The Abdus Salam International Centre for Theoretical Physics

United Nations Educational, Scientific and Cultural Organization

IAEA International Atomic Energy Agency

2253-14

Workshop on Synergies between Field Theory and Exact Computational Methods in Strongly Correlated Quantum Matter

24 - 29 July 2011

Exotic Critical Phenomena in Classical Systems - Loops and strings on lattices

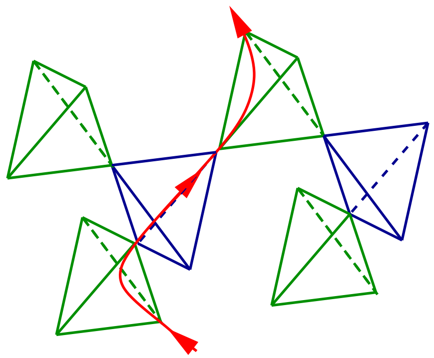
J. Chalker

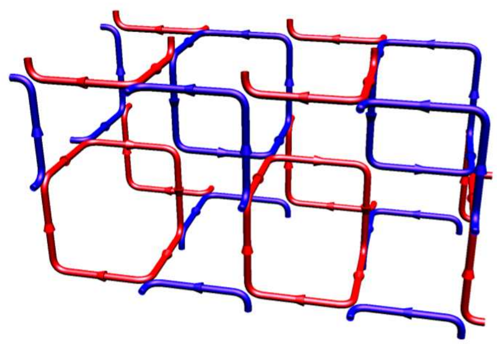
Physics Department
Oxford University
Oxford, U.K.

EXOTIC CRITICAL PHENOMENA IN CLASSICAL SYSTEMS

Loops and strings on lattices

John Chalker
Physics Department, Oxford University





Work with

Ludovic Jaubert & Peter Holdsworth (ENS Lyon), & Roderich Moessner (Dresden)

Adam Nahum (Oxford), Miguel Ortuño, Andres Somoza, & Pedro Serna (Murcia)

Outline

Statistical mechanics with extended degrees of freedom

Coulomb phases

Geometrically frustrated magnets, dimer models

Correlations from constraints

Close-packed loop models

Loop colours as non-local degrees of freedom

See also poster session

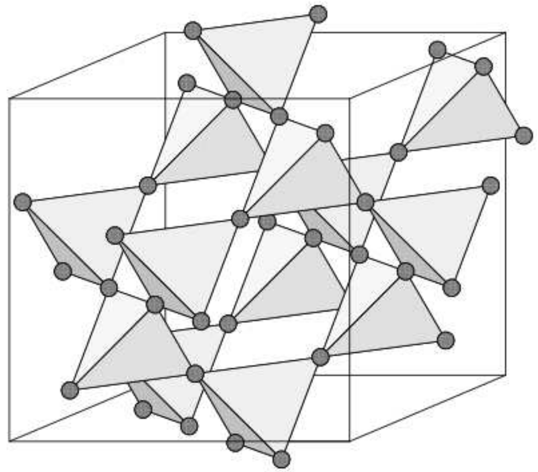
Phase transitions

Ordering transitions from the Coulomb phase

Transitions between extended-loop and short-loop phases

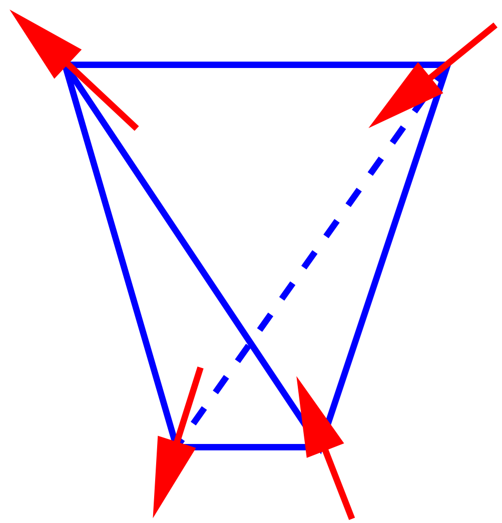
Spin Ice

$Ho_2Ti_2O_7$ and $Dy_2Ti_2O_7$



'Two-in, two-out'

ground states



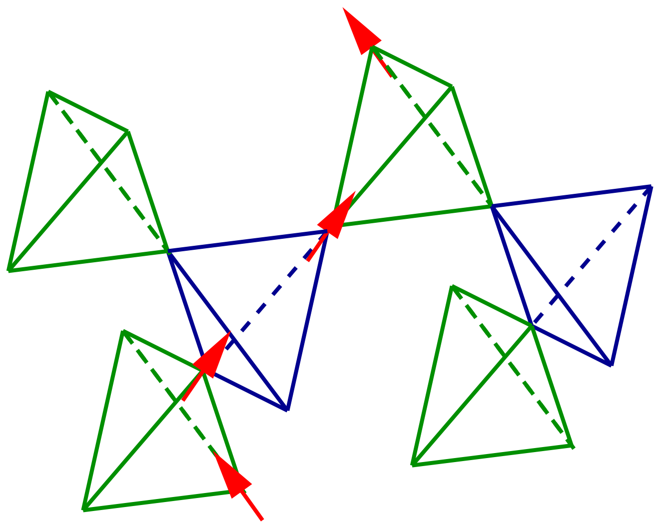
Pyrochlore ferromagnet with single-ion anisotropy

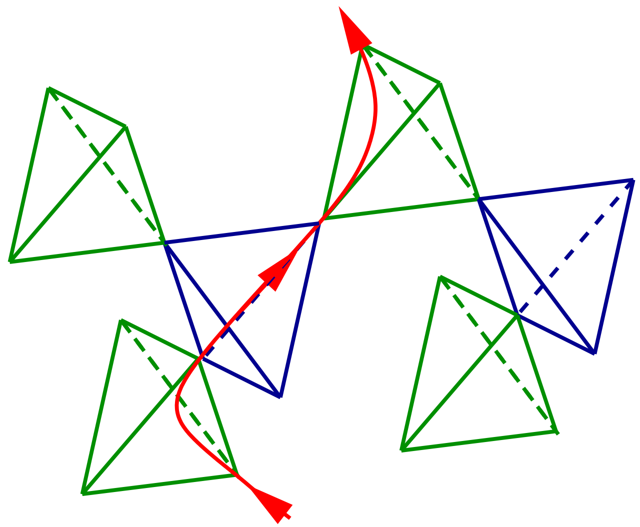
$$\mathcal{H} = -J \sum_{\langle ij \rangle} \mathbf{S}_i \cdot \mathbf{S}_j - D \sum_i (\hat{\mathbf{n}}_i \cdot \mathbf{S}_i)^2 - \mathbf{h} \cdot \sum_i \mathbf{S}_i$$

Gauge theory of ground state correlations

Youngblood *et al* (1980), Huse *et al* (2003), Henley (2004)

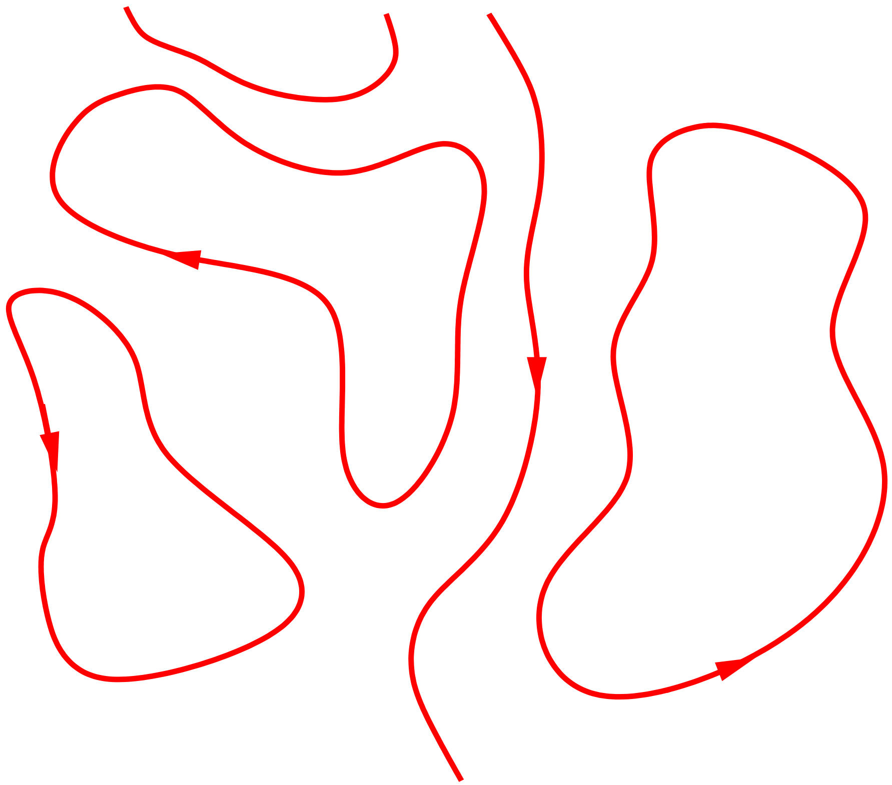
Map spin configurations . . . **. . . to vector fields** $\mathbf{B}(\mathbf{r})$





