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Geometric t-Designs and T-Cubature and their Relations to Sphere Packigns and Coverings

Greg KUPERBERG

University of California at Davis, Dept.of Mathematics
1 Shields Avenue, Davis, CA 95616
U.S.A.

Geometric t-designs and t-cubature and their relations to sphere packings and coverings

Greg Kuperberg

UC Davis

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What is numerical cubature?

Given $X \subseteq \mathbb{R}^d$ and a (normalized) measure μ on X, we want to estimate:

$$\int_X f(\vec{x})d\mu \approx f(F) \stackrel{\text{def}}{=} \sum_k w_k f(\vec{p}_k).$$

We want to choose F so that the formula is exact for polynomials of degree $\leq t$.

Example

Simpson's rule.

$$\int_0^1 f(x)dx \approx \frac{1}{6}f(0) + \frac{2}{3}f(\frac{1}{2}) + \frac{1}{6}f(1).$$

Simpson's rule is exact if $f \in \mathbb{R}[x]_{\leq 3}$. One says quadrature in one dimension and cubature in higher dimensions.

The cubature problem

Given X, μ and $t \in \mathbb{N}$, find F so that

$$\int_X P(\vec{x}) d\mu = P(F)$$

for $P \in \mathbb{R}[\vec{x}]_{\leq t}$. F is a *t*-cubature formula. We want positive and interior (PI) formulas: $w_k > 0$ and $F \subseteq X$. These are also called weighted *t*-designs (Delsarte).

The basic t-cubature or t-design problem is to minimize the number of points.

Cubature in 1D (quadrature) was solved by Gauss and Christoffel. But in $\geq$2D, cubature is an open-ended problem, just like the sphere packing problem.

An exact duality

- Delsarte found that t-designs are dual to sphere packings. First, an exact duality.
- Suppose that the domain X is a compact abelian group. Then an embedding $X \subseteq \mathbb{R}^N$ can be viewed as a "polynomial structure" on X. In math-speak, X is an affine algebraic variety (or a subset of one).
- If the affine algebraic structure is compatible with the group action, the structure is equivalent to an integer-valued metric on its Pontryagin dual $\hat{X}$. (*I.e.*, $\hat{X}$ is the Fourier space of X.)
- If $F \subseteq X$ is a subgroup (or "lattice"), then it has a dual $F^* \subseteq \hat{X}$. Fact: F is a t-design if and only if F^* has minimum distance $t+1$.
- Packings at radius $r \leftrightarrow$ sets with min distance $t+1 = 2r+1$.

An exact duality

Example

$X = (\mathbb{R}/2\pi\mathbb{Z})^d$ = the d-torus with trigonometric polynomials

$$P(\vec{\theta}) = P(\cos\theta_1, \sin\theta_1, \dots, \cos\theta_d, \sin\theta_d),$$

using the usual degree. Then $\hat{X} = \mathbb{Z}^d$ with the ℓ^1 or taxicab metric.

