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Random Spring Networks vs. Soft Sphere Packings: Jamming Meets Percolation

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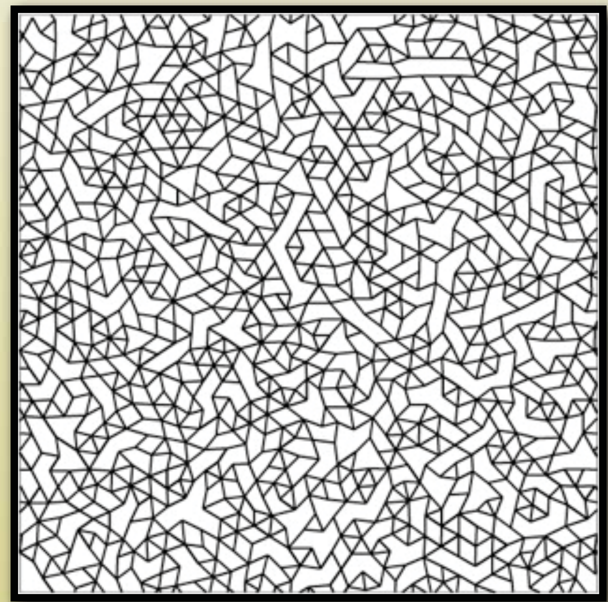

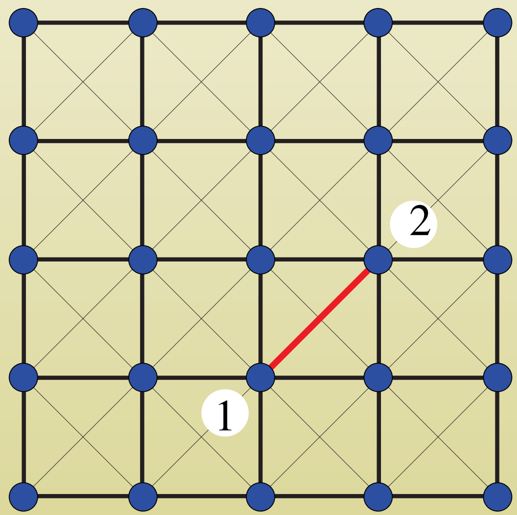

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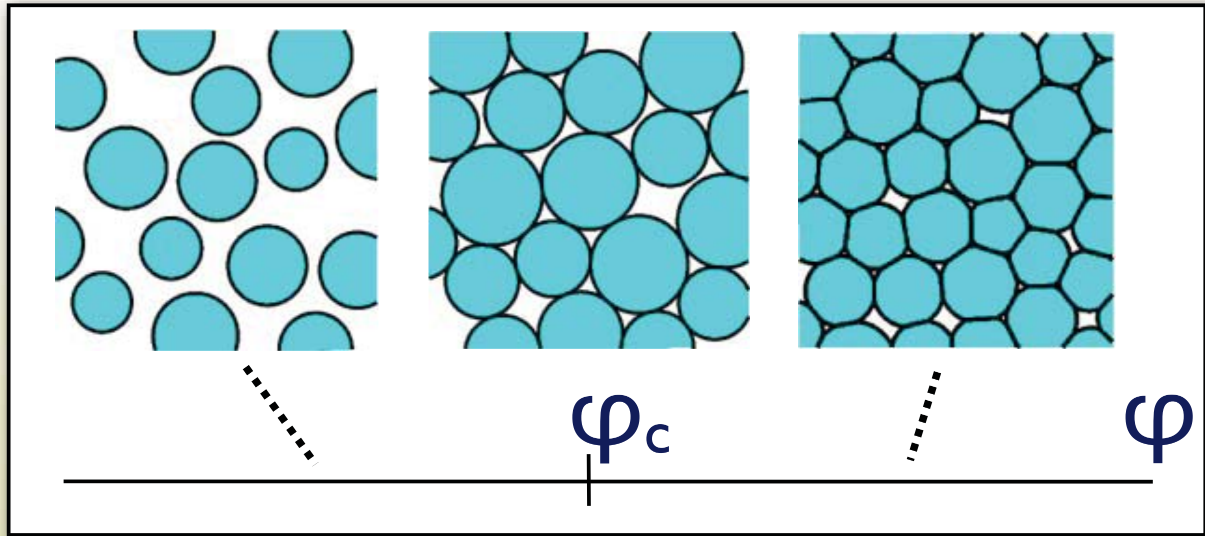
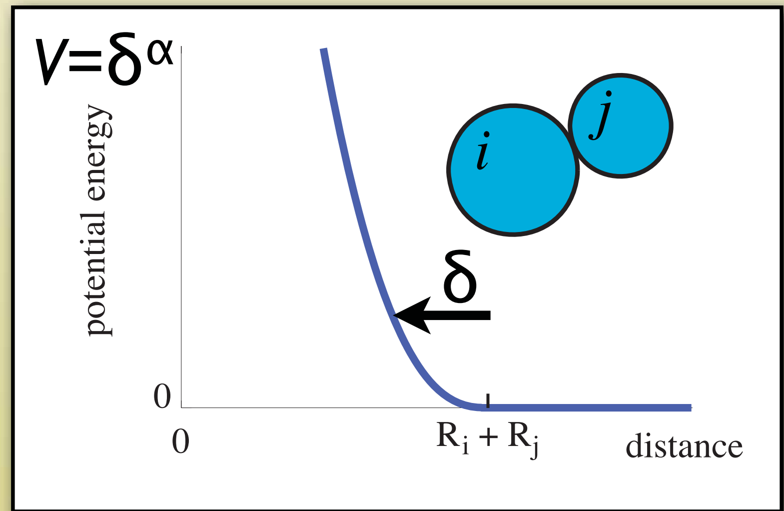
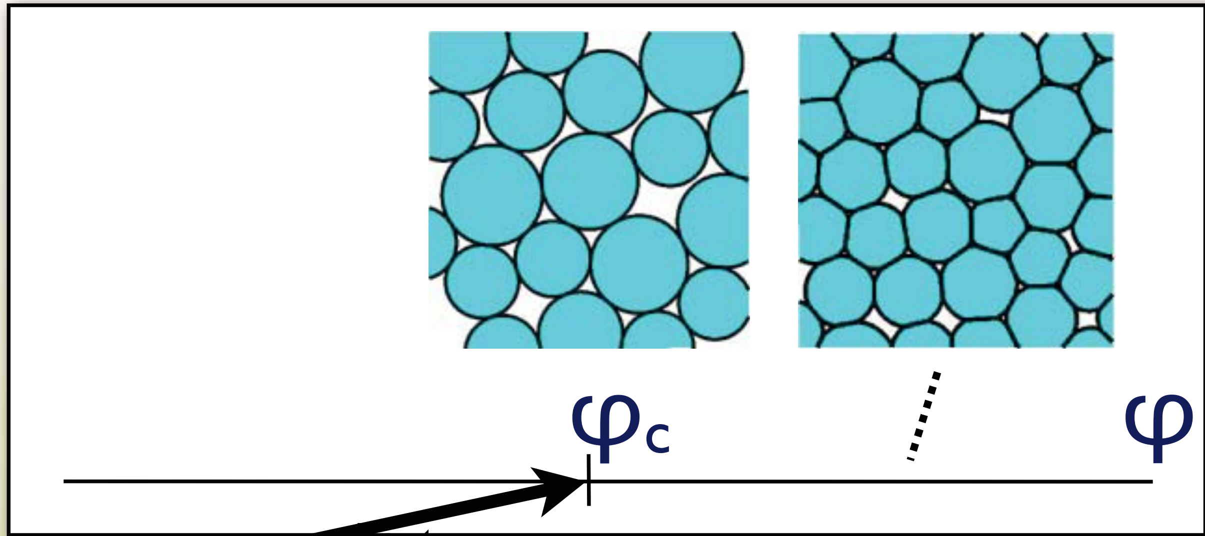
Soft sphere packings





2D polydisperse disk packings: disordered solids with properties determined by $\Delta\varphi=\varphi-\varphi_c$

Marginal stability and scaling away from it



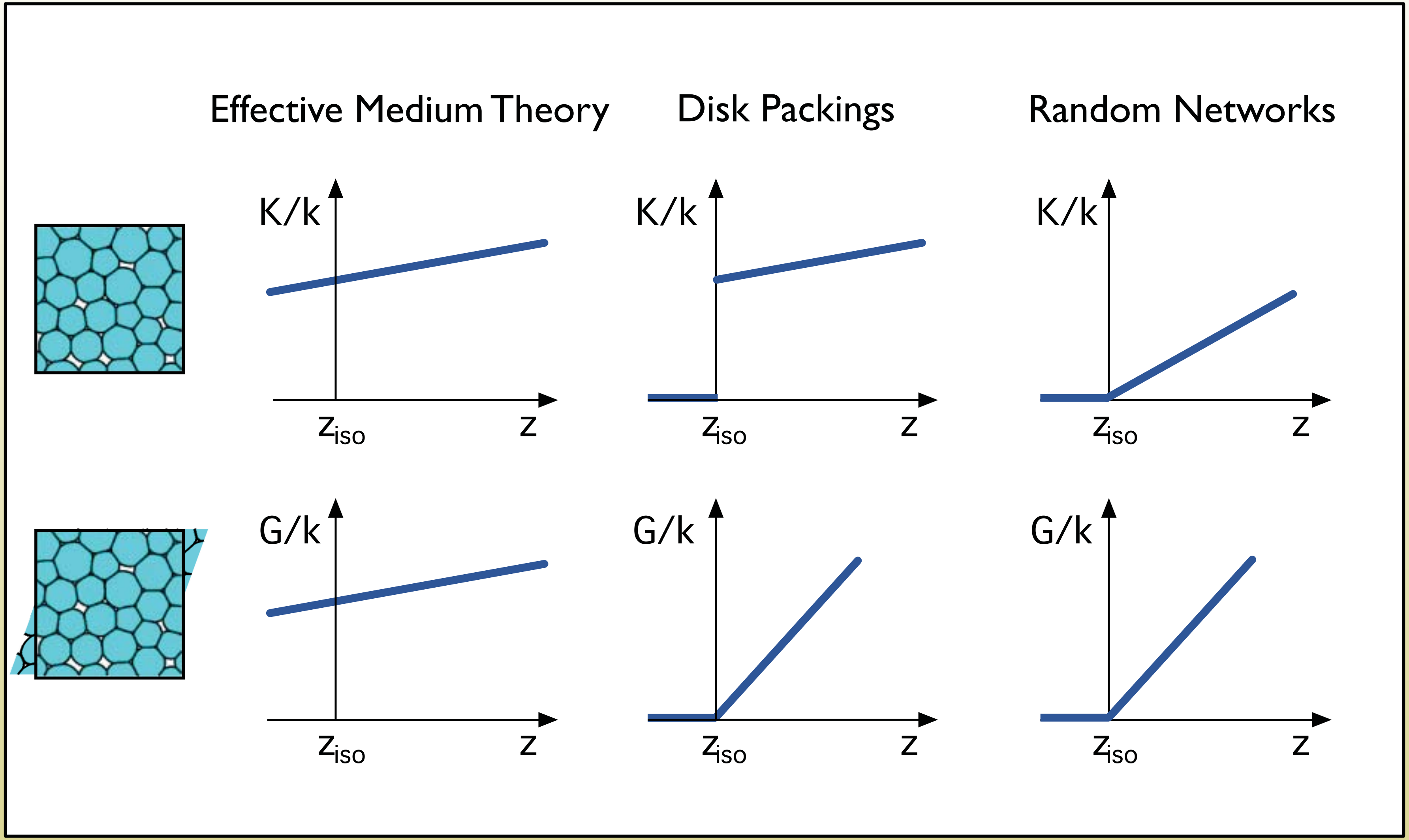
at $\varphi=\varphi_c$:

