BIFURCATIONS IN ONE-DIMENSIONAL DISCRETE SYSTEMS

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In this lesson we will study the simplest dynamical systems. We will see, however, that even in this case the scenario of different possible dynamics is very rich. In particular, we will consider dynamical systems depending on a parameter. When the value of the parameter changes continuously, the behaviour of the system may change in a discontinuous way. One says that a *bifurcation* occurs for an isolate value of the parameter at which the type of dynamic changes.

1. Discrete one-dimensional dynamical systems

A discrete one-dimensional dynamical system is a system subjected to a single equation of this type

$$x(t+1) = f(x(t)) \tag{1}$$

where $x \in I \subseteq \mathbb{R}$ and f is a function of x. The variable t is in general considered as the *time*, but in discrete systems the time takes only discrete values, so that it is possible to take $t \in \mathbb{Z}$.

A *trajectory* is a set $\{x(t)\}_{t=0}^{\infty}$ of points satisfying the above equation. It is evident that the initial point $x_0 = x(0)$ determines the entire trajectory.

The *behaviour* of the dynamical system is therefore given by *all* the trajectories $\{x(t) : x(0) = x_0\}$ for all initial values $x_0 \in I$.

A dynamical system depending on a parameter is described by a family $\{f_a\}$ of functions parametrized by a, where $a \in A \subseteq \mathbb{R}$.

$$x(t+1) = f_a(x(t)).$$

1.1. **Fixed points and their stability.**

Let $\bar{x} \in I$ be a point of the dynamical system (1) satisfying $f(\bar{x}) = \bar{x}$. Consider a trajectory starting at $x_0 = \bar{x}$. It is evident that the entire trajectory is formed by the unique point $\bar{x}$, i.e. $x(t) = \bar{x}\ \forall t \geq 0$.

A point $\bar{x}$ satisfying $f(\bar{x}) = \bar{x}$ is called a *fixed point* or a *equilibrium point* of the system (1).

Definition. The trajectory of the system (1) starting at x_0 is the set $\{x_0, f(x_0), f(f(x_0), \dots\}$, i.e., the succession of points $\{x(t)\}_{t=0}^{\infty}$ determined by the recurrence (1) with the initial condition $x(0) = x_0$.

We are now interested in the trajectories starting at points which are near $\bar{x}$.

In order to better understand the trajectories of the one-dimensional dynamical system we introduce the *graphical solution*. Consider the graph of the function $f(x)$. The abscissa represents $x(t)$ and the ordinate $x(t+1)$.

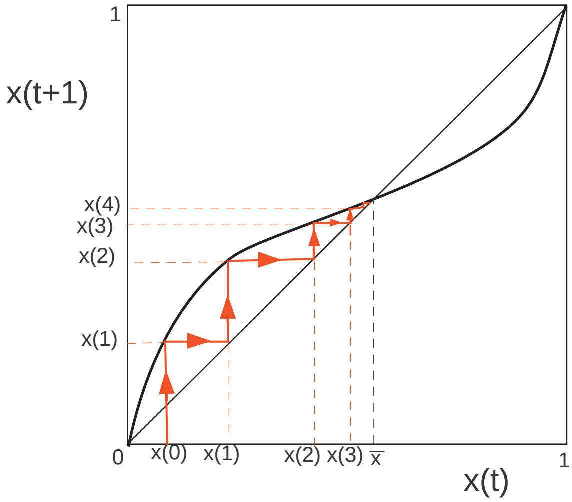

Figure 1. Graphical method to obtain a trajectory

Consider now a fixed point $\bar{x}$ and suppose that $f(x)$ be smooth at $\bar{x}$. Then there is a neighbourhood U of the point $\bar{x}$ where all trajectories starting at a point of U remain in U and approach $\bar{x}$ or all trajectories starting at a point of U move away from $\bar{x}$ and exit from U.