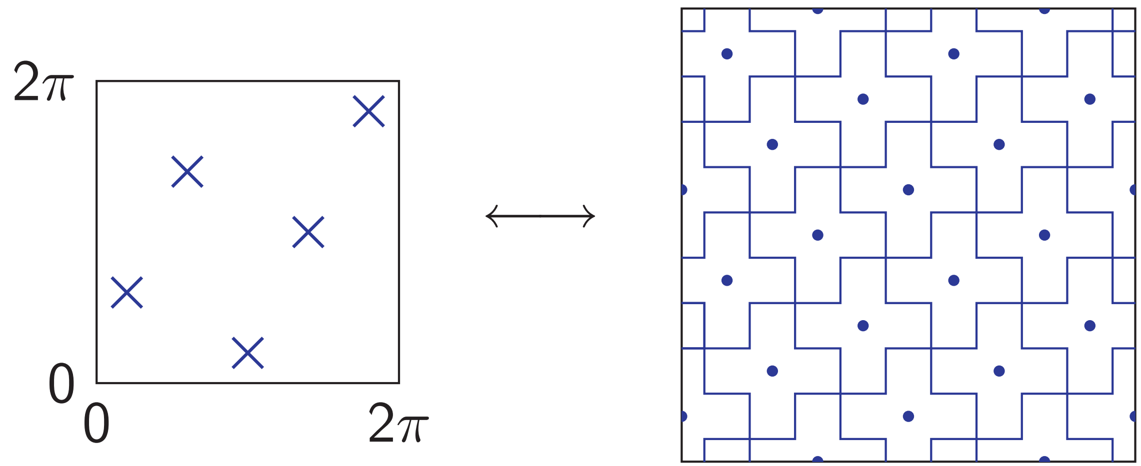
Example

Let $X = (\mathbb{Z}/2)^d$ be bit strings of length d, and define t-designs to be orthogonal arrays of strength t. Then $\hat{X} = (\mathbb{Z}/2)^d$ with the Hamming metric.

An exact duality

Example (Noskov)

Let $X = (\mathbb{R}/2\pi\mathbb{Z})^2$ and let F be the five points drawn below. Then F is a trigonometric 2-design. It is dual to a discrete sphere packing (in fact a tiling) of radius 1.



The discrete ℓ^1 spheres are "Aztec diamonds", or "plus signs" when $r = 1$.

Delsarte's duality

Delsarte found another duality. Suppose that X is a 2-point symmetric metric space. This means that X has a symmetry group which is transitive on pairs of points x and y at fixed distance $\mathrm{dist}(x, y)$. Then symmetry induces a polynomial structure on X using harmonic functions.

Examples

The sphere $X = S^{d-1}$. Hamming space $X = (\mathbb{Z}/2)^d$.

Delsarte's method: Given $F \subseteq X$ with minimum distance r, write down linear relations that the radial pair correlation function σ must satisfy. Namely, $\sigma(s) \geq 0$, $\int \sigma(s)ds = 1$, $\sigma(s) = 0$ for $0 < s < r$, and the transform $\hat{\sigma}(k) \geq 0$. By linear programming, these relations yield an upper bound on $|F| = 1/\sigma(0)$.

Delsarte's duality

Delsarte, McEliece, Rodemich, Rumsey, Welch, Odlyzko, Sloane, Kabatiansky, Levenshtein, etc., found that the Delsarte method yields excellent bounds, sometimes optimal. Cohn and Elkies generalized the method to $X = \mathbb{R}^d$; it is thought to be optimal when $d \in \{2, 8, 24\}$.

The duality is that similar equations yield a lower bound on $|F|$, where F is a PI t-design; sometimes F is optimal for both.

packings	designs
$\sigma(s) \geq 0$	$\sigma(s) \geq 0$
$\int \sigma(s) ds = 1$	$\int \sigma(s) ds = 1$
$\hat{\sigma}(k) \geq 0$	$\hat{\sigma}(k) \geq 0$
$\sigma(s) = 0, 0 < s < r$	$\hat{\sigma}(k) = 0, 0 < k \leq t$
$\min \sigma(0)$	$\max \sigma(0)$