$p=0$

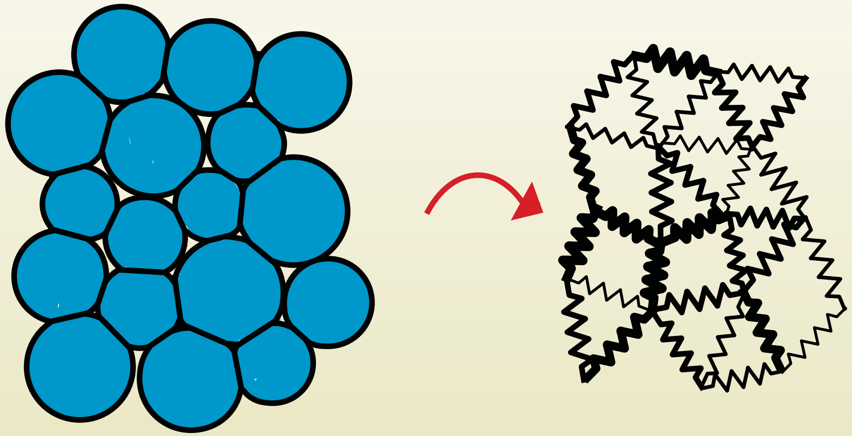
$z=z_c=2d$

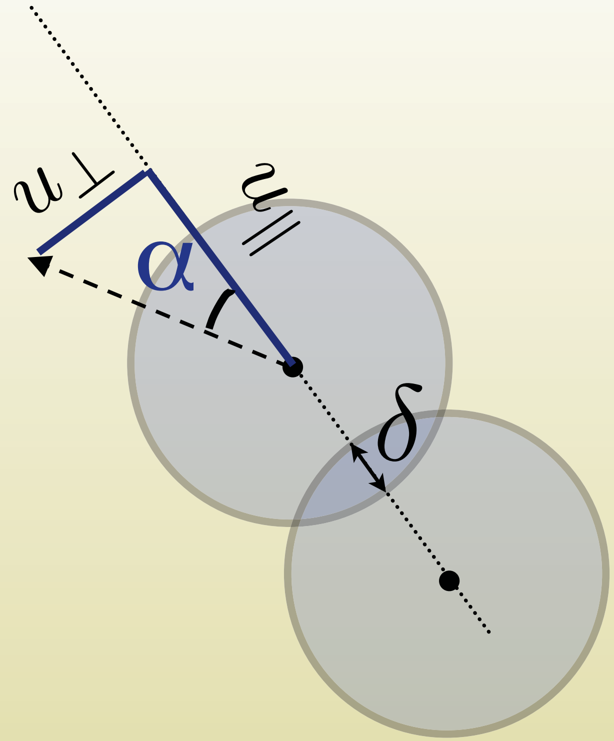
overlap	$\delta \sim \phi - \phi_c$
pressure	$p \sim \delta^{\alpha-1}$
contact number	$z - z_c \sim \delta^{1/2}$
elastic moduli	$G/K \sim \delta^{1/2}$
length scale	$\ell^* \sim \delta^{-1/2}$

Which is the odd one out?



Elastic network description of packings





Total elastic energy

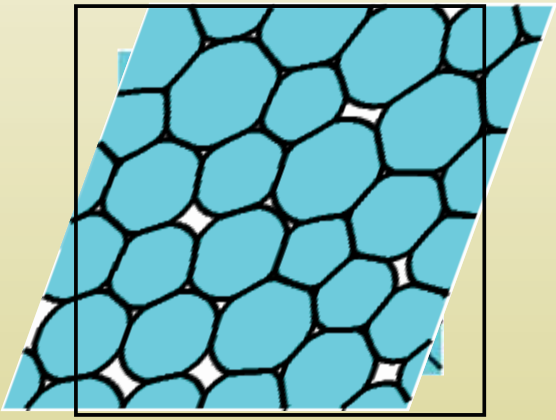
$$E = \sum_{i \neq j} V(r_{ij})$$

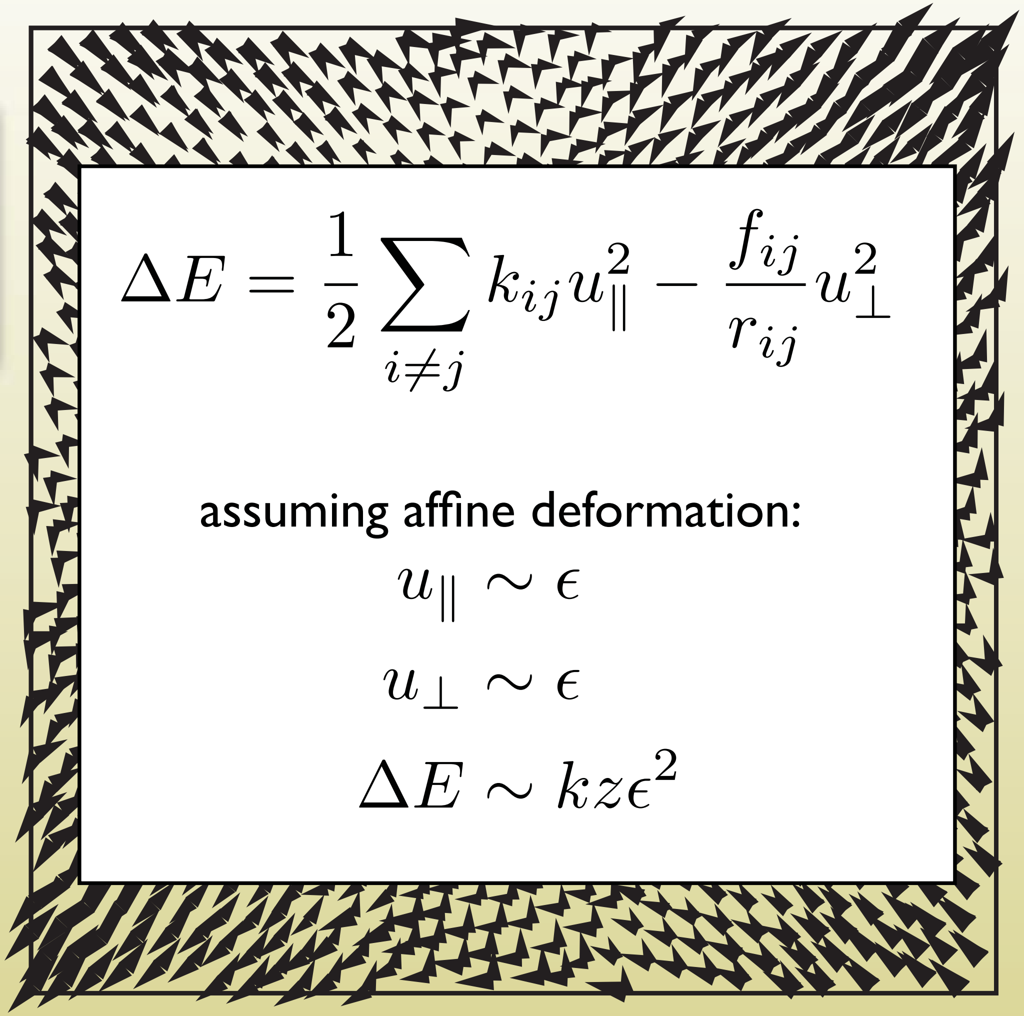
Change in elastic energy due to displacements *u*

$$\Delta E = \frac{1}{2} \sum_{i \neq j} k_{ij} u_\parallel^2 - \frac{f_{ij}}{r_{ij}} u_\perp^2$$

Effective Medium Theory?

EMT assumes that the map from old to new positions is *affine*





$$\Delta E = \frac{1}{2} \sum_{i \neq j} k_{ij} u_{\parallel}^2 - \frac{f_{ij}}{r_{ij}} u_{\perp}^2$$