'two-in two out' groundstates . . . **. . . map to divergenceless** $\mathbf{B}(\mathbf{r})$

Ground states as flux loops

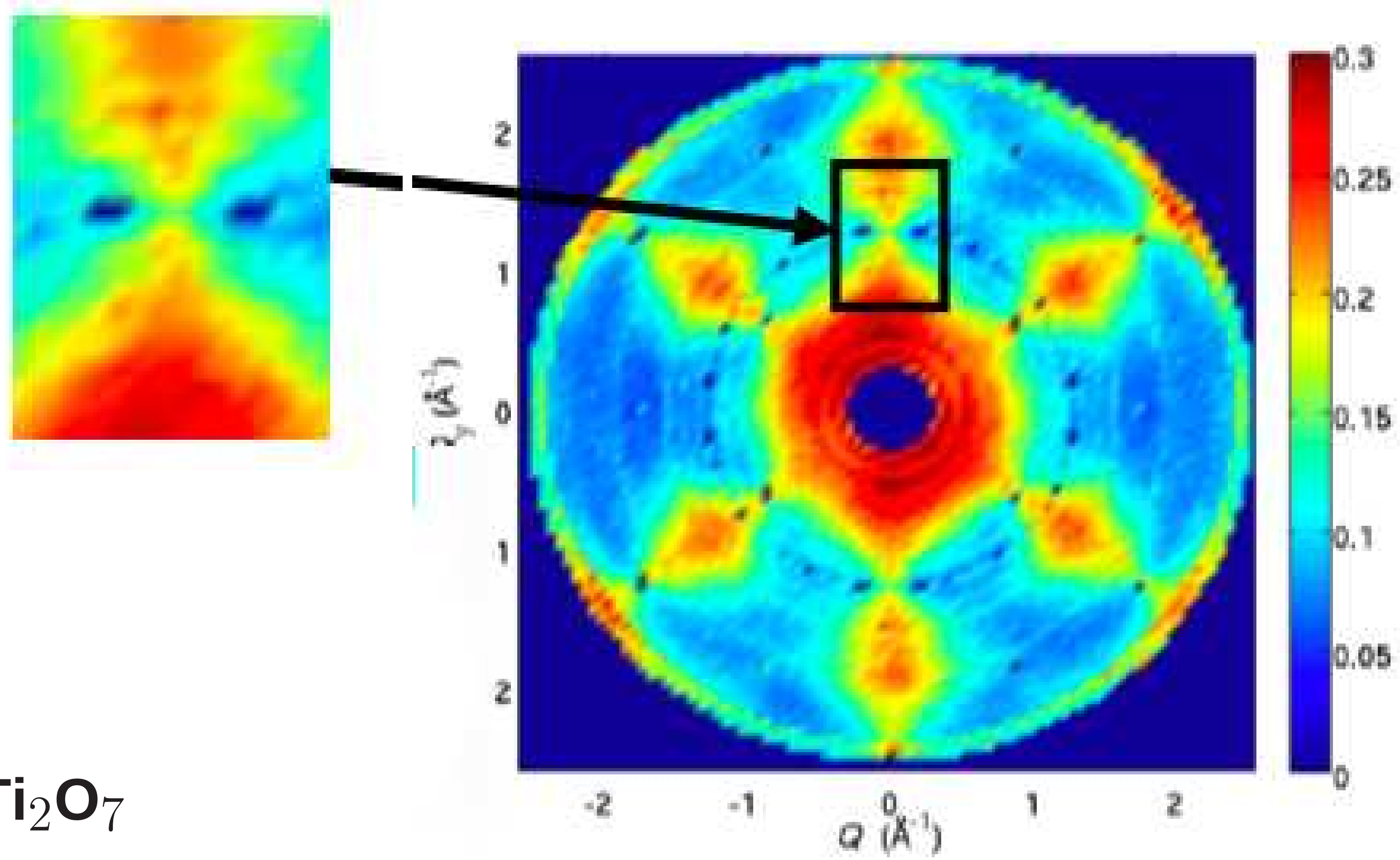


Entropic distribution: $P[\mathbf{B}(\mathbf{r})] \propto \exp(-\kappa \int \mathbf{B}^2(\mathbf{r}) \mathrm{d}^3\mathbf{r})$

Power-law correlations: $\langle B_i(\mathbf{r}) B_j(\mathbf{0}) \rangle \propto r^{-3}$

Low T correlations from neutron diffraction

Fennell *et al*   Science 326, 415 (2009)



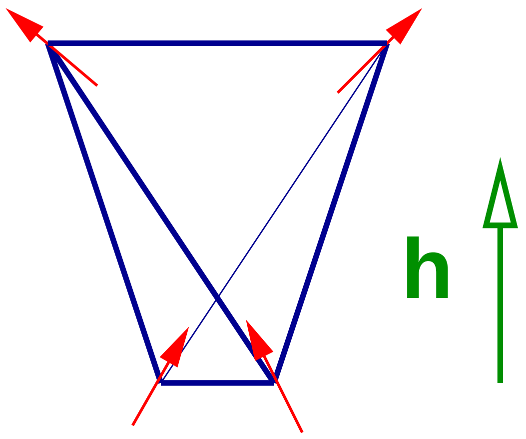
$Ho_2Ti_2O_7$

Engineering transitions in spin ice

Select ordered state with Zeeman field or strain

Kasteleyn transition

in staggered field



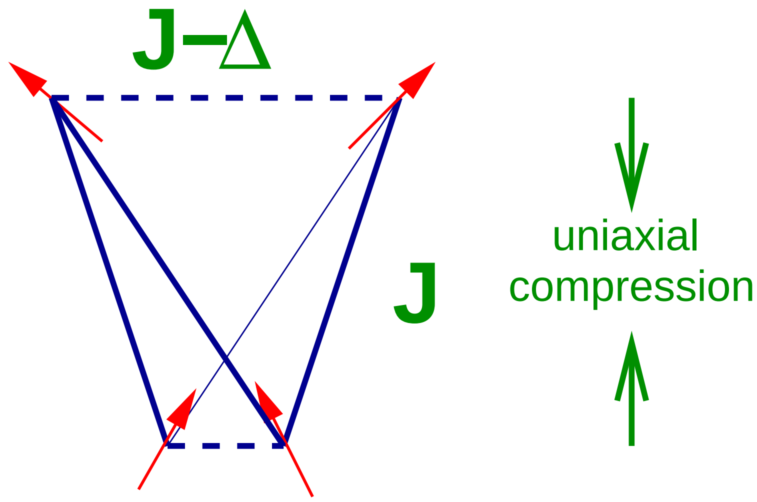
Magnetisation

vs h^{eff}/T

for $h^{\text{eff}}, T \ll J$

Ferromagnetic ordering

strain + magnetoelastic coupling



Magnetic order for

$T \ll \Delta$

Coulomb phase for

$\Delta \ll T \ll J$

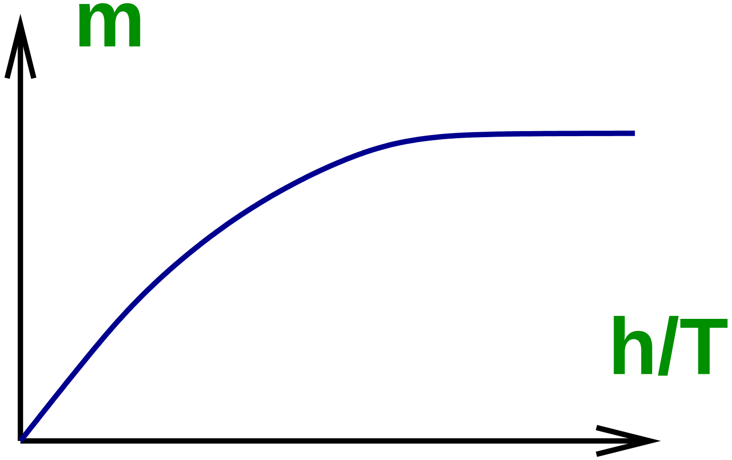
Jaubert, JTC, Holdsworth + Moessner, PRL (2008) + (2010)