Trigonometric cubature and discrete sphere packings

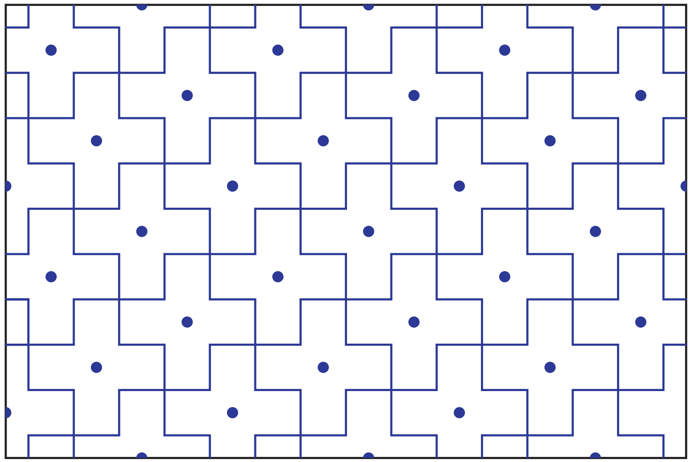
Problem

Trigonometric t-cubature F on $(\mathbb{R}/2\pi\mathbb{Z})^d$ for fixed t and $d \to \infty$.
Or, packings F^* in $\mathbb{Z}^d$ at radius r with $t = 2r$.

For $r = 1$, let F^* be the set of $\vec{x}$ with

$$x_1 + 2x_2 + 3x_3 + \cdots + dx_d \equiv 0 \pmod{2d+1}.$$

Then this is a tiling of d-dimensional plus signs.



Trigonometric cubature and discrete sphere packings

What about for higher r? Suppose that $p \geq 2d+1$ is prime. Define a group homomorphism $\phi : \mathbb{Z}^d \to (\mathbb{Z}/p)^r$ by

$$\phi(\vec{x}) = \sum_{k=1}^{r} x_k (k, k^3, k^5, \ldots, k^{2r-1}) \in (\mathbb{Z}/p)^r.$$

Then

$$F^* = \Lambda \stackrel{\mathrm{def}}{=} \ker \phi$$

is a sphere packing with density $\to 1/r!$ as $d \to \infty$, *i.e.*, within a constant factor of the volume bound.

Theorem (K.)

For each t, trigonometric t-designs exist with $O(d^{\lfloor t/2 \rfloor})$ points.

It is easy to boost t to $2r+1$ by restricting to even points.

Trigonometric cubature and discrete sphere packings

Theorem (Stroud)

A volume bound holds for any t-cubature formula.

We match Stroud's bound of $O(d^{\lfloor t/2 \rfloor})$ points, for each fixed t.

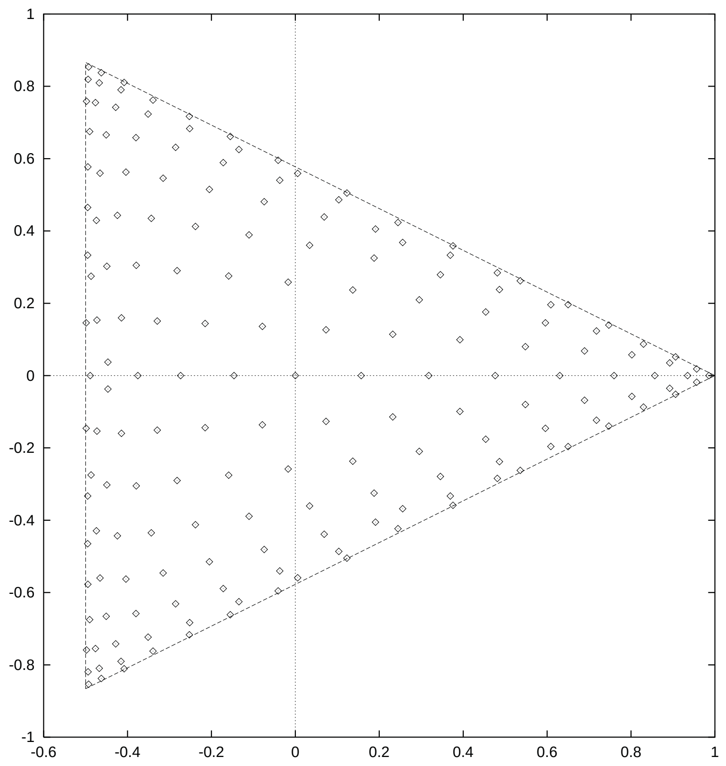
- Λ is a modified Craig lattice (Conway and Sloane). These lattices were described for Euclidean spheres, but they are even better for discrete ℓ^1 spheres. ($\ell^1 \approx \ell^2$ for small r.)
- An analogy:

$\mathbb{Z}/2$	$\mathbb{Z}$
Hamming code	plus lattice
BCH code	Craig lattice

- Is there a competitive statistical mechanics approach? Note: These constructions yield deeply overdetermined t-designs.