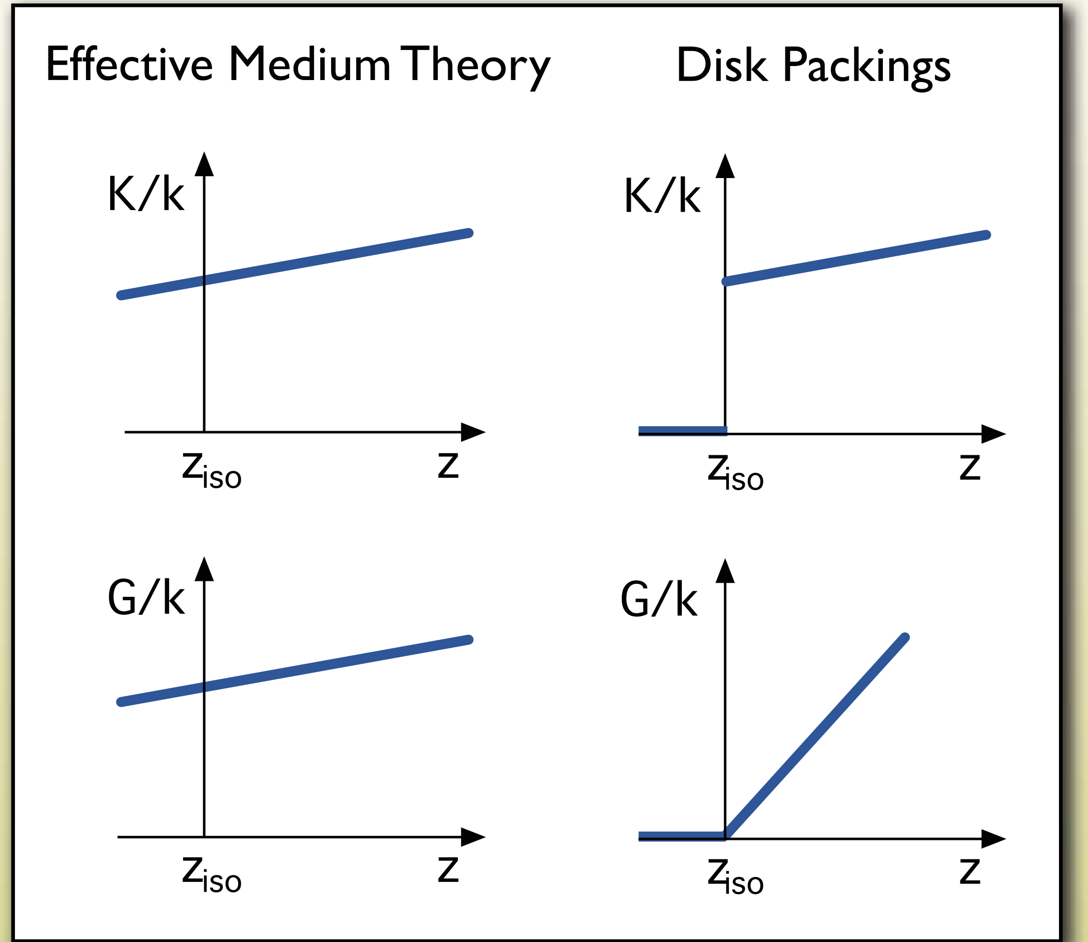
assuming affine deformation:

$$u_{\parallel} \sim \epsilon$$

$$u_{\perp} \sim \epsilon$$

$$\Delta E \sim kz\epsilon^2$$

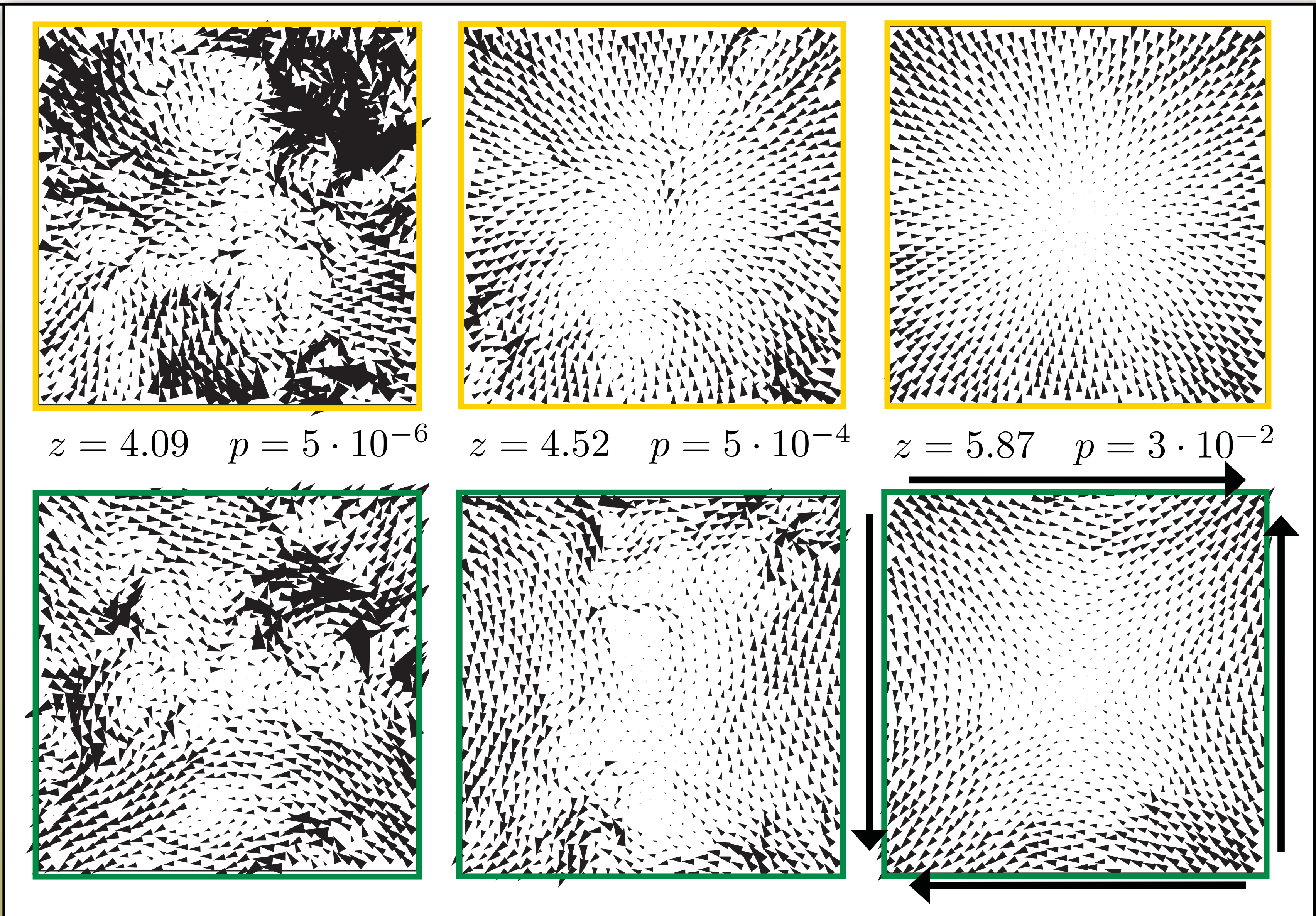
Effective Medium Theory?



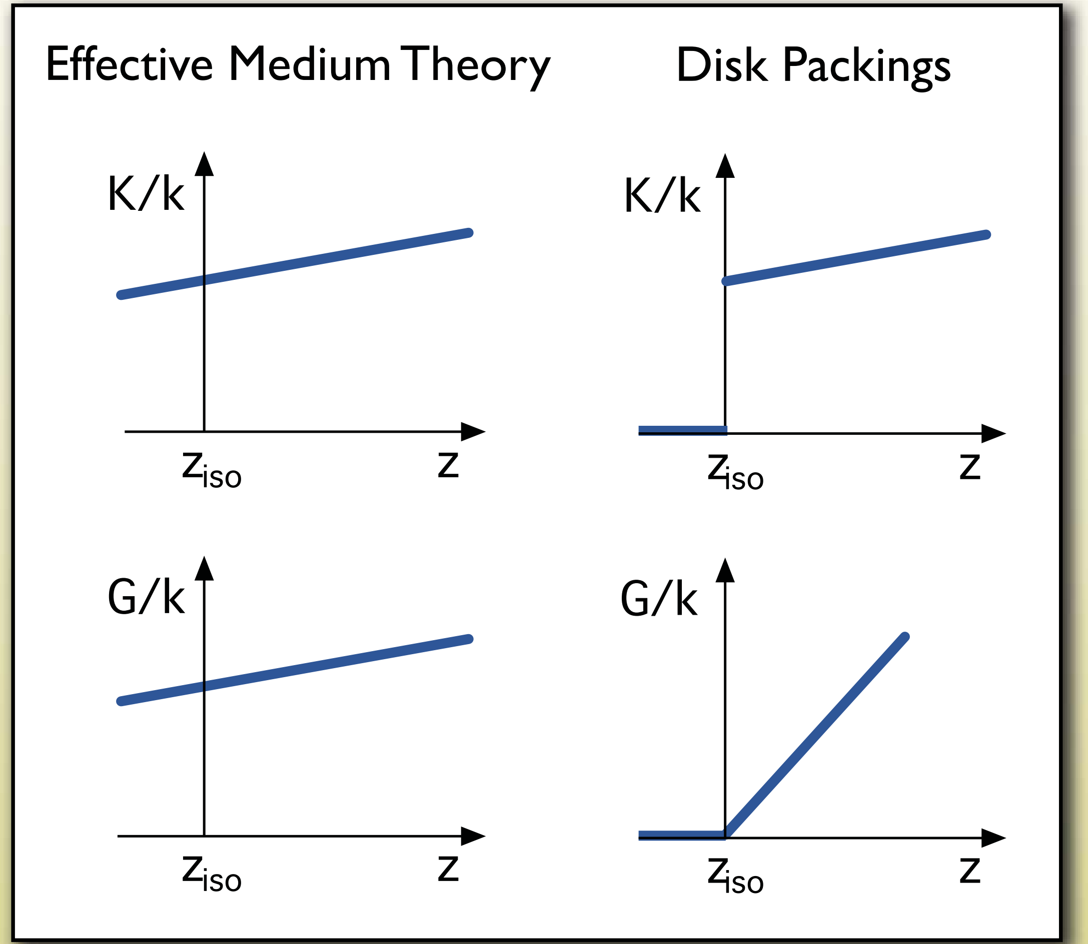
Why does EMT fail?

Makse *et al.*, PRL 1999

Non-affinity!



