The Abdus Salam International Centre for Theoretical Physics

United Nations Educational, Scientific and Cultural Organization

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2254-14

Workshop on Sphere Packing and Amorphous Materials

25 - 29 July 2011

Sphere Packing in the Hamming Space: Cavity Approach

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Sphere packing in the Hamming space: Cavity approach

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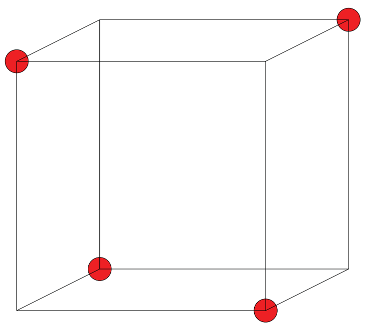
July 26, 2011

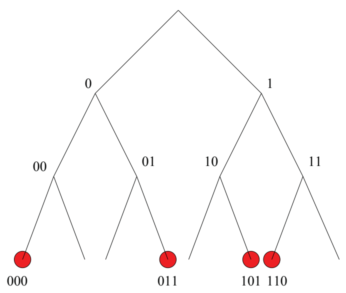
Outline

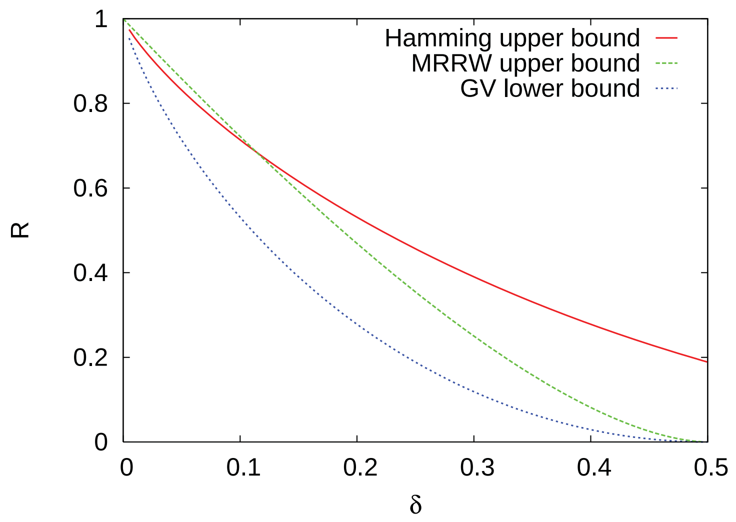
Introduction

Motivations

- Represent N symbols in binary strings of length n: after transmitting through a noisy channel, flipping at most $\frac{d-1}{2}$ bits, recover the original massage.
- A packing problem in the binary Hamming space $\Lambda \equiv \{0,1\}^n$ with hard spheres of diameter d and rate of packing $R \equiv \lim_{n,d\to\infty,\delta=d/n} \frac{1}{n} \log_2(N_{max})$.
- Hamming upper bound: $N_{max} \leq \frac{|\Lambda|}{V_{\frac{(d-1)}{2}}}$, $R \leq R^H \equiv 1 - H(\frac{\delta}{2})$.
- Gilbert-Varshamov lower bound: $N_{max} \geq \frac{|\Lambda|}{V_{d-1}}$, $R \geq R^{GV} \equiv 1 - H(\delta)$.