A Kasteleyn transition

Magnetisation induced by applied field

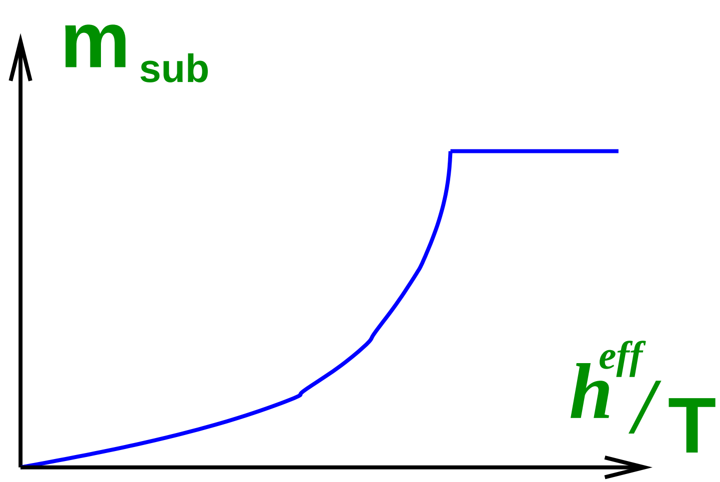
Magnetisation vs temperature

In a paramagnet



No transition

From the Coulomb phase

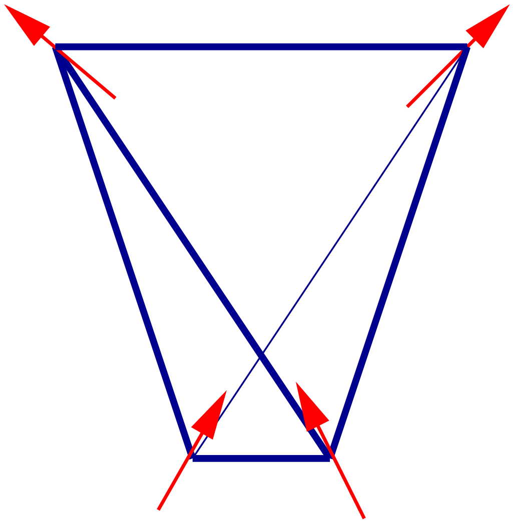


One-sided transition

- Continuous from low-field side
- First-order from high-field side

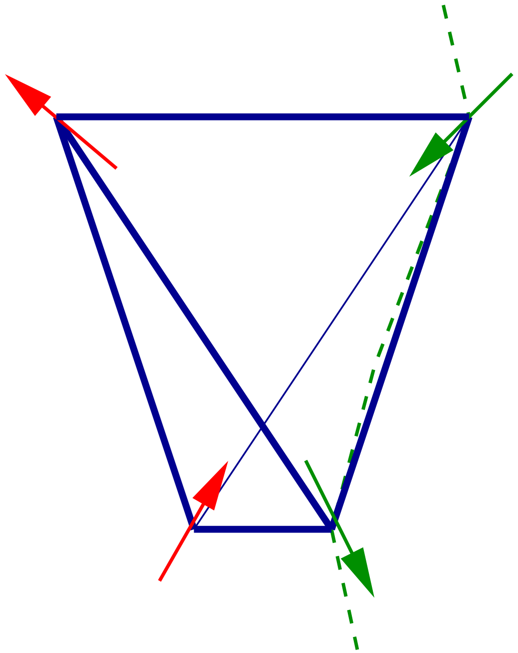
Description of the transition

Reference state: fully polarised



'Vacuum'

Excitations: spin reversals



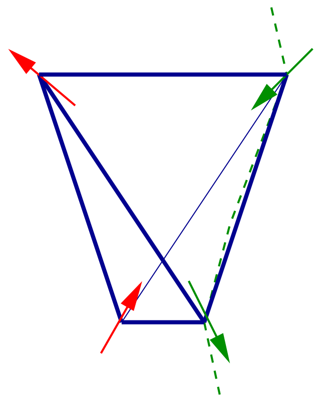
String excitation

Thermodynamics of isolated string, length L:

Energy $L \cdot h$ **Entropy** $L \cdot k_{\mathrm{B}} \ln(2)$ **Free energy** $L \cdot [h - k_{\mathrm{B}} T \ln(2)]$

String density: finite for $h/k_{\mathrm{B}}T < \ln(2)$ **zero for** $h/k_{\mathrm{B}}T > \ln(2)$

Classical to quantum mapping



View strings as boson world lines

3D classical $\equiv (2+1)D$ **quantum**

$$Z = \mathrm{Tr}\left(T^L\right) \qquad T \equiv \mathrm{e}^{\mathcal{H}}$$

$\mathcal{H}$ **hard core bosons hopping on** $\langle 100 \rangle$ **plane**

magnetic field $\Leftrightarrow$ **boson chemical potential**

Coulomb phase correlations $\Leftrightarrow$ **Goldstone fluctuations of condensate**

monopole deconfinement $\Leftrightarrow$ **off-diagonal long range order**

Quantum Description as XY ferromagnet

Kasteleyn transition

$$\mathcal{H} = -\mathcal{J} \sum_{\langle ij \rangle} [S_i^+ S_j^- + S_i^- S_j^+] - \mathcal{B} \sum_i S_i^z$$

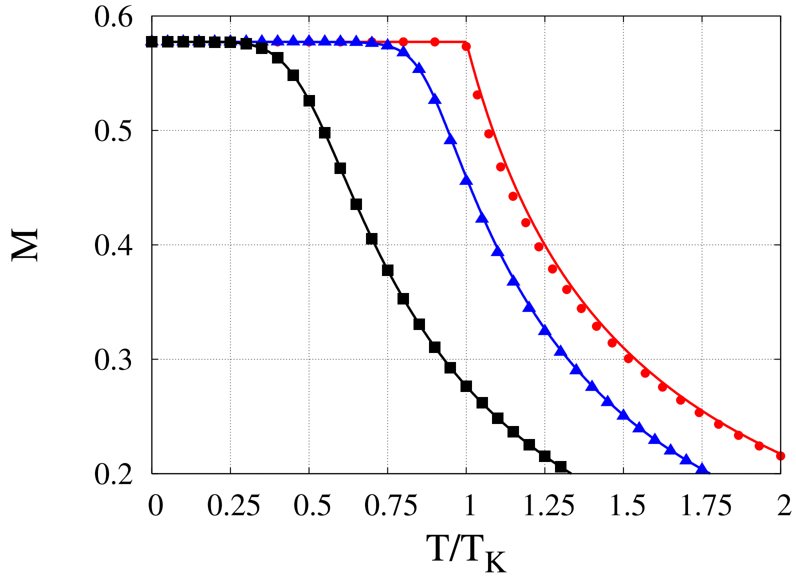
Correspondence with classical description: $\mathcal{B} \equiv h/T$