EMT's main assumption fails horribly near φ_c



Why does EMT fail?
Makse *et al.*, PRL 1999

Why does EMT seem to work?
Ellenbroek *et al.*, EPL 2009

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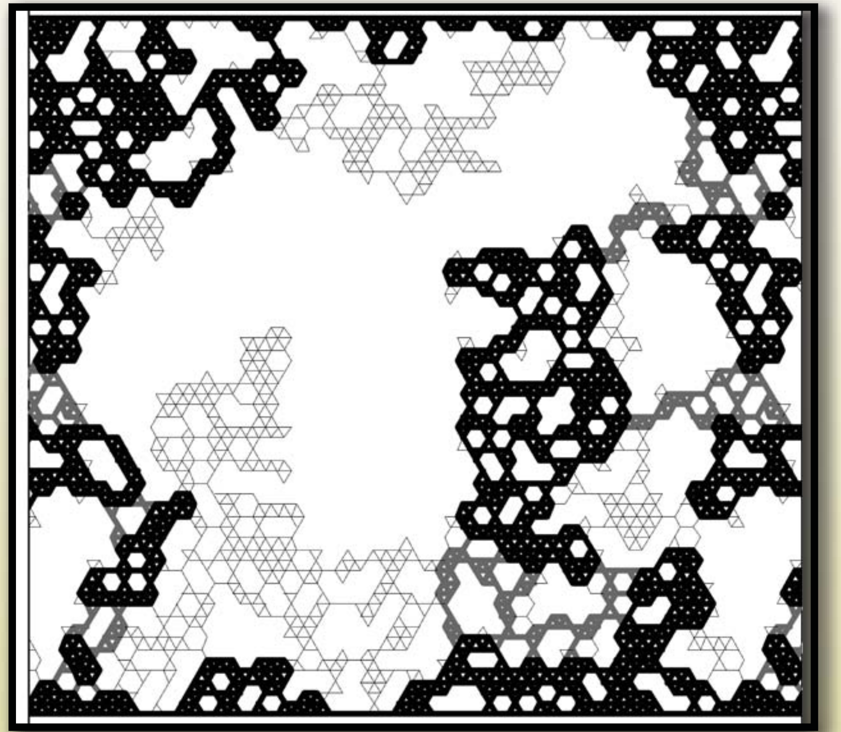
Rigidity percolation



Moukarzel, 1999
Jacobs and Thorpe, 1995

Traditional example:
Diluted triangular lattice
Each bond present with probability p
Threshold value p_c
Fractal rigid backbone
Second order transition
Elastic moduli vanish at the transition

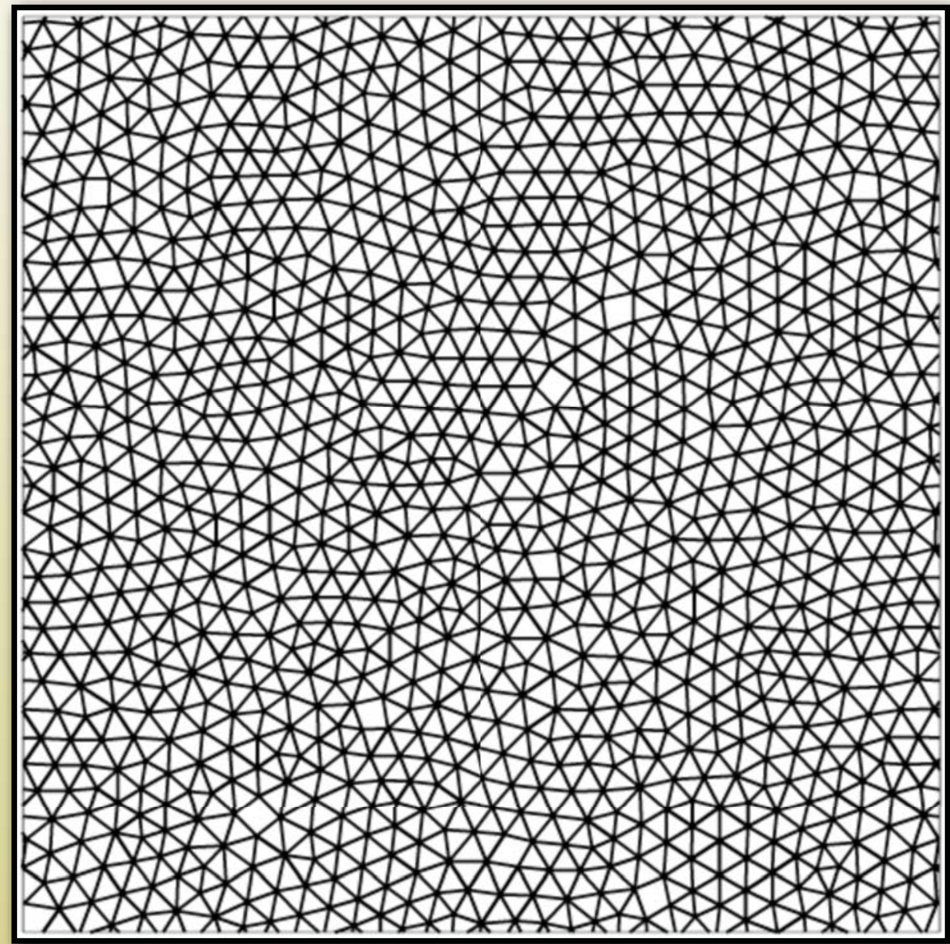
What can we learn from rigidity percolation models that are closer to soft disk packings?

Random Networks

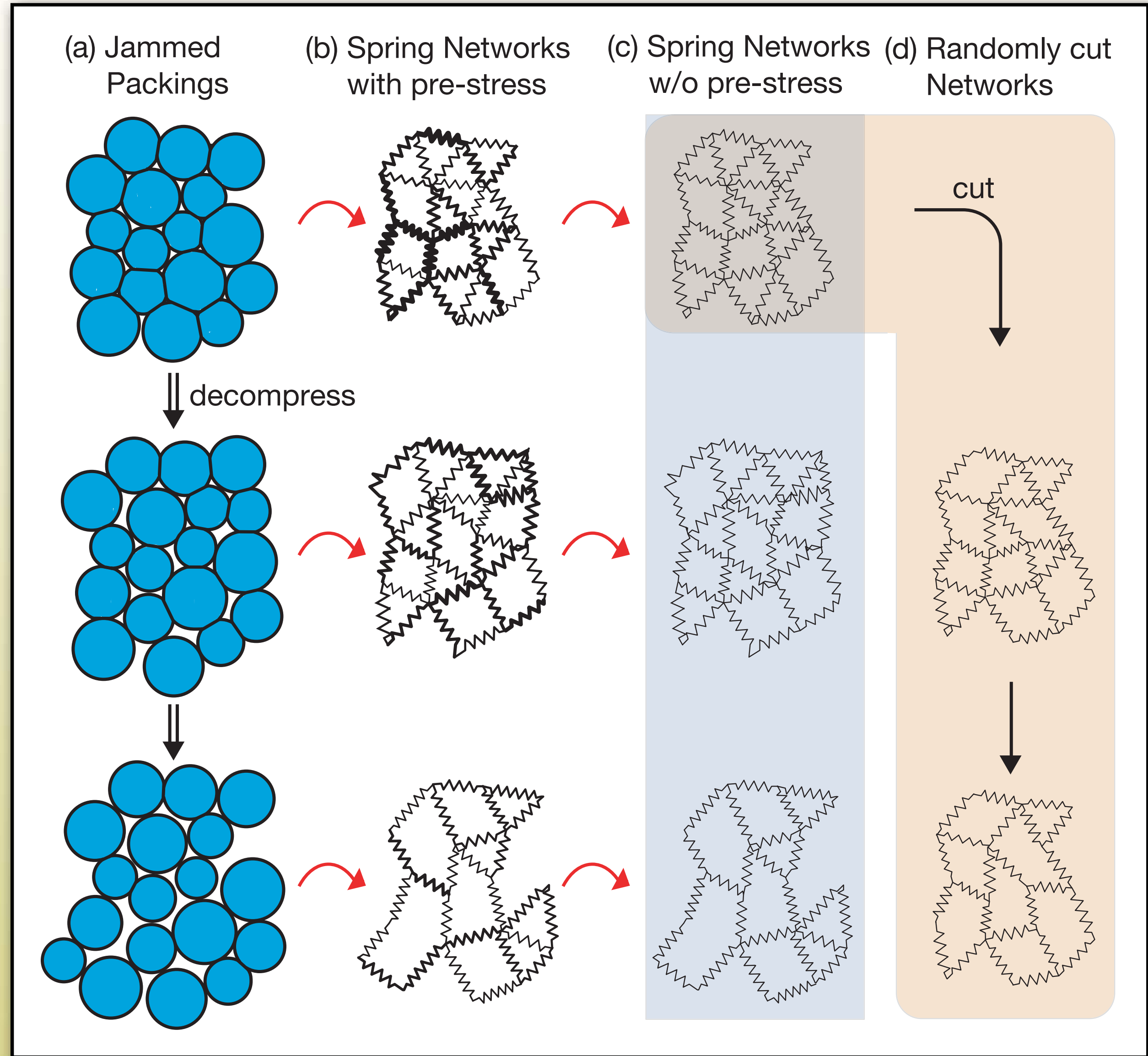
Packings are *almost* like random networks... but not quite!

Start from high density packing

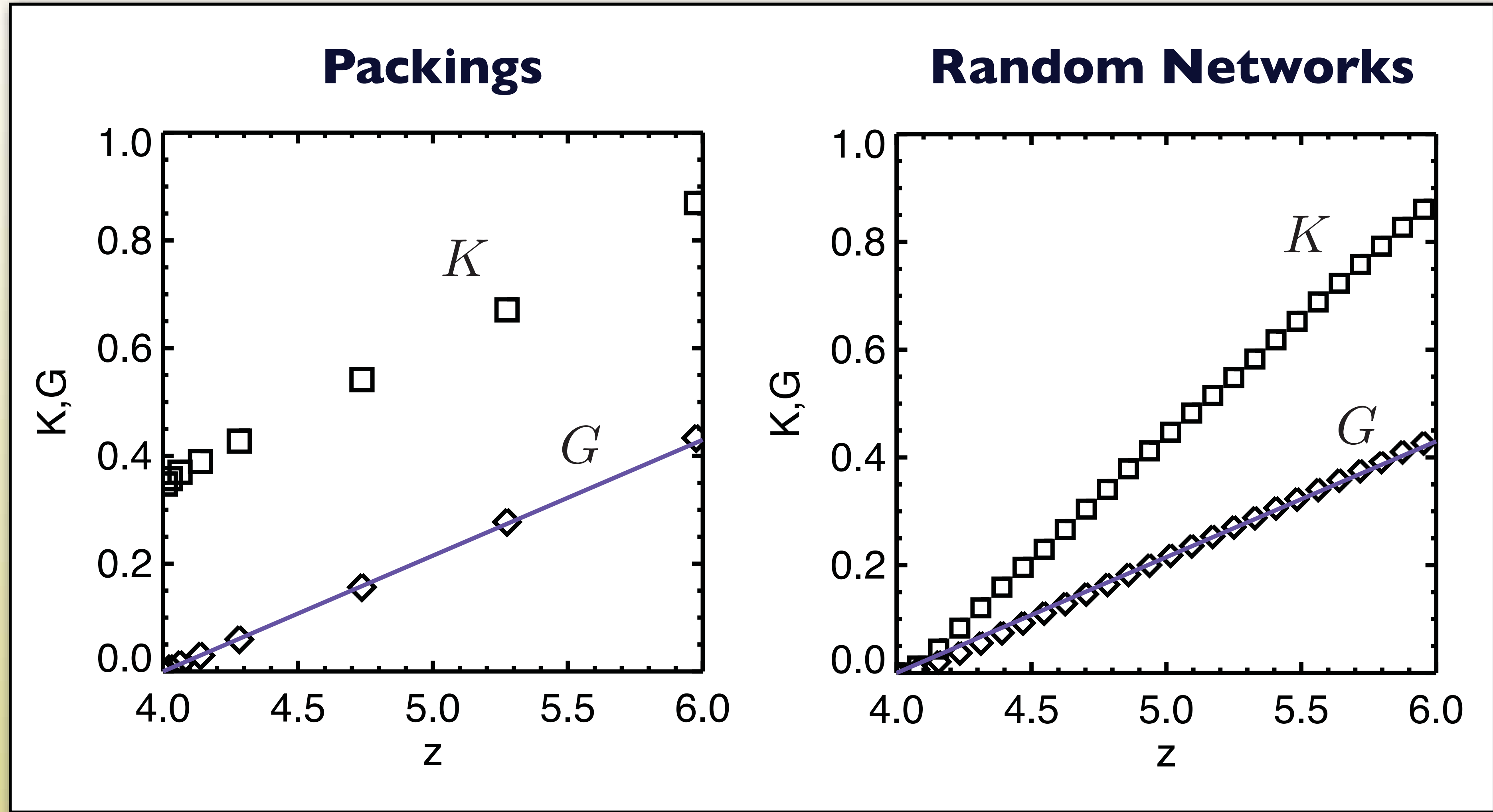
Randomly delete/cut bonds while keeping at least 3 bonds per node