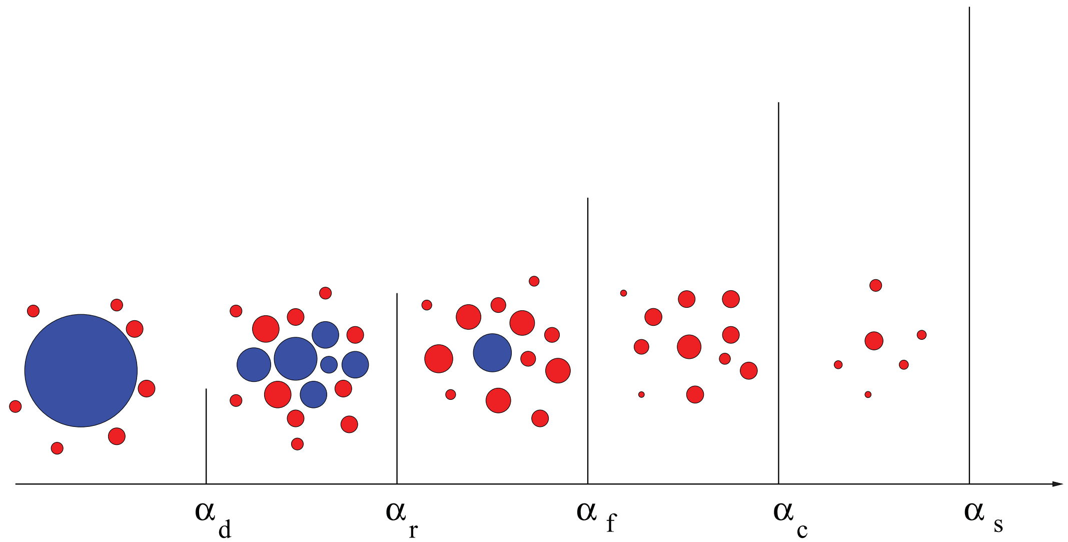


R. W. Hamming (1950), E. N. Gilbert (1952), R. R. Varshamov (1957)

Introduction

Motivations

- What can statistical physics say about this problem?
- A lower bound for the average number of spheres in a grand canonical ensemble results to the GV lower bound.
- The point that liquid entropy in the Hyper-Netted-Chain approximation vanishes results to the GV lower bound.
- Consider the packing problem as a constraint satisfaction problem: N variables in Λ and $M \equiv \frac{N(N-1)}{2}$ constraints.



A. Procacci and B. Scoppola (1999), G. Parisi and F. Zamponi (2006), M. Mezard, G. Parisi and R. Zecchina (2002), F. Krzakala et al. (2007)

Introduction

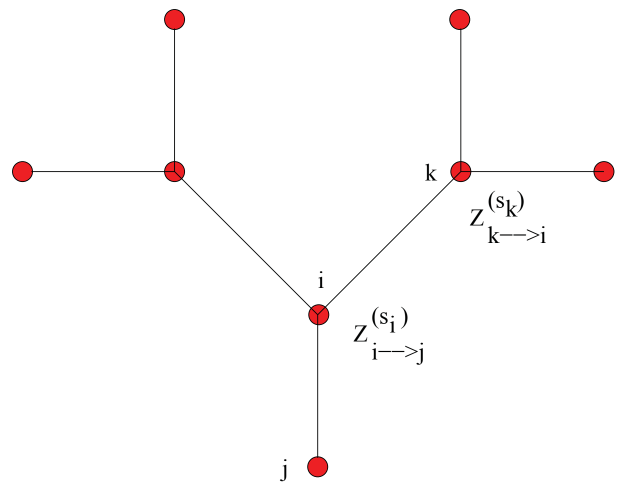
cavity method

- Bethe approximation: asymptotically exact on locally tree-like graphs.
- In some problems it provides a lower bound for the free energy.
- A message passing algorithm; replica symmetry (RS) and replica symmetry breaking (RSB).

$$\mu(\underline{s}) = \frac{1}{Z} \prod_{(ij)} I_{ij}(s_i, s_j) \simeq \prod_i \mu_i(s_i) \prod_{(ij)} \frac{\mu_{ij}(s_i, s_j)}{\mu_i(s_i)\mu_j(s_j)},$$

$$\mu_{i\to j}(s_i) \propto \prod_{k\in\partial i\setminus j} \left(\sum_{s_k} I_{ki}(s_k, s_i)\mu_{i\to j}(s_i)\right),$$