$\mathcal{B} > \mathcal{B}_\mathrm{c}$ **Quantum spins polarised along** z

$\mathcal{B} < \mathcal{B}_\mathrm{c}$ **Quantum spins have** xy **order**

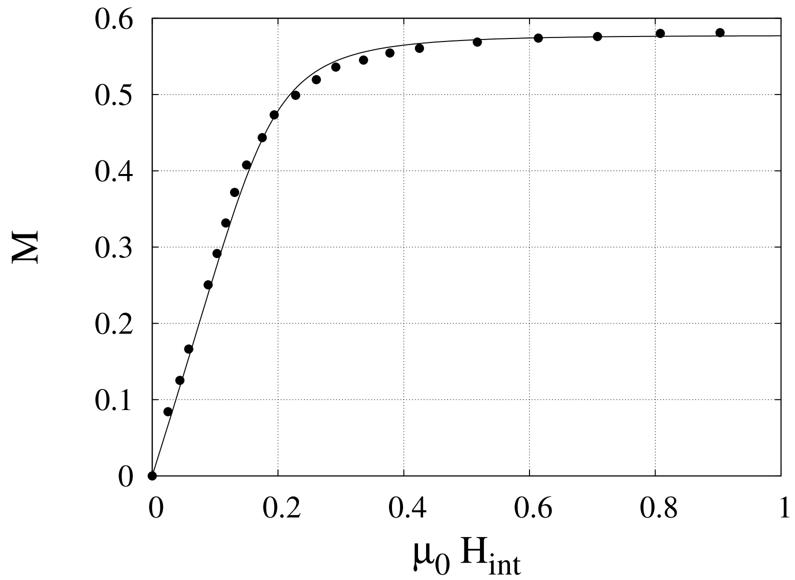
Kasteleyn: Simulation and Experiment

Magnetisation vs T



$$h/J = 0.58,\ 0.13,\ 10^{-3}$$

Magnetisation vs H



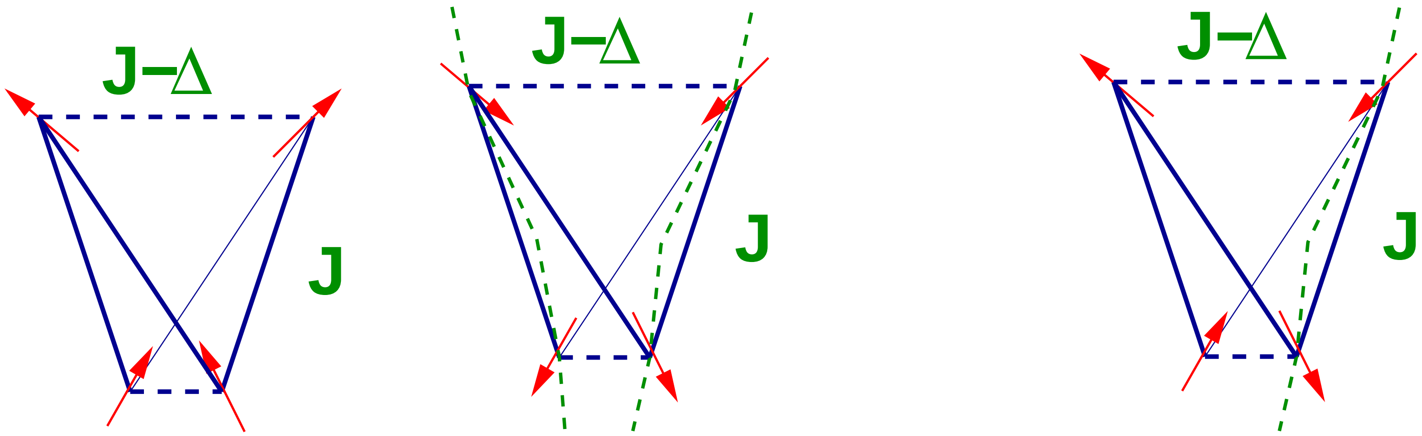
Data for $Dy_2Ti_2O_7$ at 1.8K

(Fukazawa *et al.* 2002)

$$k_B T \simeq 1.6 J_{\text{eff}}$$

Ferromagnetic ordering in strained spin ice

Classical-quantum mapping: ordering as reorientation of quantum spins



Low energy configurations **High energy configuration**

$$\mathcal{H} = -\mathcal{J} \sum_{\langle ij \rangle} [S_i^+ S_j^- + S_i^- S_j^+] - \mathcal{D} \sum_{\langle ij \rangle} S_i^z S_j^z$$

$\mathcal{D} < \mathcal{J}$ **quantum spins in xy plane** $\mathcal{D} \equiv \Delta / T$

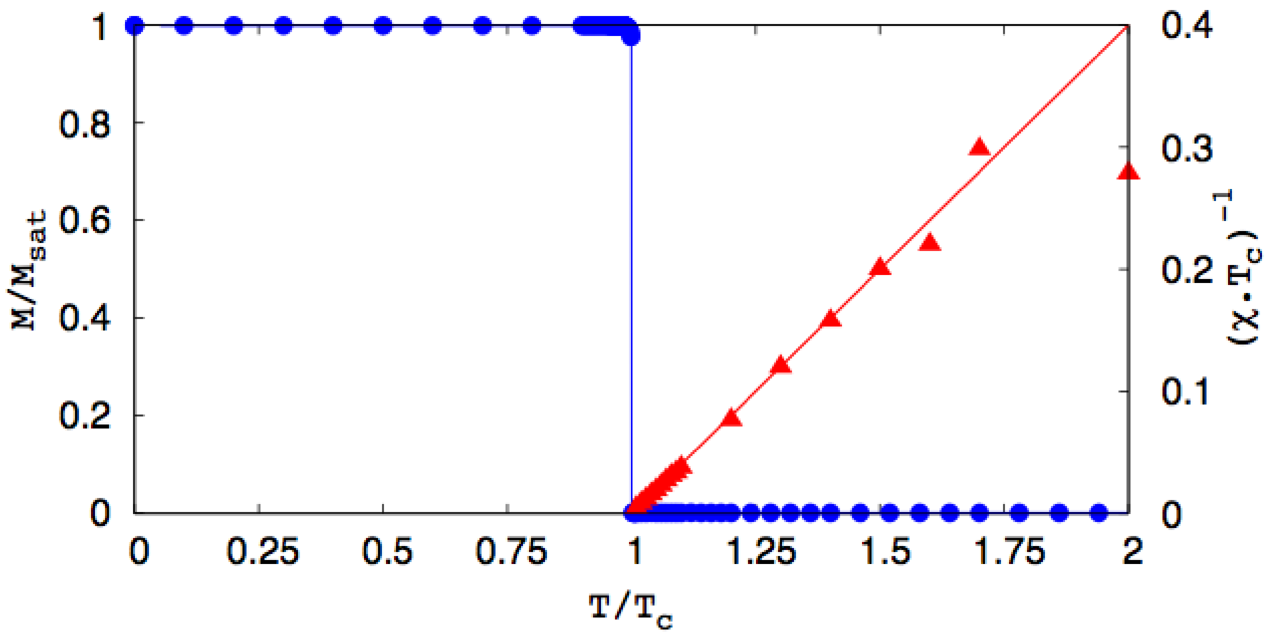
$\mathcal{D} > \mathcal{J}$ **quantum spins along z**

$\mathcal{D} = \mathcal{J}$ **emergent SU(2) symmetry at critical point**

Exotic features of ordering in strained spin ice

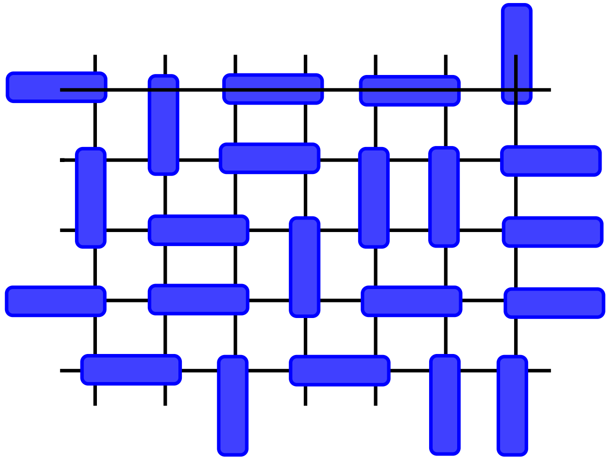
Transition is ‘infinite order’ multicritical point

- Magnetisation (maximally) discontinuous
- Susceptibility divergent as $T \longrightarrow T_{\mathrm{c}}^{+}$
- Domain wall width divergent as $T \longrightarrow T_{\mathrm{c}}^{-}$



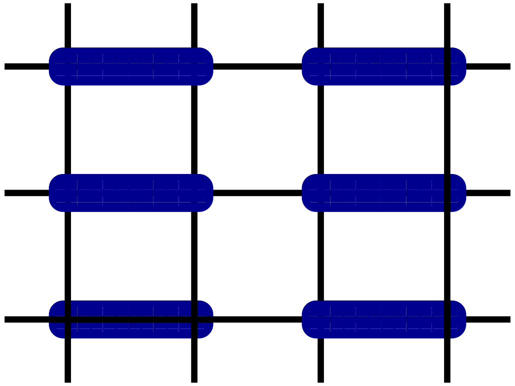
Ordering from the Coulomb phase of dimer models

Allowed states of close-packed dimer models



Dimer crystallisation

favour parallel pairs



$$\mathcal{H} = -J(n_{||} + n_{//} + n_{=})$$

Crystal for $T \ll J$

Coulomb phase for $T \gg J$

Simulations:

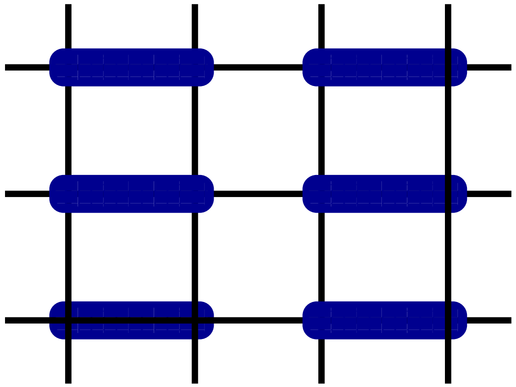
- **continuous transition possible**
- **classical non-LGW critical point**