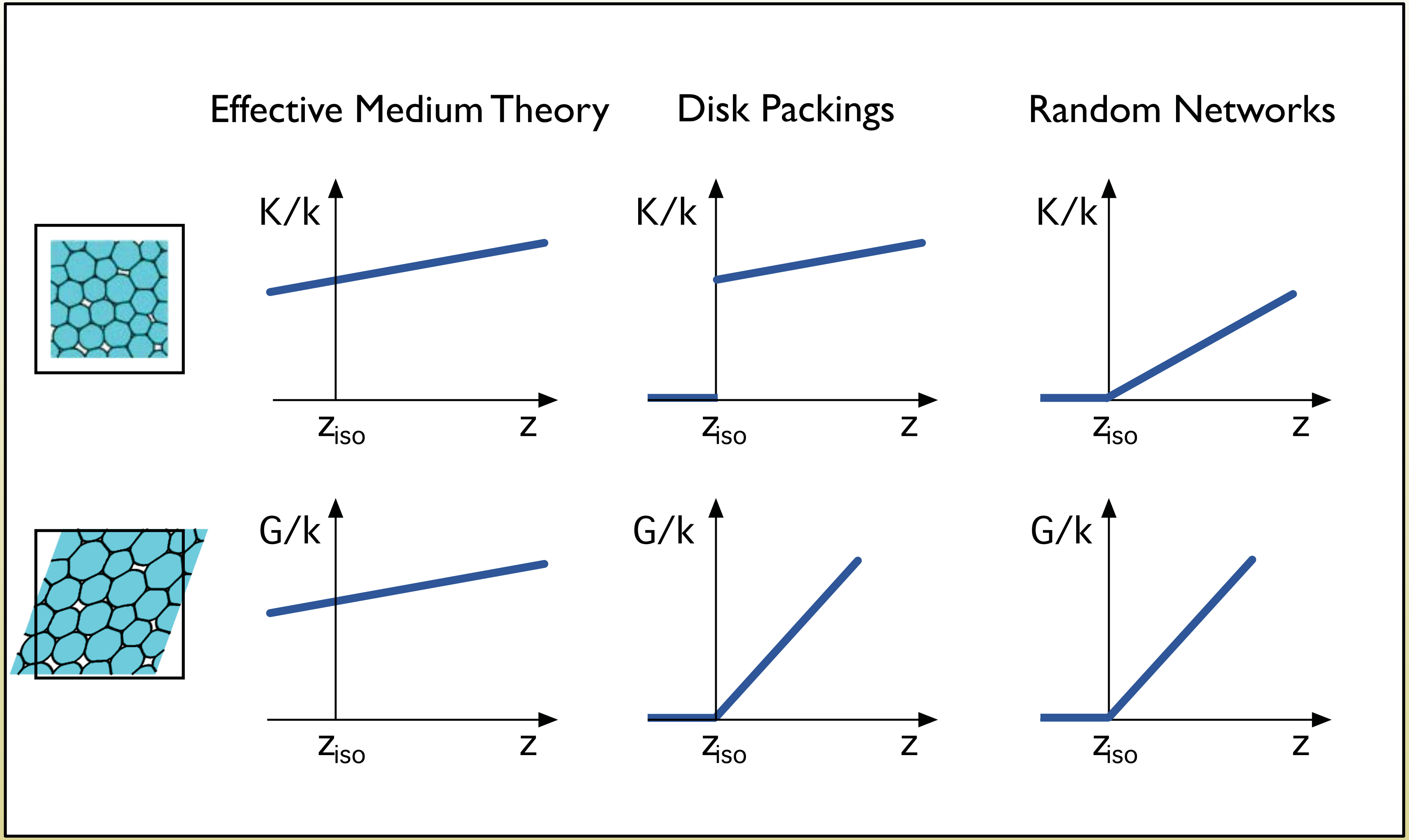
Families of networks



Comparing the elastic moduli



What else can we learn from this?

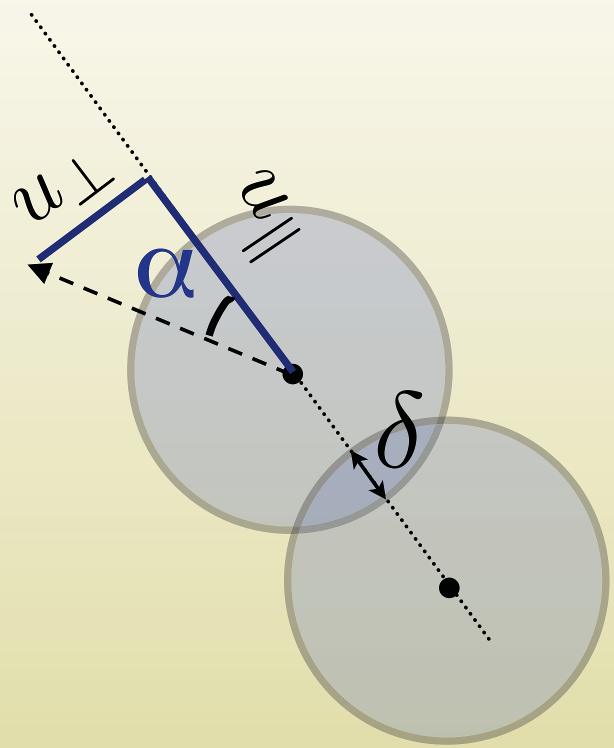


Characterizing non-affinity

Change in elastic energy due to displacements *u*

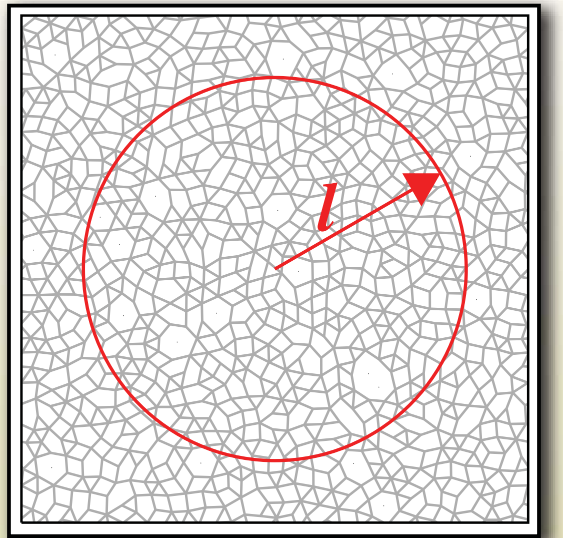
$$\Delta E = \frac{1}{2}\sum_{i\neq j} k_{ij}u_{\parallel}^2 - \frac{f_{ij}}{r_{ij}}u_{\perp}^2$$

Note that we can vary dN coordinates to minimize zN/2 energy contributions



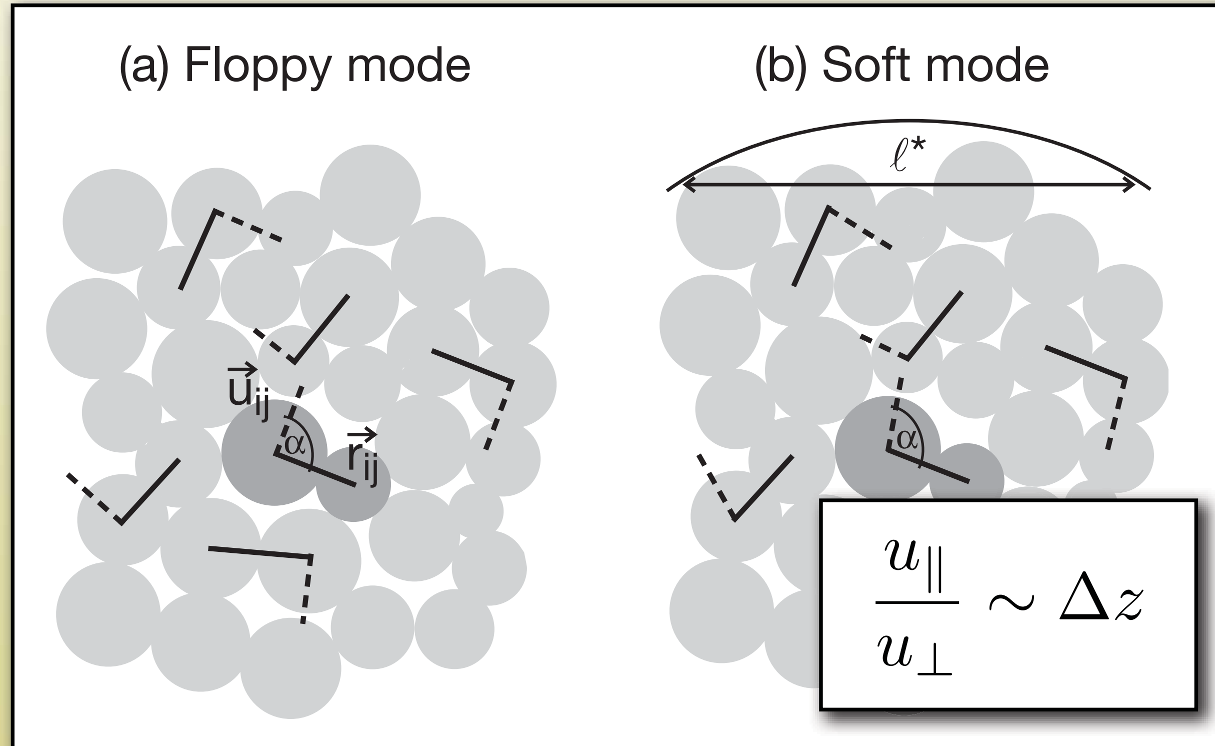
Study statistics of the "displacement angle" α while varying z