A statistical mechanics result

Wandzura and Xiao (2001) used simulated annealing to find good t-cubature on the triangle Δ_2 for large t:



$$t = 30$$
$$|F| = 175$$

It looks like a sphere covering with anisotropy near the boundary.

t-designs are coverings

Theorem (K.)

A t-design on the simplex Δ_d has covering radius $O(1/t)$ when pulled back to an orthant section of S^d under the map

$$\pi : (x_0, x_1, \ldots, x_d) \mapsto (x_0^2, x_1^2, \ldots, x_d^2)$$

in barycentric coordinates.

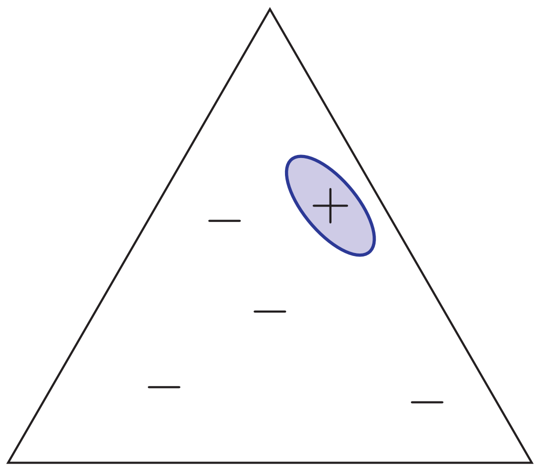
This is for weighted t-designs. For unweighted t-designs, the result is even stronger, because crowding at the edges forces more points in the middle just to make the weights equal.

Positive islands

The proof uses a polynomial $P(\vec{x})$ of degree t with

$$\int_{\Delta_d} P(\vec{x})d\vec{x} > 0,$$

but which is only positive on a small positive island:

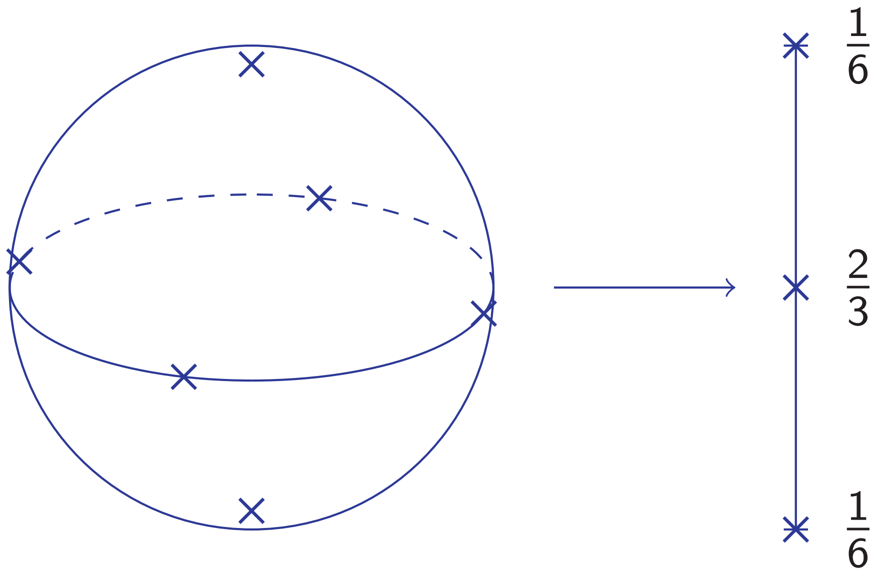


Any PI formula has a point in the island, so it is an sphere covering in a metric in which the islands are approximately round.