$$\mu_{i\to j}(s_i) \to \mu^{\alpha}_{i\to j}(s_i).$$

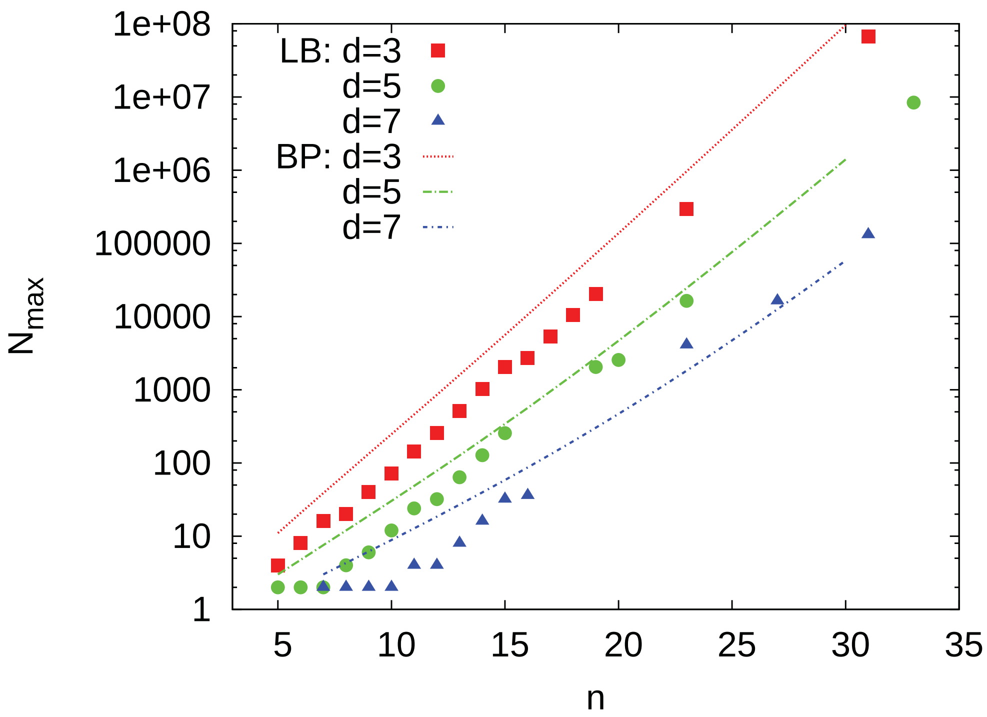


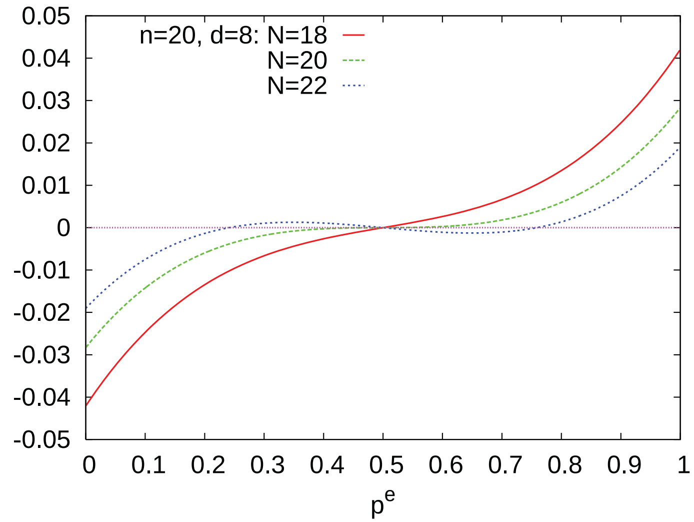
F. Guerra (2003), S. Franz and M. Leone (2003), P. Contucci et al. (2011), M. Mezard and A. Montanary (2009)

Replica symmetric solutions

BP equations

- Liquid solution: $\mu_{i\to j}(s_i) = \frac{1}{|\Lambda|}$, $\qquad N_{max}^{BPL}(\frac{V_{d-1}}{2^n}) \simeq (2\ln 2)n + o(1)$.
- Crystalline solution: $\mu_{i\to j}(s_i) = p_e \delta_{s_i\in\Lambda_+} + (1-p_e)\delta_{s_i\in\Lambda_-}$.





http://www2.research.att.com/njas/codes/And, G. Parisi and F. Zamponi (2006)