Alet et al: 2006, 2010

Classical dimer ordering in 3d and bosons in 2d

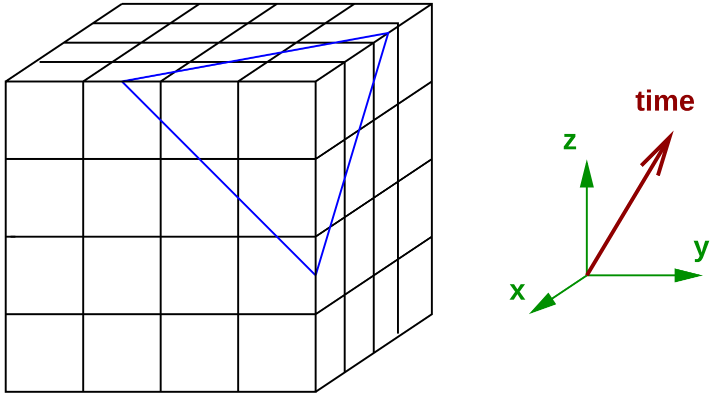
Dimer crystallisation

favour parallel pairs

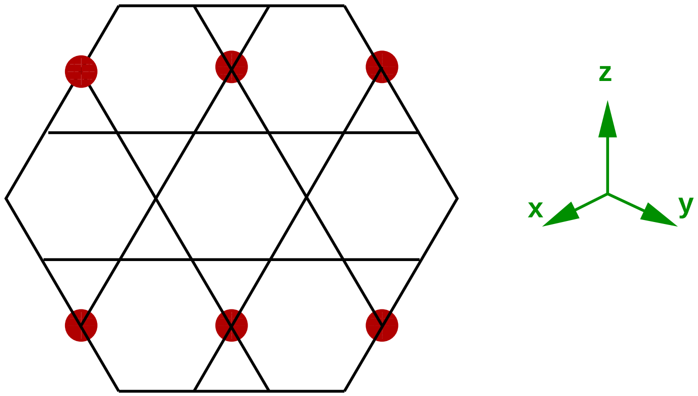


Expect non-LGW critical point

From 3d classical to (2+1)d quantum



Map to bosons on kagome lattice



1/6 filling with hard-core repulsion

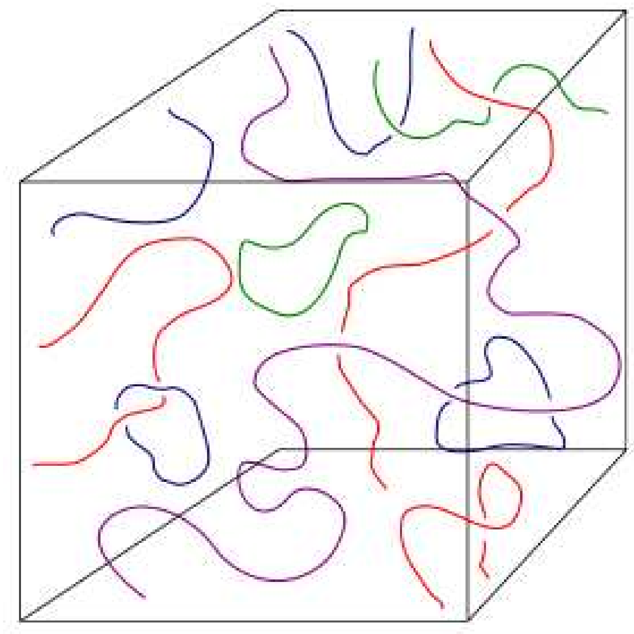
Dimer liquid maps to superfluid

Dimer crystal maps to boson crystal

Powell + JTC, 2009

Loop models

Continuum problem



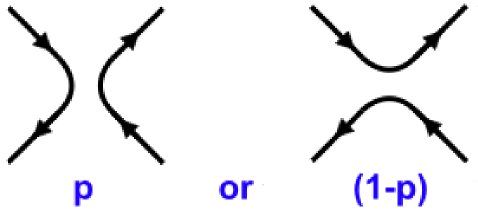
Lattice formulation

Close-packed loops with n colours

on lattice of (directed) links

Nahum, JTC, Serna, Ortuño, and Somoza, arXiv:1104.4096

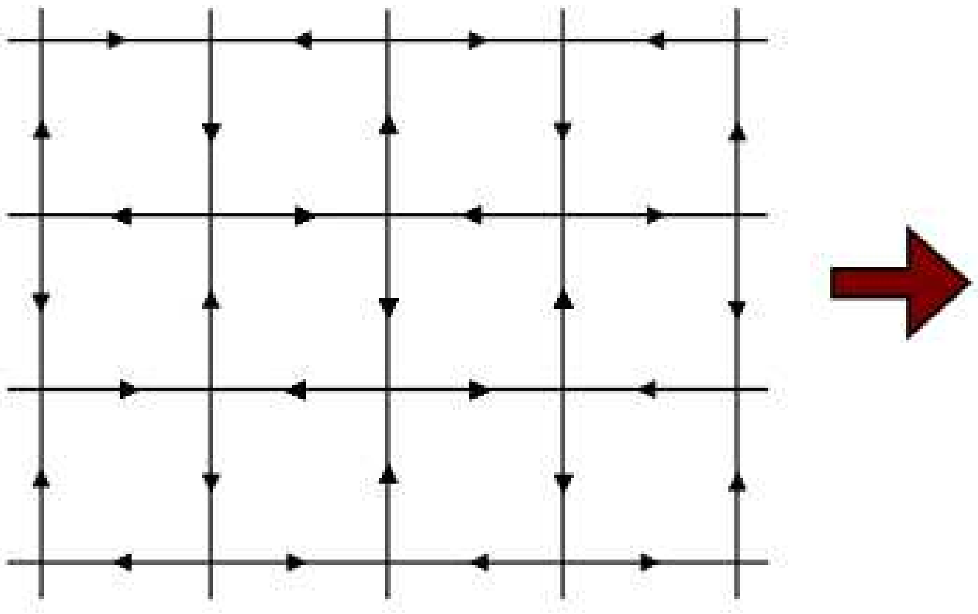
Phase transitions in loop models



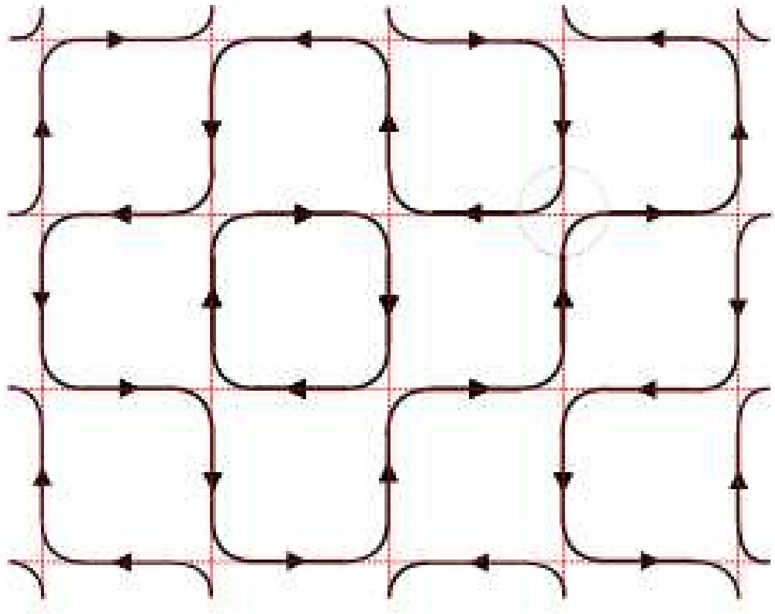
$$Z = \sum_{\text{configs}} p^{n_{\text{p}}} (1-p)^{n_{1-\text{p}}} n^{n_{\text{loops}}}$$

To define model: specify lattice, link directions and nodes

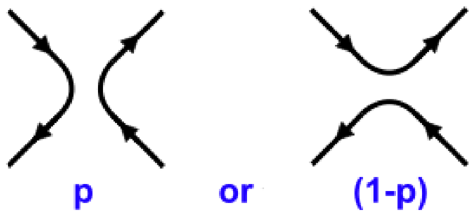
2D model



Sample configuration



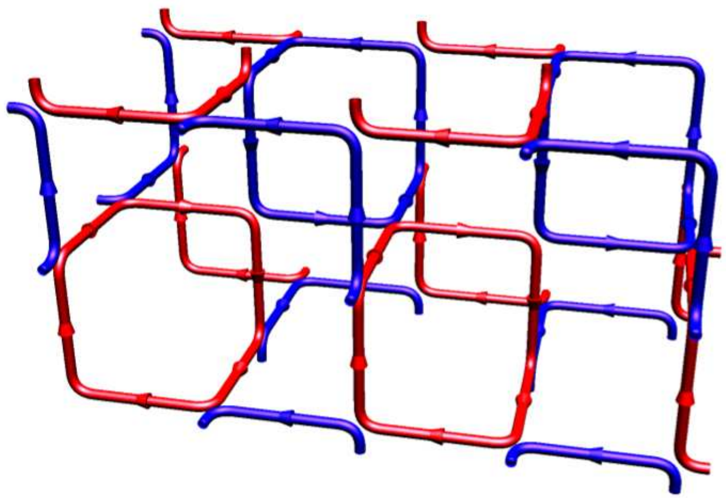
Phase transitions in loop models



$$Z = \sum_{\text{configs}} p^{n_{\text{p}}} (1-p)^{n_{1-\text{p}}} n^{n_{\text{loops}}}$$