Archimedes' theorem and t-designs

Theorem (Archimedes)

An axis projection of S^2 preserves normalized volume.



This explains Simpson's rule: It is the projection of a set in S^2 which is a 3-design by symmetry. But that is another story.

Archimedes' theorem and t-designs

- Archimedes' map generalizes to the moment map

$$\pi : \mathbb{C}P^d \to \Delta_d \qquad \mathbb{C}P^1 = S^2.$$

- Here $\mathbb{C}P^d$ is an affine real algebraic variety in coordinates $\mathrm{Re}\, z_j\overline{z_k}$ and $\mathrm{Im}\, z_j\overline{z_k}$, and

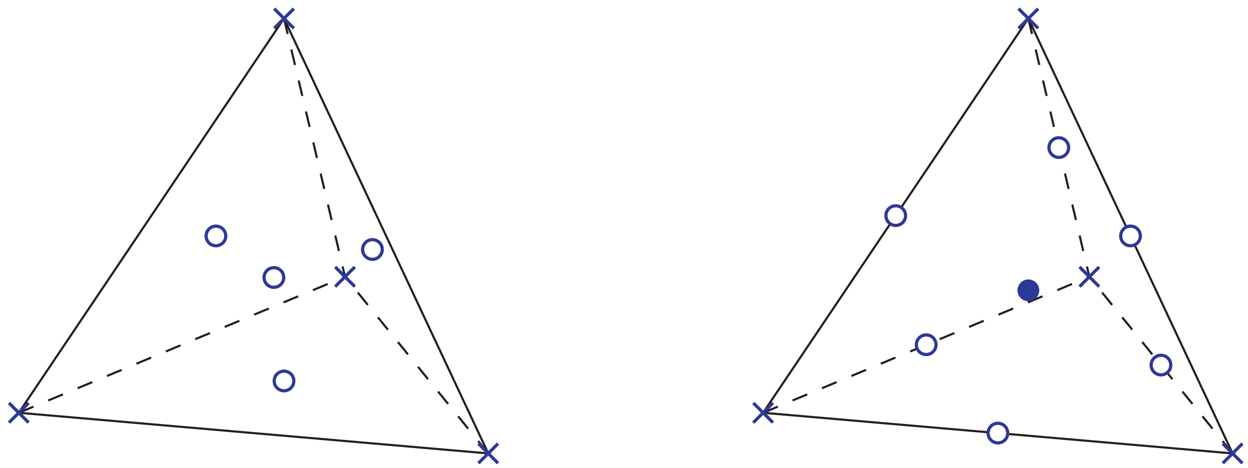
$$\pi(\vec{z}) = (|z_0|^2, |z_1|^2, \ldots, |z_d|^2)$$

  in barycentric coordinates on Δ_d. Because $\mathbb{C}P^d$ is a projective toric variety, its moment map π preserves volume and is linear.
- In physics-speak, $\mathbb{C}P^d$ is the space of quantum states in the Hilbert space $\mathbb{C}^{d+1}$. It is also a classical phase space, and π is a vector of conservation laws from d commuting symmetries.