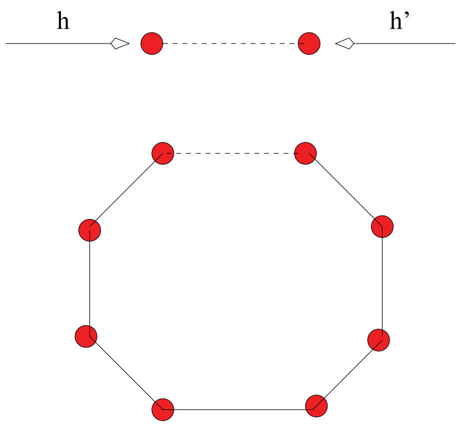
Replica symmetric solutions

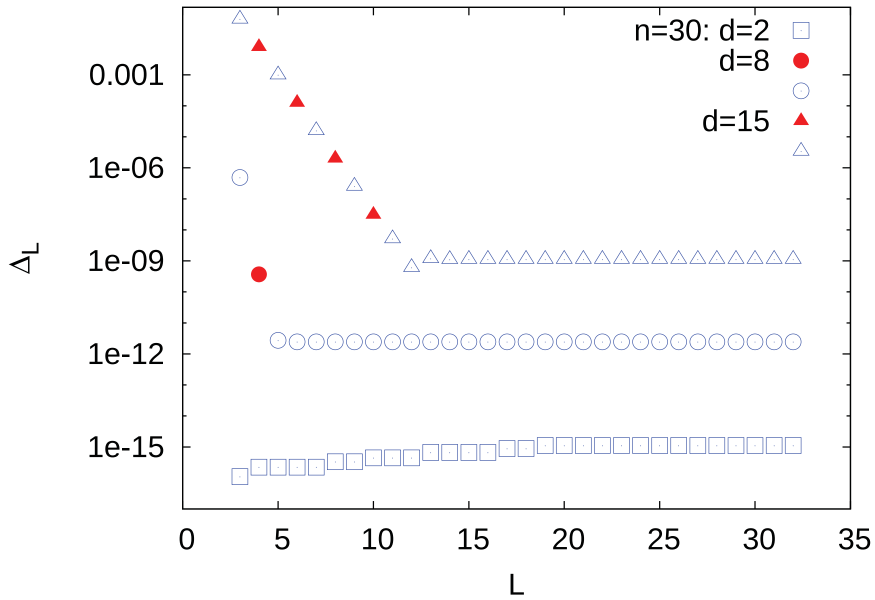
An interpolation

- Label the edges of the interaction graph by $t = 1, \cdots, M$.

$$\log Z = \log Z^{BPL} + \sum_{t=1}^{M} \log(1 + \Delta_t), \qquad \Delta_{t+1} = \frac{\langle I_{t+1} \rangle_t - \langle I_{t+1} \rangle_0}{\langle I_{t+1} \rangle_0},$$

- $\langle I_{t+1} \rangle_t$: the probability of satisfying constraint I_{t+1} when the interaction set is given by the first t interactions.





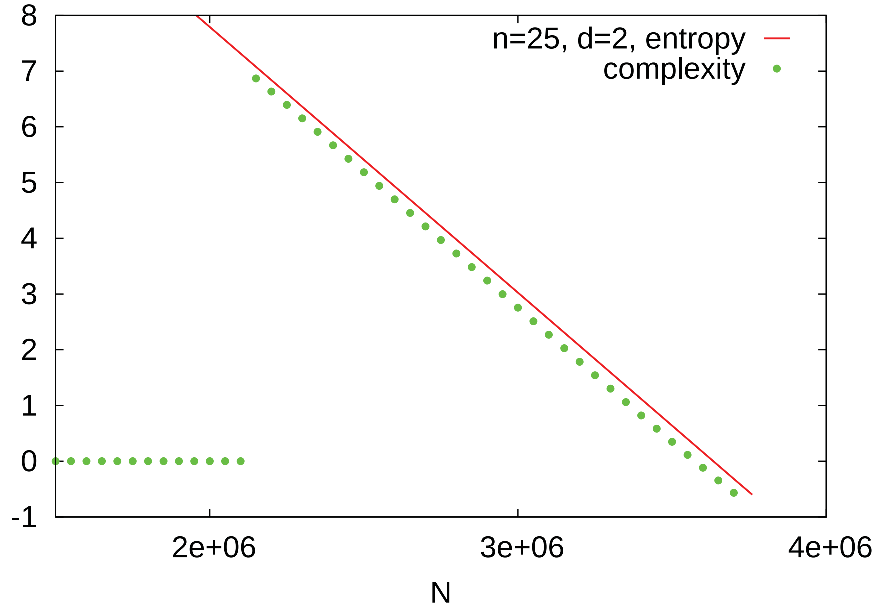
1-step Replica symmetry breaking

SP equations

- Configurational entropy or complexity: $\mathcal{N}_c \sim e^{N\Sigma}$.
- $\eta_{i\to j}^{s_1,\dots,s_m}$: cavity probability that sphere i in absence of j is frozen on points $\{s_1,\dots,s_m\}$.