To define model: specify lattice, link directions and nodes

Configuration of 3D model



Lattice designed so that:

$p = 0$ **only short loops**

$p = 1$ **all curves extended**

(**Alternative has symmetry**

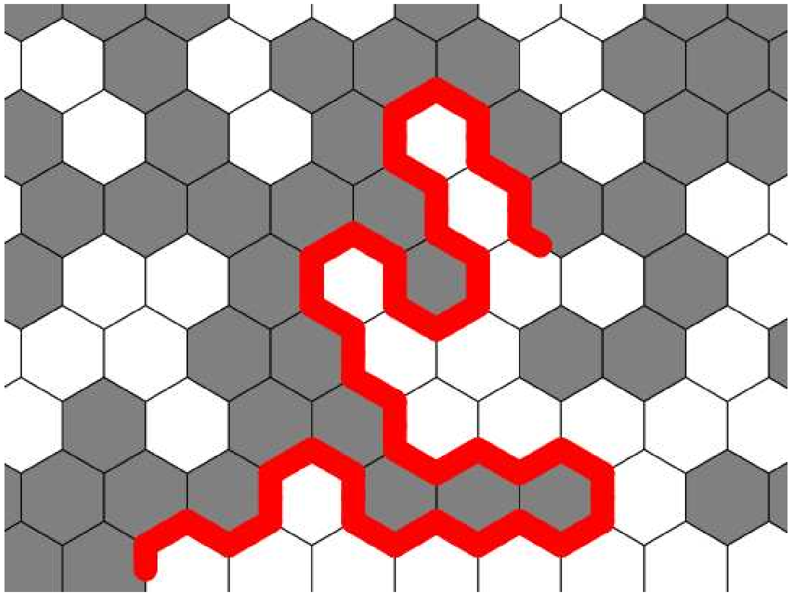
under $p \leftrightarrow [1-p]$)

Loop models and non-intersecting random curves in 3D

Random curves appear in many contexts

2D random curves

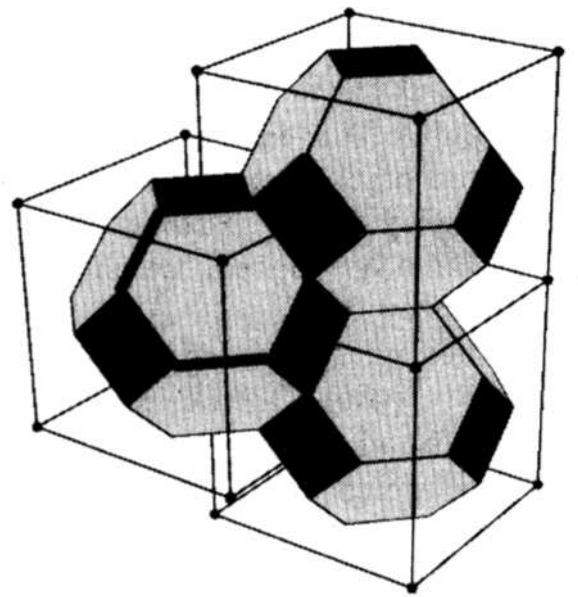
– zero-lines of random scalar field



Lattice version – percolation hulls

3D random curves

– zero-lines of random 2-cpt field



Lattice version – tricolour percolation

Scaling properties match

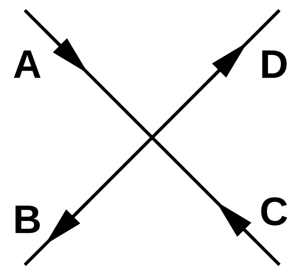
$n = 1$ **loop model**

Local Description and Continuum Theory

$$Z = \sum_{\text{configs}} p^{n_{\text{p}}} (1-p)^{n_{1-\text{p}}} n^{n_{\text{loops}}}$$

Introduce n **component complex**

unit vector $\vec{z}_l$ **on each link** l



Calculate $\mathcal{Z} = \mathcal{N} \prod_l \int \mathrm{d}\vec{z}_l \; \mathrm{e}^{-\mathcal{S}}$

with $\mathrm{e}^{-\mathcal{S}} = \prod_{\text{nodes}} \left[p(\vec{z}_A^{\dagger} \cdot \vec{z}_B)(\vec{z}_C^{\dagger} \cdot \vec{z}_D) + (1-p)(\vec{z}_A^{\dagger} \cdot \vec{z}_D)(\vec{z}_C^{\dagger} \cdot \vec{z}_B) \right]$

Expand $\prod_{\text{nodes}}[\ldots]$ **Loops contribute factors**

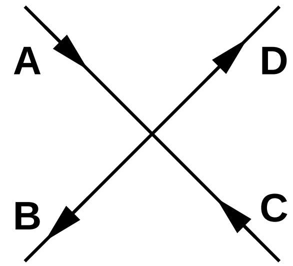
$$\sum_{\alpha,\beta,\ldots\gamma} \int \mathrm{d}\vec{z}_1 \ldots \int \mathrm{d}\vec{z}_L \; z_1^{*\alpha} z_2^{\alpha} z_2^{*\beta} \ldots z_L^{*\gamma} z_1^{\gamma}$$

Hence: **(i) factor of** n **per loop** **(ii) invariance under** $\vec{z}_l \longrightarrow \mathrm{e}^{\mathrm{i}\varphi_l}$

Local Description and Continuum Theory

$$Z = \sum_{\text{configs}} p^{n_\mathrm{p}} (1-p)^{n_{1-\mathrm{p}}} n^{n_{\text{loops}}}$$

Introduce n **component complex unit vector** $\vec{z}_l$ **on each link** l



Calculate $\mathcal{Z} = \mathcal{N} \prod_l \int \mathrm{d}\vec{z}_l \, \mathrm{e}^{-\mathcal{S}}$

with $\mathrm{e}^{-\mathcal{S}} = \prod_{\text{nodes}} \left[p(\vec{z}_A^\dagger \cdot \vec{z}_B)(\vec{z}_C^\dagger \cdot \vec{z}_D) + (1-p)(\vec{z}_A^\dagger \cdot \vec{z}_D)(\vec{z}_C^\dagger \cdot \vec{z}_B) \right]$

Continuum limit CP(n-1) model

$$S = \frac{1}{g} \int \mathrm{d}^d \mathbf{r} \, |(\nabla - \mathrm{i}A)\vec{z}|^2 \quad \textbf{with} \quad A = \frac{\mathrm{i}}{2}(z^{*\alpha} \nabla z^{\alpha} - z^{\alpha} \nabla z^{*\alpha})$$

with $|\vec{z}|^2 = 1$ **and invariance under** $\vec{z} \longrightarrow \mathrm{e}^{\mathrm{i}\varphi(\mathrm{r})} \vec{z}$

see also: Candu, Jacobsen, Read and Saleur (2009)

Phase transitions in CP^{n-1} model

Gauge-invariant degrees of freedom: 'spins' $Q \equiv zz^\dagger - 1/n$

$\big($**Mapping to Heisenberg model for** $n = 2$ **via** $S^\alpha = z^\dagger \sigma^\alpha z\big)$

Correlations

$\langle \mathrm{tr}\, Q(\mathbf{0}) Q(\mathbf{r}) \rangle \propto G(r)$ **– prob. points $\mathbf{0}$ and $\mathbf{r}$ lie on same loop**