What α do we expect?



$$\ell^* \sim \frac{1}{\Delta z}$$

Cutting out this piece will give something floppy if there are more boundary bonds than excess bulk bonds



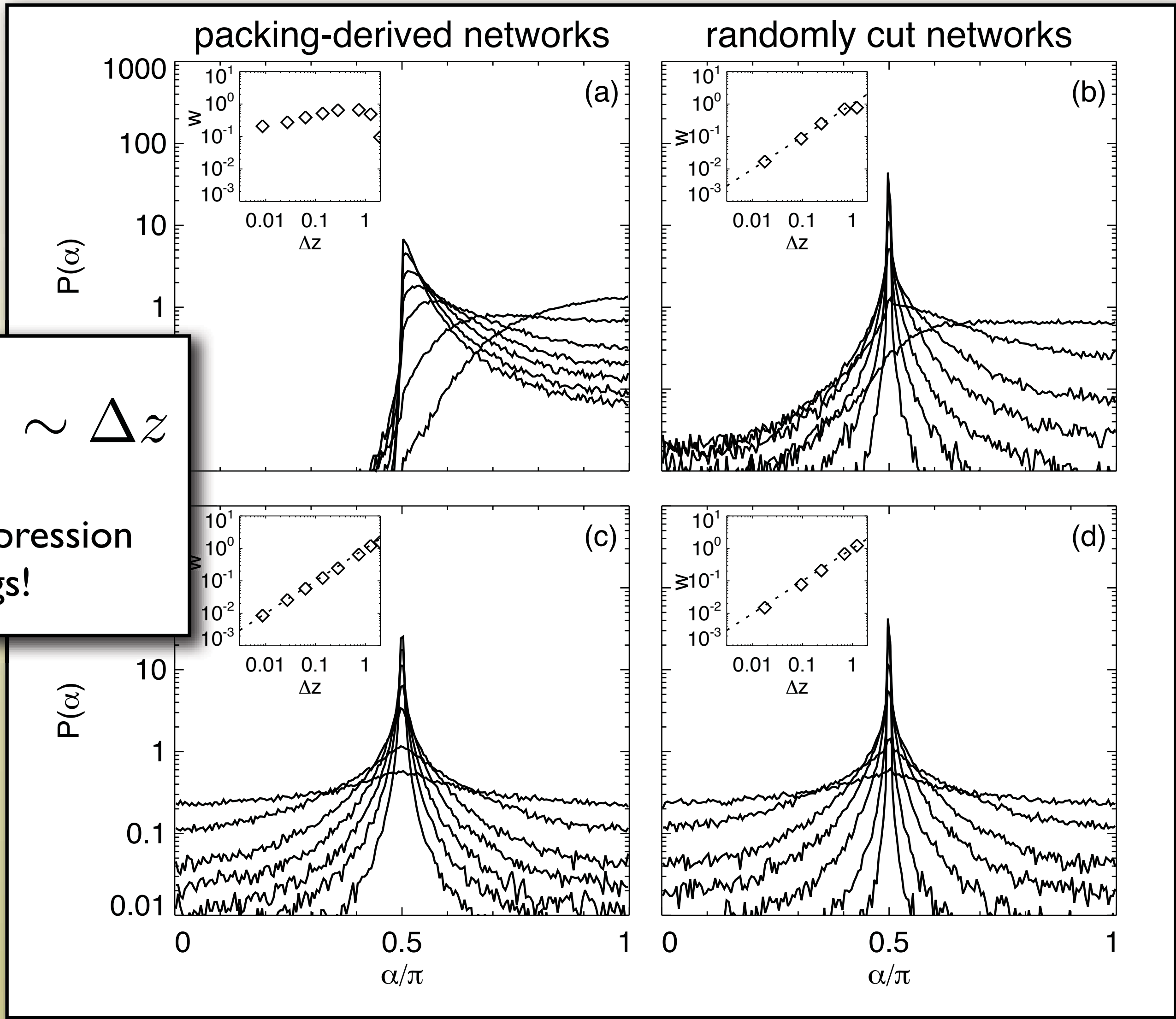
$$\frac{u_\parallel}{u_\perp} \sim \Delta z$$

Wyart *et al.*, EPL (2005), PRL (2008)
Ellenbroek *et al.*, EPL (2009)

Probability densities of α

Compression

Shear



$$\text{width} \sim \frac{u_{\parallel}}{u_{\perp}} \sim \Delta z$$

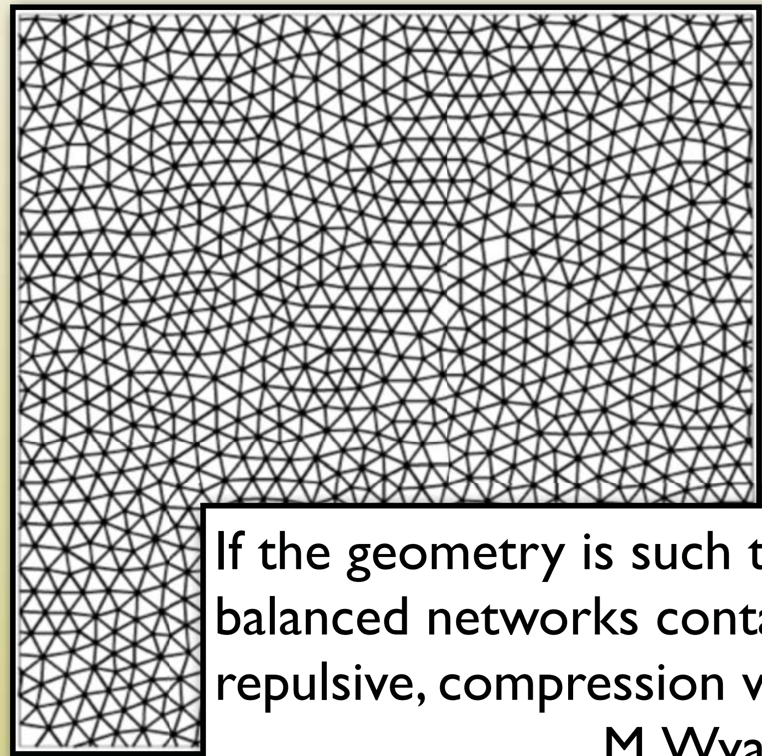
Except for compression of packings!

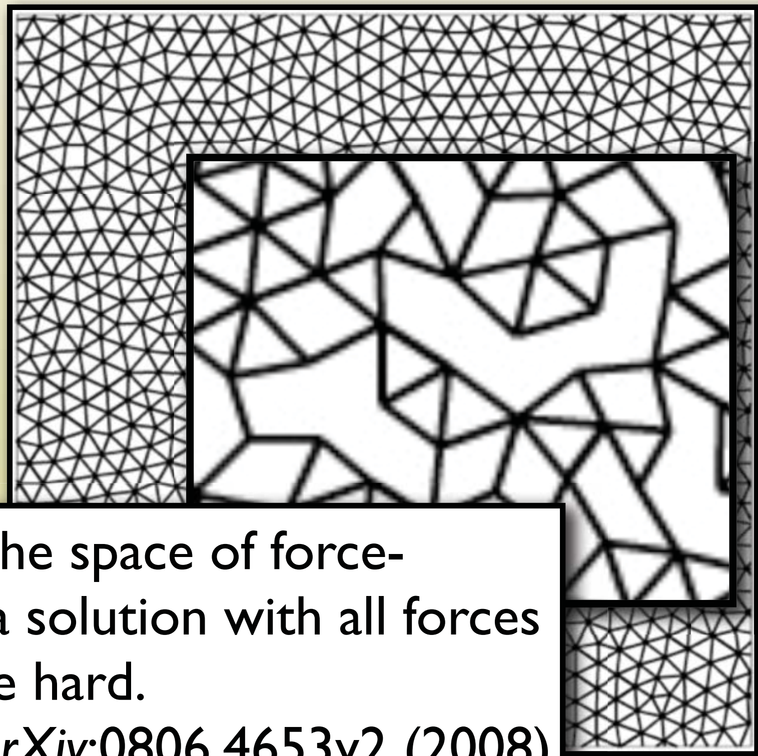
Ellenbroek *et al.*, EPL (2009)

What’s behind this?

$$\Delta E = \frac{1}{2}\sum_{i\neq j} k_{ij} u_{\parallel}^2 = \frac{1}{2}\sum_{i,j=1}^{dN} u_i M_{ij} u_j$$

With all spring constants identical, the dynamical matrix M is *purely geometric*





If the geometry is such that the space of force-balanced networks contains a solution with all forces repulsive, compression will be hard.

M. Wyart, *arXiv*:0806.4653v2 (2008)

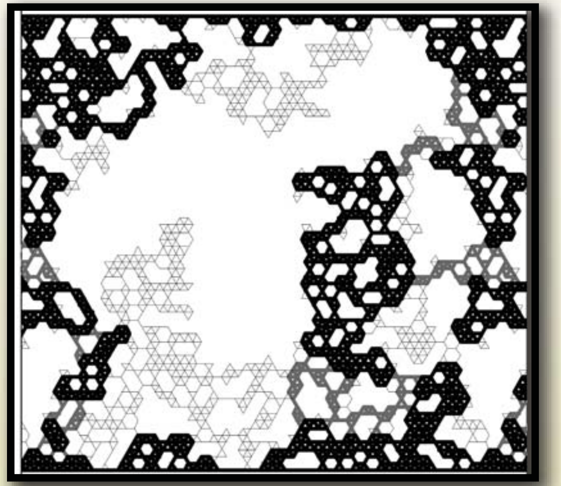
Summary (moduli)

Non-affinity diverges as unjamming is approached.

Elastic behavior of random networks is the same as that of sheared packings.