$$N_c^{SP}\left(\frac{V_{d-1}}{2^n}\right) \simeq (\ln 2)n + \ln n,$$

$$N_{max}^{SP}\left(\frac{V_{d-1}}{2^n}\right) \simeq (2\ln 2)n + o(1).$$



M. Mezard and R. Zecchina (2002)

A packing algorithm based on the BP equations

- BP equations: $\mu_{i\to j}(s_i) \propto \prod_{k\in\partial i\setminus j}\left(\sum_{s_k} I_{ik}(s_i,s_k)\mu_{k\to i}(s_k)\right)$.

- Use BP marginals to decimate the spheres, or reinforce the messages to converge to a packing.

- rBP equations: $\mu_{i\to j}(s_i) \propto [\mu_i(s_i)]^r e^{\beta w_i(s_i)} \prod_{k\in\partial i\setminus j}\left(\sum_{s_k} I_{ik}(s_i,s_k)\mu_{k\to i}(s_k)\right)$

- Finding some maximum packings: $(n=10, d=4, N=40)$, $(n=11, d=3, N=144)$, $(n=11, d=5, N=24)$, $(n=15, d=7, N=32)$

- Time complexity $(N|\Lambda|)^2$, and memory complexity $N^2|\Lambda|$

A. Braunstein and R. Zecchina (2006)

A packing algorithm based on the BP equations

Restricted search space

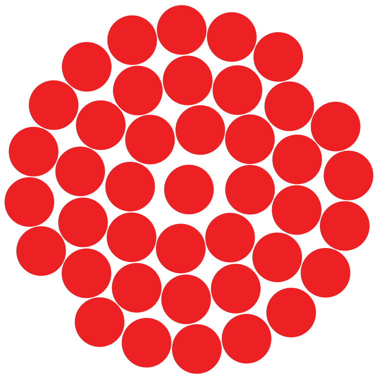
- Restrict the domain of each sphere to a small search space $R_i = \{s_i^1, \ldots, s_i^R\}$.

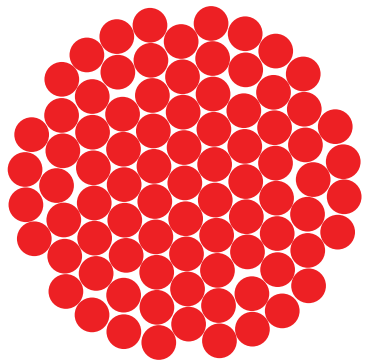
$$\mu_{i\to j}(s_i) \propto [\mu_i(s_i)]^r \prod_{k\in\partial i\setminus j} \left(\sum_{s_k\in R_k} I_{ik}(s_i, s_k)\mu_{k\to i}(s_k)\right).$$

- Update the search spaces; replacing some of the less probable states with some copies of the most probable state.

- Packings $(n = 10, d = 4, N = 40)$, $(n = 11, d = 3, N = 144)$ are obtained with $R = 2^3, 2^5$. Denser packings (larger d) are found for $(n = 48, d = 32, N = 4)$, $(n = 51, d = 30, N = 6)$.

- It can be used in continuous spaces.