Paramagnetic phase
— only finite loops

$$G(r) \sim \frac{1}{r} \mathrm{e}^{-r/\xi}$$

Critical point
— fractal loops

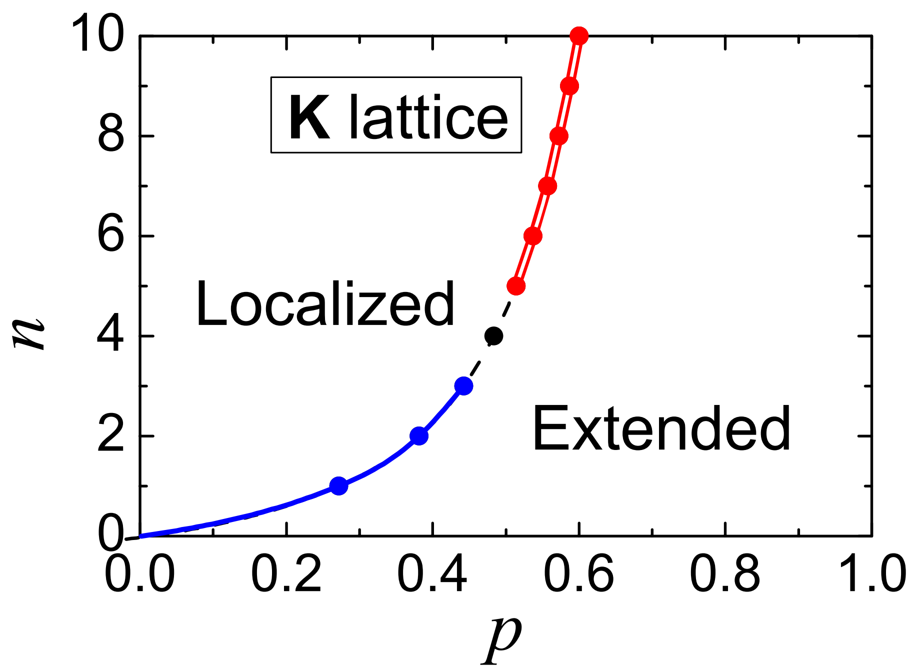
$$G(r) \sim r^{-(1+\eta)} \qquad d_f = \frac{5-\eta}{2}$$

Ordered phase
— Brownian loops escape to infinity

$$G(r) \sim r^{-2}$$

Results from simulations

Phase diagram



Critical exponents

n	ν	γ
1	0.9985(15)	2.065(18)
2	0.708(6)	1.39(1)
3	0.50(2)	1.01(2)

$n \geq 5:$ **1st order**

— **consistent at** $n = 2$ **with best estimates for classical**

Heisenberg model: $\nu = 0.7112(5)$ $\gamma = 1.3960(9)$

Summary

Two classes of system having non-local degrees of freedom:

- **Coulomb phases in spin Ice + dimer models**
- **Loop models**

Exotic critical behaviour at ordering transitions:

- **Symmetry-sustaining:**

  one-sided Kasteleyn transition

- **Symmetry-breaking:**

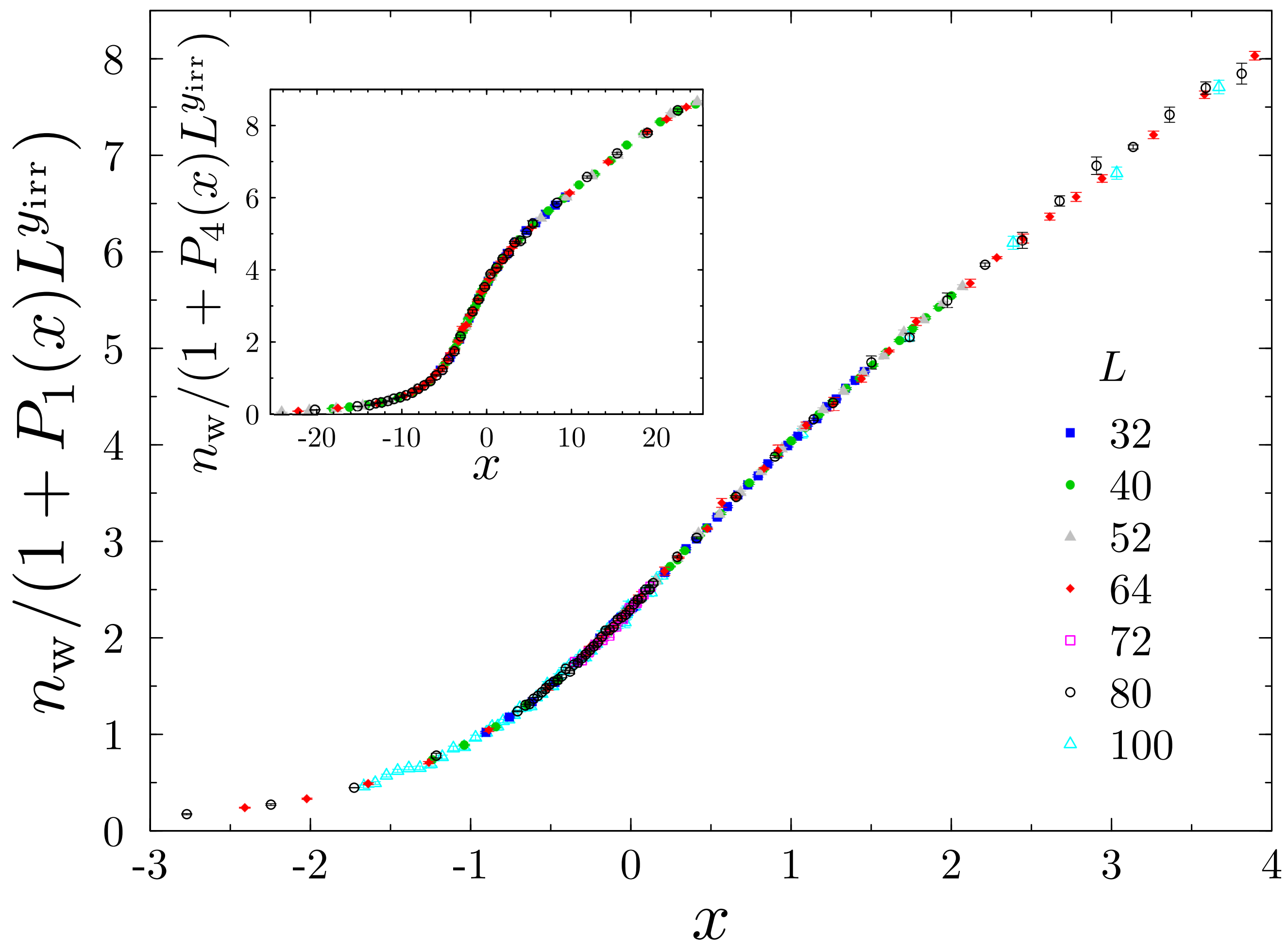
  non-standard critical behaviour at Curie transition

  non-LGW critical point for dimer ordering transition

- **Symmetry-breaking:**

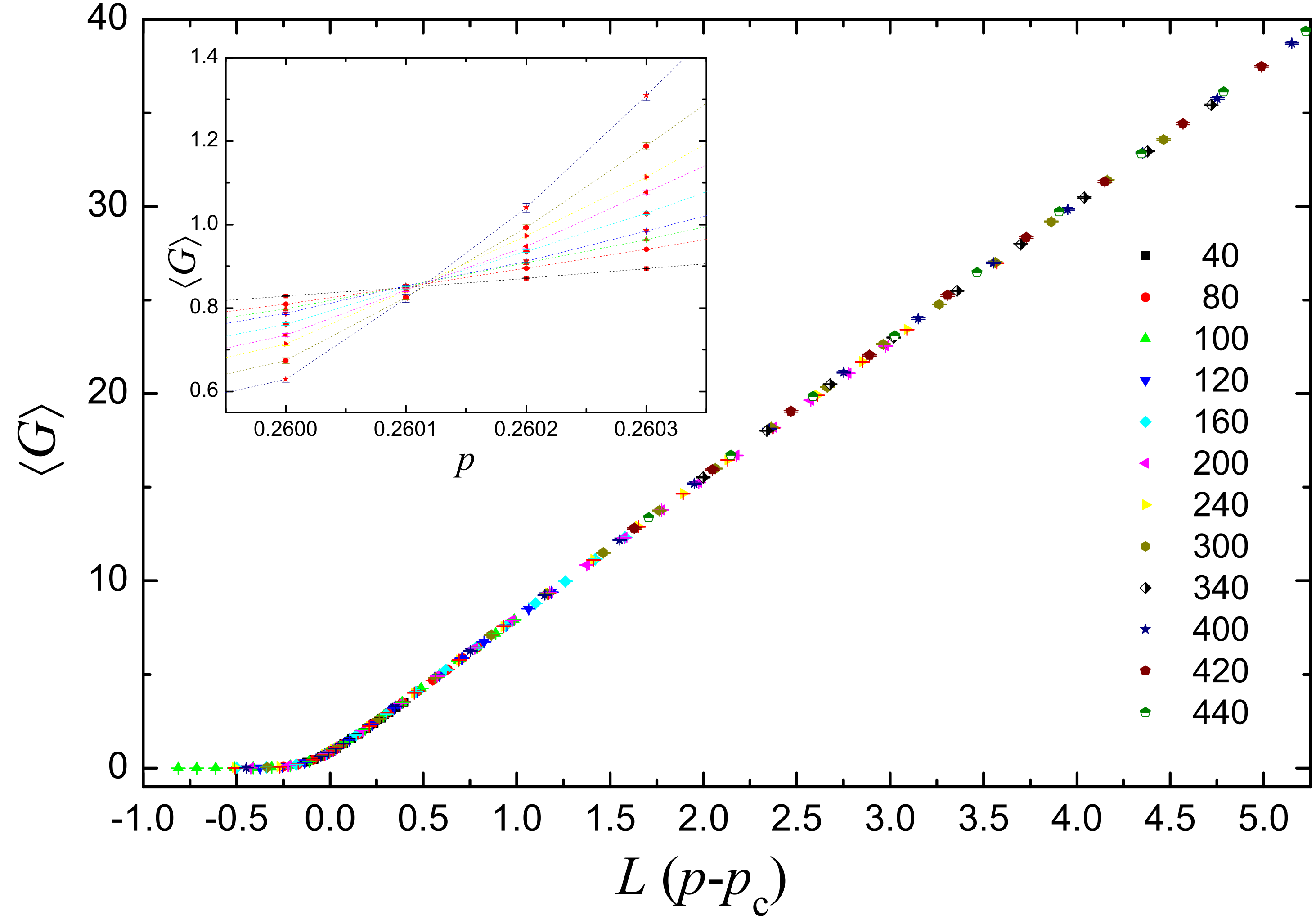
  loops as representation of CP^{n-1} model