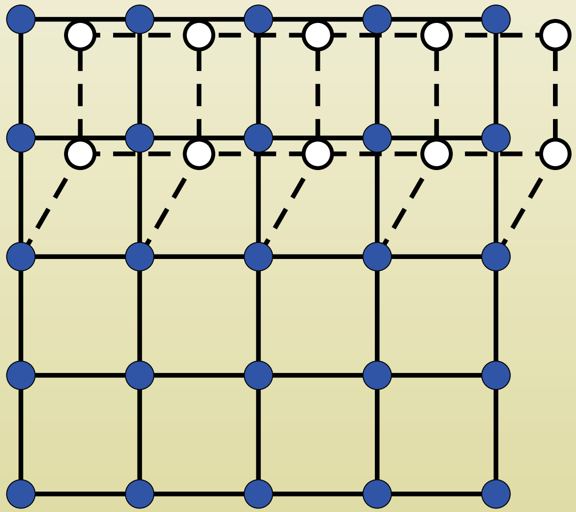
The compression response of packings is anomalous.

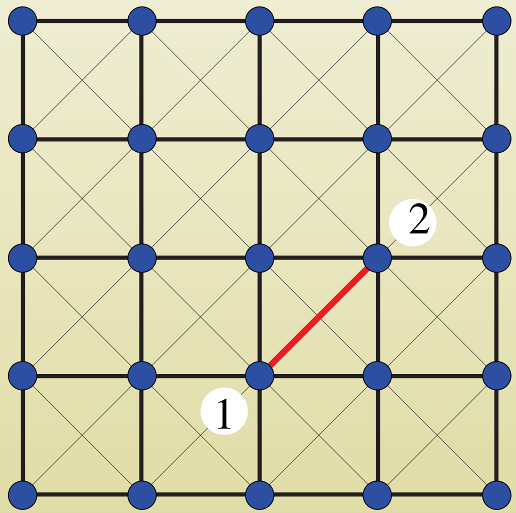
How to theorize more?



Learning about jamming from rigidity percolation:
What are suitable models?

- we want a non-fractal structure at the transition
- we want isostaticity at the transition

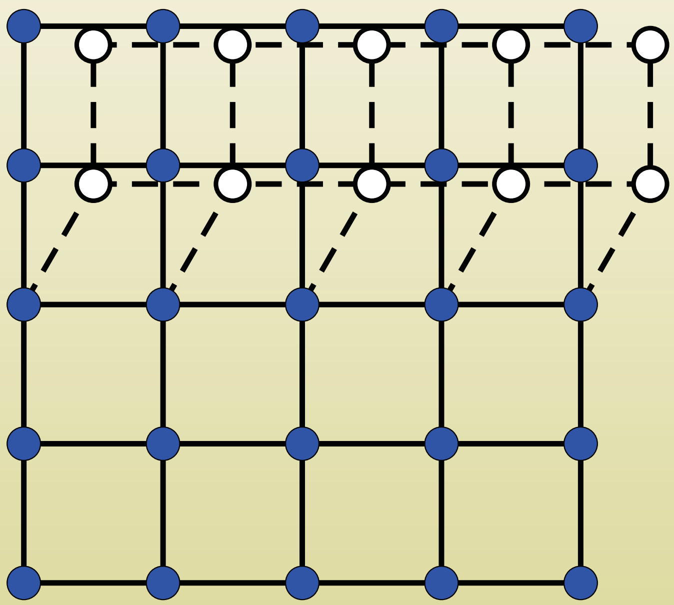


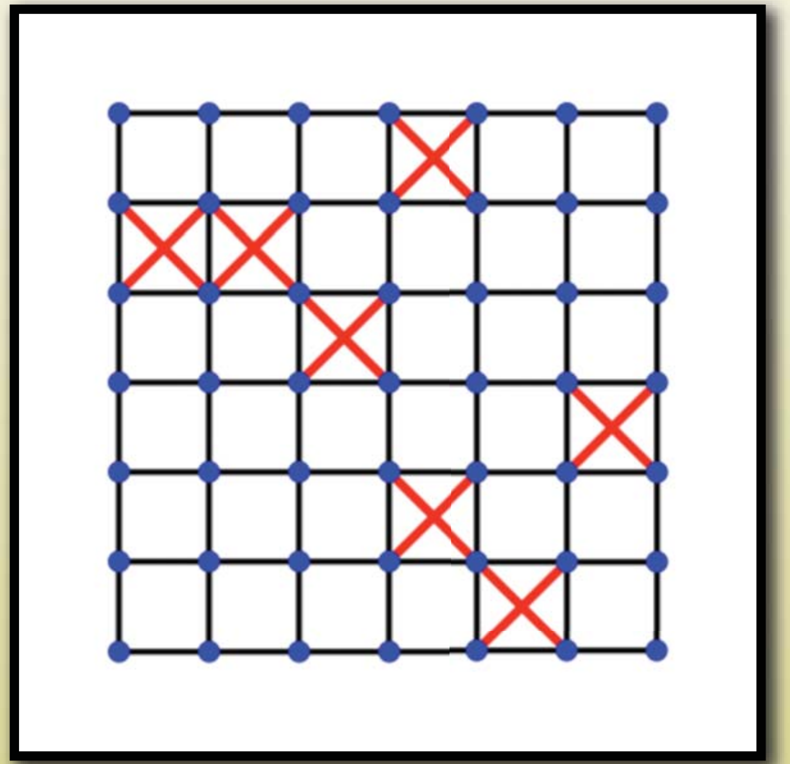


Square lattice with randomly added next-nearest-neighbor bonds

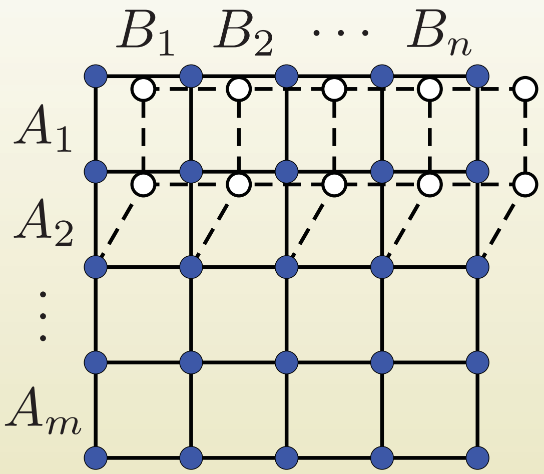
Rigidity transition

For what p is the resulting structure rigid?
How does this p depend on system size?

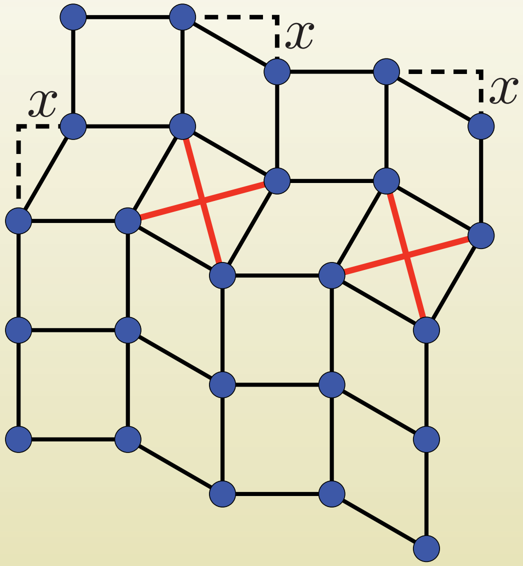




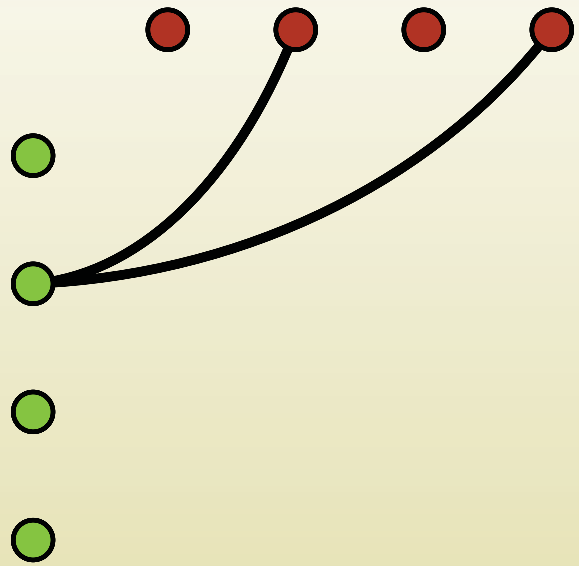
Defining variables and mapping



(n+m)-dimensional space
of floppy modes



Each crosslink sets two
coordinates to be equal



A connected graph represents
a rigid configuration

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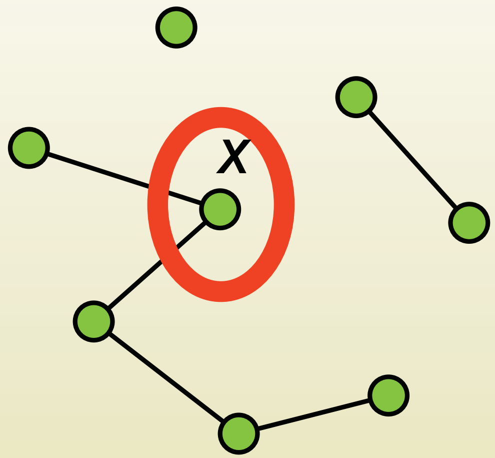
Connectivity of simple random graph

$$\mathcal{F}_1(n,p) =$$ P[random graph with n vertices and edge probability p is connected]



illustrating k=5 term

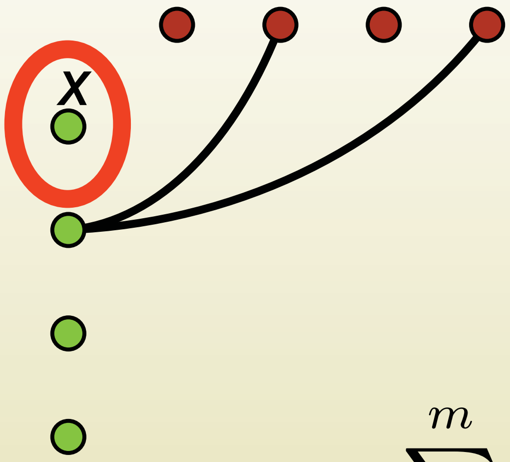
Construct recurrence relation by considering all possible sizes k of the cluster that node x belongs to

$$1 - \mathcal{F}_1(n,p) = \sum_{k=1}^{n-1} \binom{n-1}{k-1} \mathcal{F}_1(k,p) q^{k(n-k)}$$

$$q = 1 - p$$

Gilbert, *Ann. Math. Statist.* (1959)

