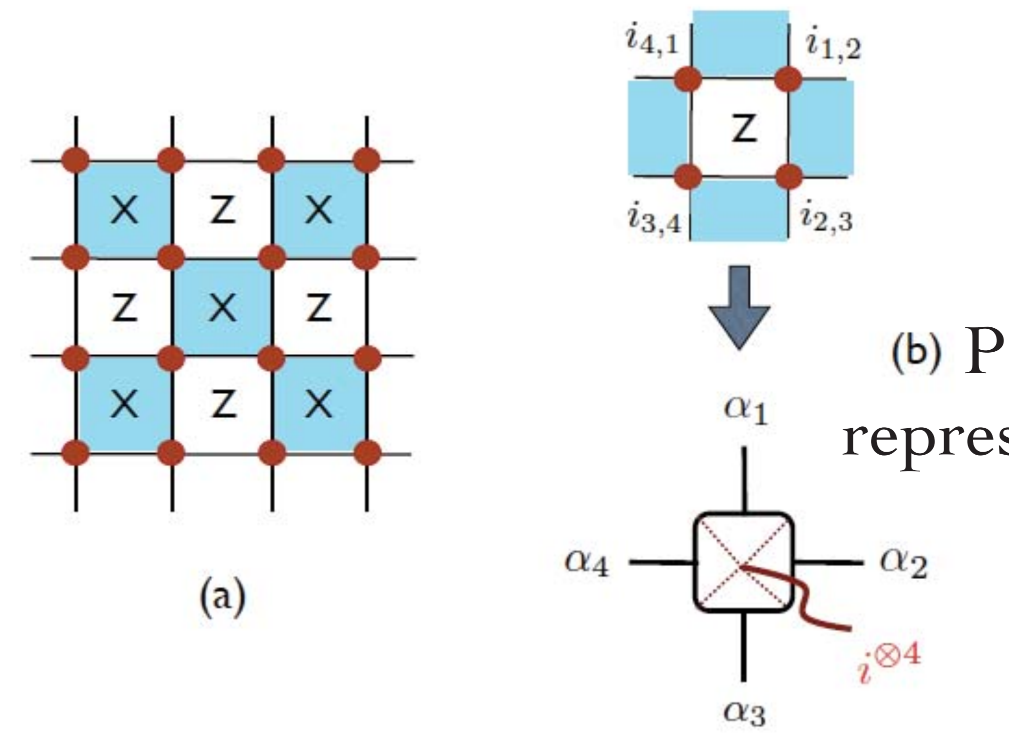
effective transverse field



$$(d_n/d_1)^{1/2} \sim \exp\left(-n/\xi_b\right)$$

What happens for topological ordered states ?

A simple example: the Kitaev code model

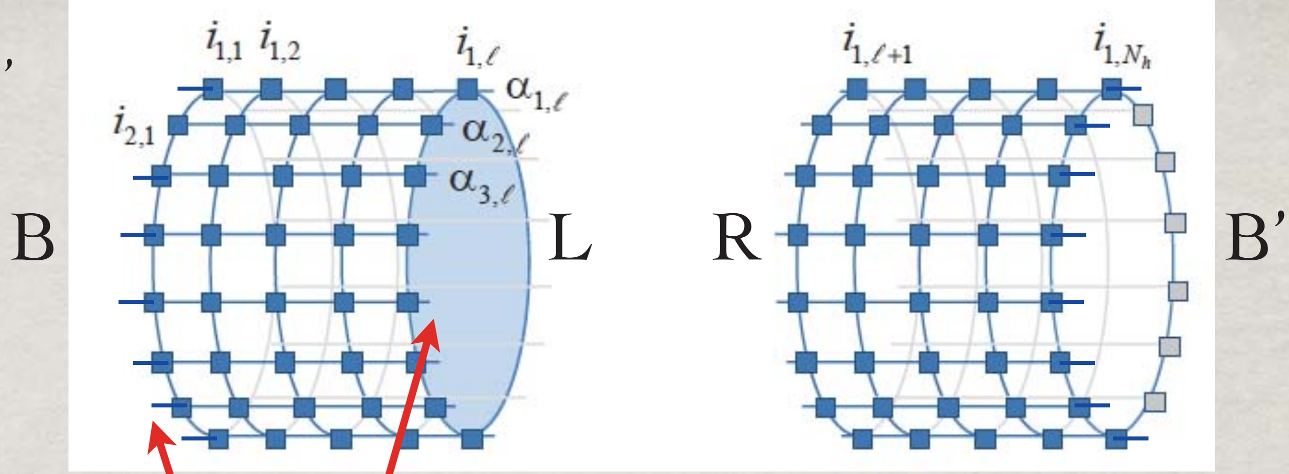


PEPS representation

D=2 !

$$A^{i_{1,2},i_{2,3},i_{3,4},i_{4,1}}_{\alpha_1,\alpha_2,\alpha_3,\alpha_4} = \begin{cases} 1 \text{ if } i_{x,x+1} = \alpha_{x+1} - \alpha_x \bmod 2 \; \forall x \\ 0 \text{ otherwise.} \end{cases}$$

open ends B & B'



$$\sigma_{BL} = \sigma_{B'R} \propto 1^{\otimes N_v} \otimes 1^{\otimes N_v} + X^{\otimes N_v} \otimes X^{\otimes N_v}$$

+ fixing the left and right BC B & B': $|\chi_\theta) = \cos\frac{\theta}{2}\,|0)^{\otimes N_v} + \sin\frac{\theta}{2}\,|1)^{\otimes N_v}$

↓

$$\rho_\ell \propto (1+\sin^2\theta)\,1^{\otimes N_v} + (2\sin\theta)\,X^{\otimes N_v}$$

↓

$$H_l = -\mathrm{sign}(\sin\theta)X^{\otimes N_v}$$

Non-local boundary Hamiltonian !

Conclusion and outlook

* natural mapping between bulk and boundary
→ properties of bulk reflected in the property of the boundary Hamiltonian H_b
→ property of the bulk can be "read off" the property of H_b

* extension to arbitrary region in the lattice possible

* extension to models leading to higher spin H_b (like S=1) e.g. RVB PEPS (D=3 on Kagome)

* extensions to fermions, anyons, ...