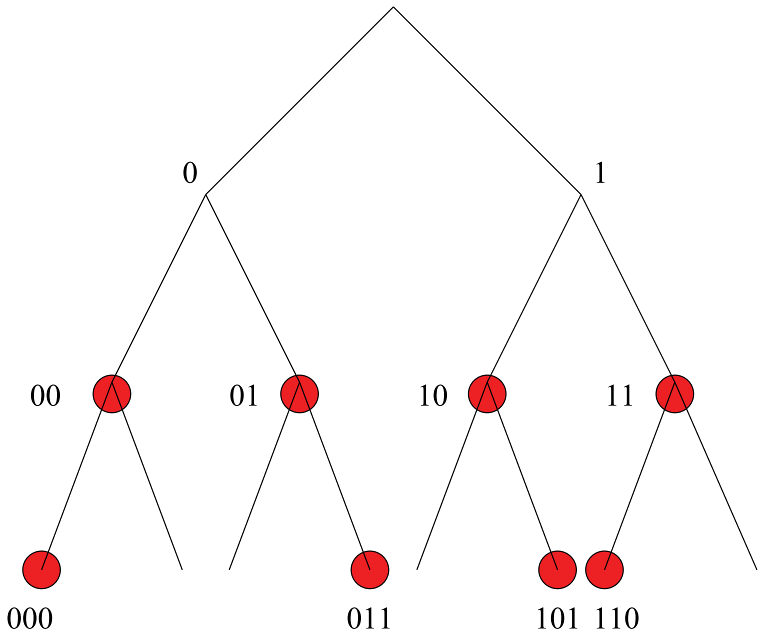


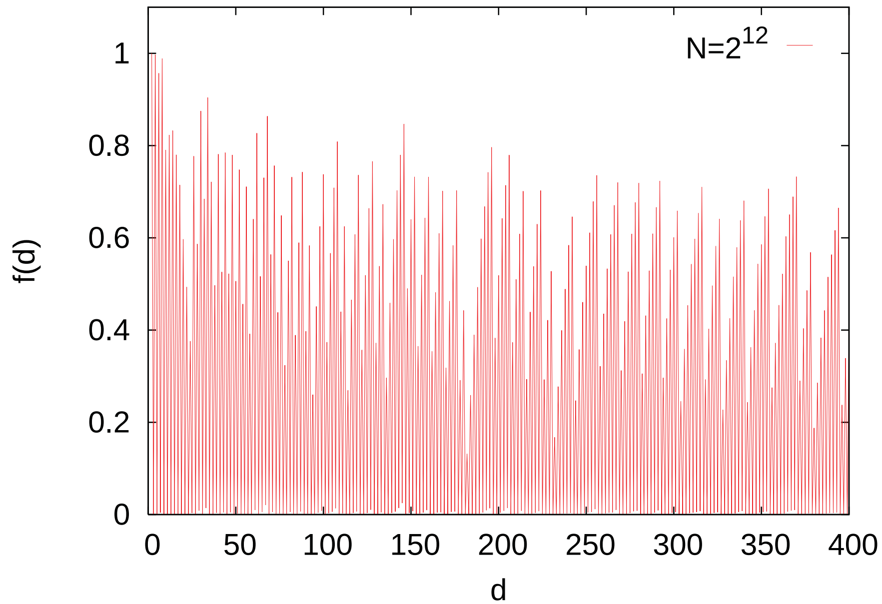


A. B. Hopkins, F. H. Stillinger and S. Torquato (2010)

Packing in an ultrametric space

- Liquid solution of the BP equations are asymptotically exact: $R^{UM} = 1 - \delta = R^{BPL}$.
- An iterative algorithm: increase d by one and find a packing in larger dimension.
- The algorithm is exact in an ultrametric space, and for even d in the Hamming space gives the crystalline packings.





Packing in q-ary Hamming space

- Liquid solution of the BP equations: $R^{BPL} = 1 - H_q(\delta) = R^{GV}$.
- Algebraic-geometry codes for square q: $R^{TV} = 1 - \delta - \frac{1}{\sqrt{q} - 1}$ can be larger than R^{GV} for $q \geq 49$.
- Perhaps a crystalline solution localized on a quasi ultrametric subspace.
- An upper bound for the size of an ultrametric subspace in the Hamming space is exponentially large only for $q \geq 3$.

M. A. Tsfasman and S. G. Vladut (1991)

Summary

- Both the BP and SP equations give asymptomatically the same rate of packing as the GV one.
- A message passing algorithm to find dense packings in discrete and continuous spaces.
- Is the replica symmetric solution asymptotically exact?
- Can we recover the algebraic-geometry packings?

Thanks to: H. Cohn, N. Elkies, S. Franz, S. Torquato, F. Zamponi

Thank You For Your Attention