Connectivity of bipartite random graph



$\mathcal{F}(m,n,p) =$ P[random graph with *m* green and *n* red vertices and edge probability *p* is connected]

Generalize recurrence relation by considering all possible sizes *k*,*l* of the cluster that node *x* belongs to

$$1 = \sum_{k=1}^{m} \sum_{l=0}^{n} \binom{m-1}{k-1} \binom{n}{l} \mathcal{F}(k,l,p) q^{k(n-l)} q^{l(m-k)}$$

Ellenbroek and Mao, *arXiv*:1107.3933 (2011)

Connectivity of bipartite random graph

$$1 = \sum_{k=1}^{m} \sum_{l=0}^{n} \binom{m-1}{k-1} \binom{n}{l} \mathcal{F}(k,l,p) q^{k(n-l)} q^{l(m-k)}$$

in the limit $m = n \to \infty$

Upper bound on ***F*** from 1-***F***(*n*,*n*,*p*) ≥ P[graph contains at least 1 isolated node]

Lower bound on ***F*** from ***F***(*k*,*l*,*p*)≤1

Bounds coincide to lowest order in 1/*n* : $\mathcal{F}(n,n,p) \to 1 - 2nq^n$

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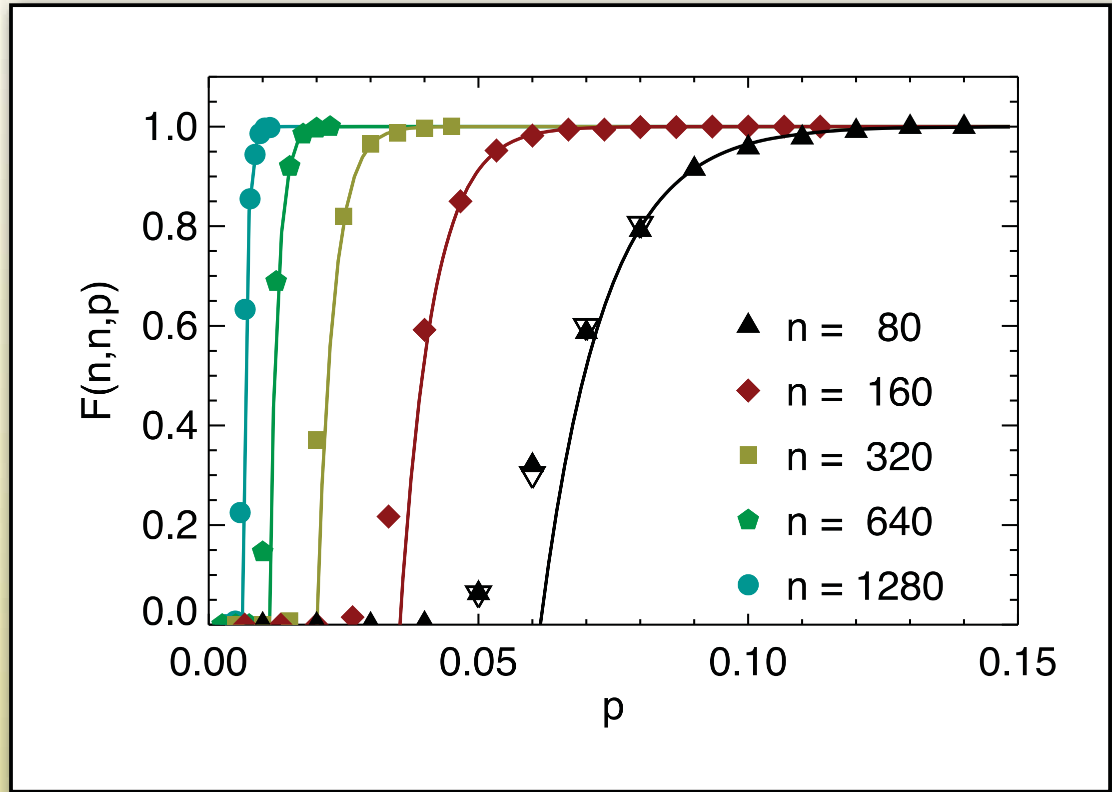
Testing the limiting form of *F*

$$\mathcal{F}(n,n,p) \to 1 - 2nq^n$$



Closed symbols: numerical test of graph connectivity
Open symbols: evaluation of recurrence formula
Lines: Limiting form of ***F***

Scaling of the threshold probability

Define critical *p* as function of system size through

$$\mathcal{F}(n, n, p_{\mathrm{R}}(n)) = 1/2$$

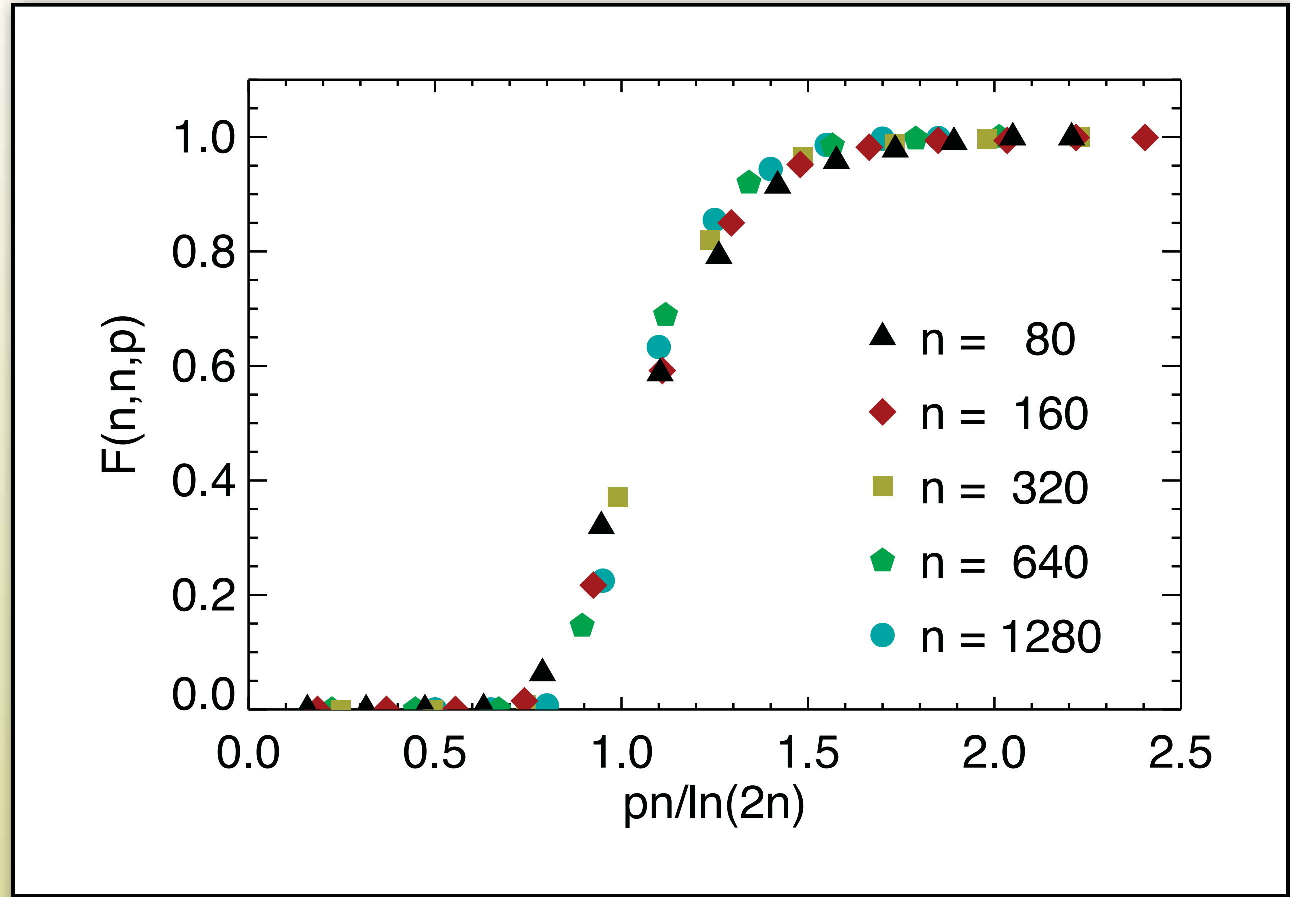
From our work, rigorously $p_{\mathrm{R}} \geq \frac{\ln 2n}{n}$

and numerically $p_{\mathrm{R}} \lesssim \frac{\ln 4.93n}{n}$

From Palásti (1963) it can be derived that $p_{\mathrm{R}} = \frac{\ln(2n/\ln 2)}{n} \approx \frac{\ln 2.89n}{n}$

$$p_{\mathrm{R}} = \frac{\ln n}{n} + \mathcal{O}(1/n) \qquad \text{as } n \to \infty$$

Finite size scaling of the numerical data



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Generic rigidity?

If we move away from the perfect square lattice to something with the same topology but with disordered positions...

- having one crossbar in each row and column is still not sufficient, and no longer necessary for rigidity
- the structure can be rigid even for non-connected graphs
- the order of the rows becomes important (not all green nodes are equivalent anymore)
- graph mapping used so far becomes pretty hopeless

...but numerically we can use the pebble game!
Jacobs and Thorpe, *PRL* (1995), *PRE* (1996)

Conclusion (square lattice)

The threshold p for NNN rigidity percolation on the square lattice goes to zero with increasing system size
⇒ transition at isostatic point

$$p_{\mathrm{R}} = \frac{\ln n}{n} + \mathcal{O}(1/n) \qquad \text{as } n \to \infty$$

Now what if we want to learn about jamming from this?
⇒

Recent work by Xiaoming Mao, Anton Souslov, Tom Lubensky, Andrea Liu

Mao *et al.*, *PRL* **104**, 085504 (2010)
Souslov *et al.*, *PRL* **103**, 205503 (2009)

Summary

Effective medium theory has nothing to say about elasticity of packings close to unjamming.

Random networks tell us that what's special about packings is that they resist compression so strongly.

It's fun to link together bits of known math to write down an exact expression, even if the relevant asymptotics were already known.

Ellenbroek, Zeravcic, Van Saarloos, Van Hecke, *EPL* **87**, 34004 (2009)
Ellenbroek, Mao, *arXiv*:1107:3933 (2011)
and references therein

Thank you so much...

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