Monte Carlo data for $n = 2$ and $n = 3$

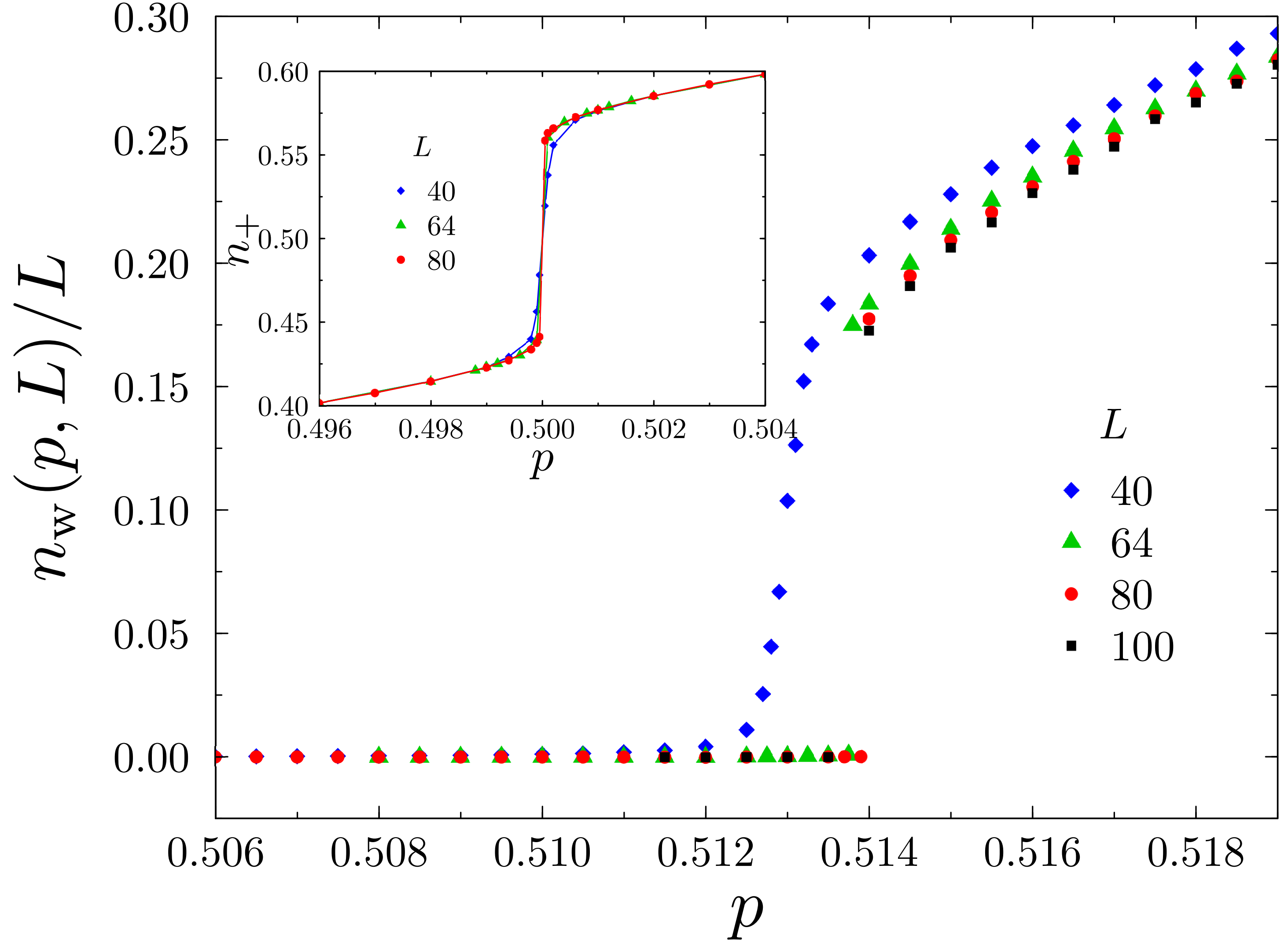


$x \approx (p - p_c)L^{1/\nu}$

Monte Carlo data for $n = 1$

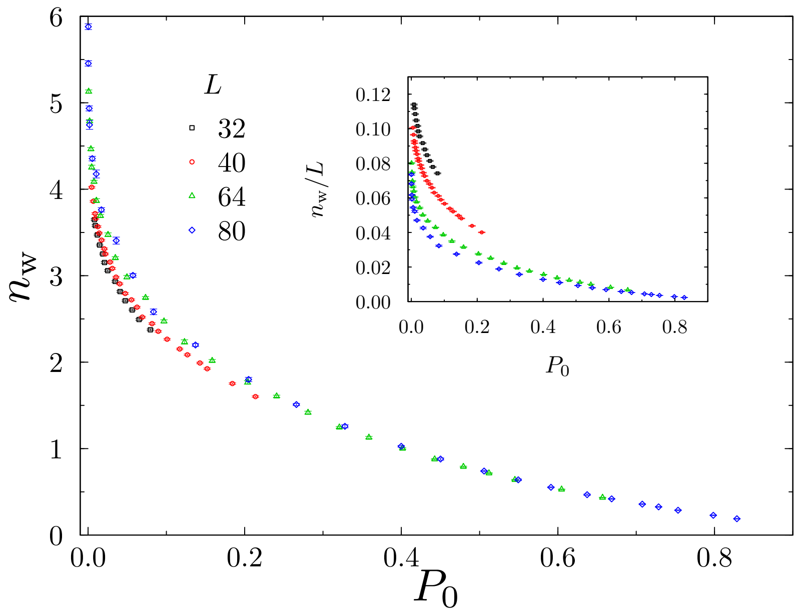


Monte Carlo data for $n = 5$

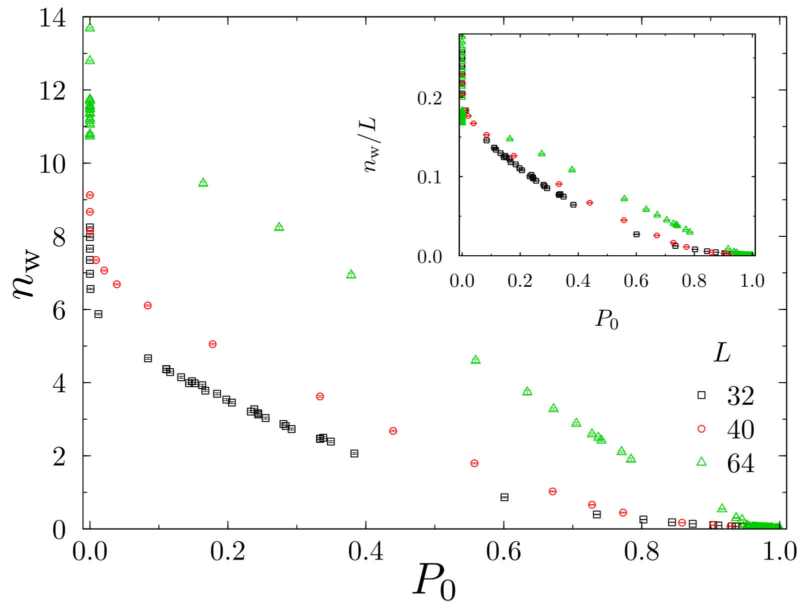


Comparing $n = 3$ and $n = 5$

[Avge No spanning curves] vs [Prob no spanning curve]



$n = 3$



$n = 5$