Archimedes' theorem and t-designs

Example

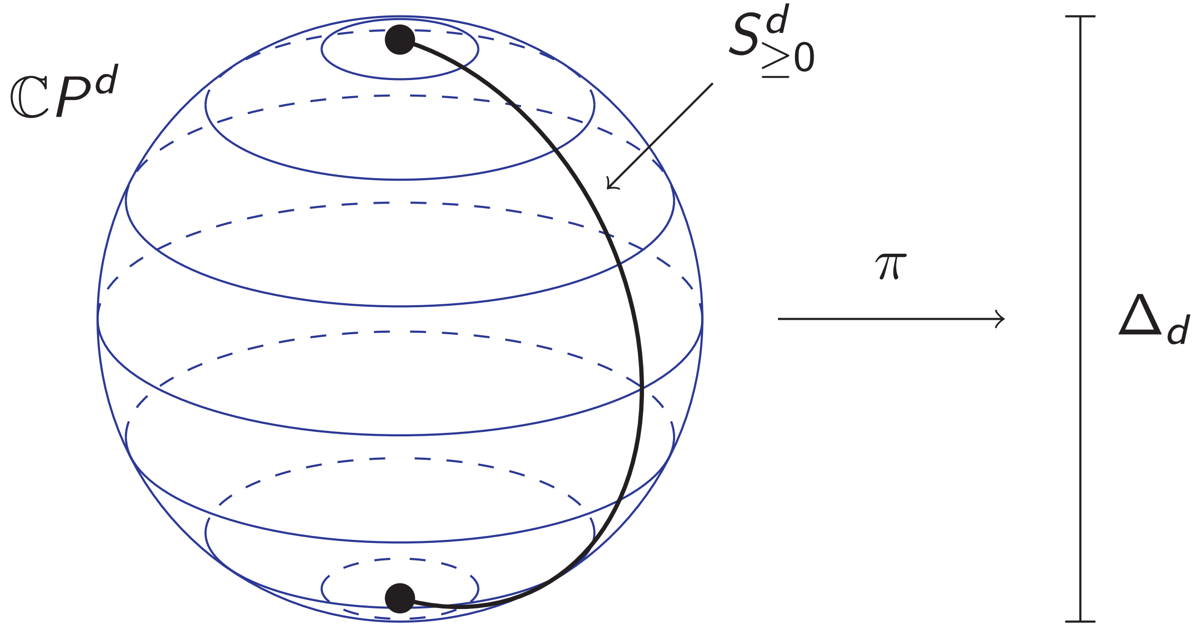
The 240 kissing points of $E_8 \subseteq S^7$ (the sphere kissing problem solution in 8 dimensions) project to 60 or 40 points on $\mathbb{C}P^3$. Those project to 3-designs on Δ_3 with 8 and 11 points.



They are described in Abramowitz and Stegun (1964)! Again, another story.

The positive island

We actually define P on $\mathbb{C}P^d$, rotate it to the desired position, and project to Δ_d by averaging over fibers. Before rotation, $P(\vec{z}) = P(|z_0|)$. It is made using numerical quadrature on $[0,1]$ with $\mu(x) = x^{d-1}$. *I.e.*, $P(|z_0|)$ comes from a Jacobi polynomial.



Tao's question

Terry Tao posed this "congestion" (conjecture or question) in MathOverflow.

Question (Tao)

Suppose that K is a symmetric convex body in $\mathbb{R}^d$ which and Λ is a lattice packing of K. Then is the reciprocal lattice a covering of rK^*, where K^* is the reciprocal convex body, with $r = d/2$ or at least $r = O(d)$?

The constant $d/2$ is from the putative worst case of a d-cube.

Tao's question

I can prove $r = O(d^{3/2})$ using the positive island method.

- First, replace K with an ellipsoid E. By John's theorem, this sacrifices a factor of $O(d^{1/2})$. Apply a linear map to make E a standard sphere.
- Let $f : \mathbb{R}^d \to \mathbb{R}$ be a band-limited function, *i.e.*, $\hat{f}(\vec{k}) = 0$ when $||\vec{k}|| > t$. We can view f as a "polynomial" of degree t. Then we can define Fourier t-designs on $\mathbb{R}^d$ with $t \in \mathbb{R}_{\geq 0}$ (*cf.*, Cohn and Elkies).
- Λ^* is a t-design with $t = 2$ by duality. Does that force it to have a good covering radius? We can let f be a band-limited positive island function using a Bessel function, with radius $O(d)$. QED.

Open problems

- In the discrete ℓ^1-ball or ℓ^2-ball packing problem, what if $d, r \to \infty$ together at some rate?
- A t-design on a simplex Δ_d has covering radius $O(1/t)$ on the orthant. Is this an optimal local density estimate?
- What is the answer to Tao's question?

Acknowledgments

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