In the first case the fixed point is said to be an *attracting point* or a *stable equilibrium point* and in the second case a *repelling point* or an *unstable equilibrium point.*

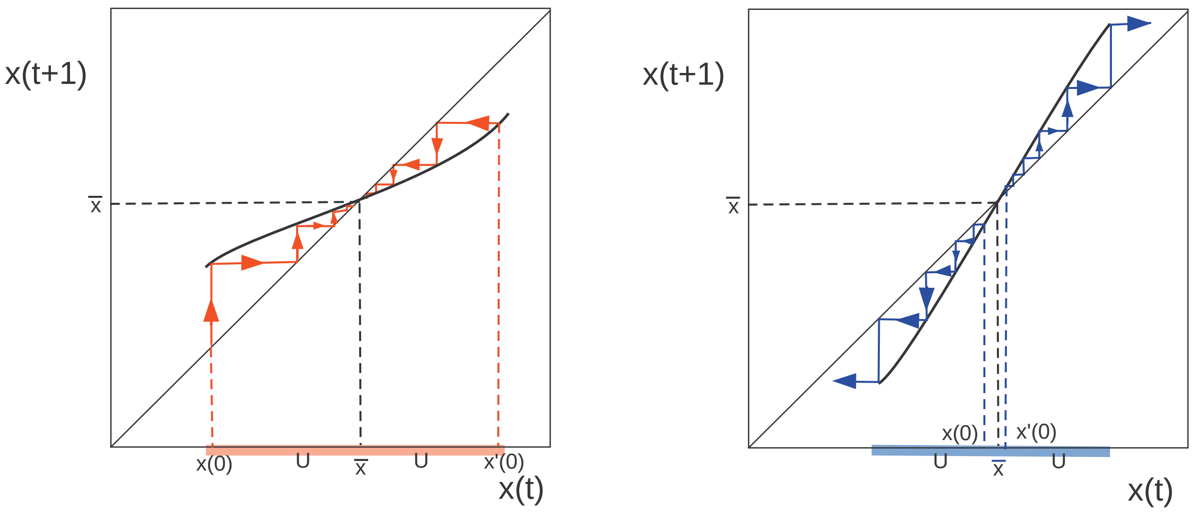

Figure 2. Attracting (left) and repelling (right) fixed point

Question. Observe Figures 2 and 3. Which property of f at the fixed point $\bar{x}$ determines whether $\bar{x}$ is attracting or repelling?

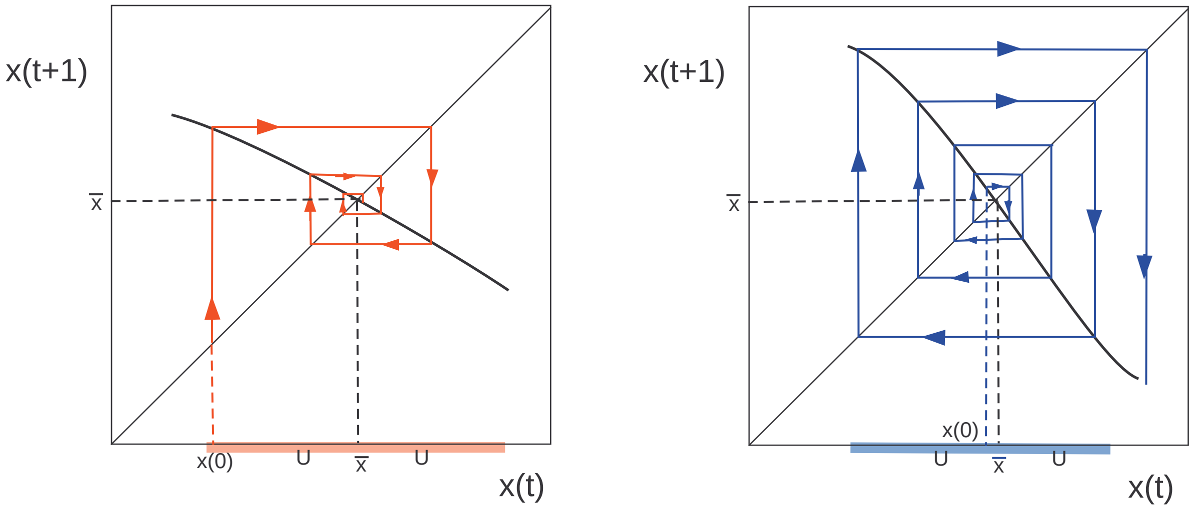

Figure 3. Attracting (left) and repelling (right) fixed point

The following theorem answers the question above.

Theorem 1. If $\bar{x}$ is a fixed point of equation $x(t+1) = f(x(t))$ and f at $\bar{x}$ is smooth then $\bar{x}$ is an attracting point if $|f'(\bar{x})| < 1$ and $\bar{x}$ is a repelling point if $|f'(\bar{x})| > 1$.

Problem 1. Give a graphical example where a fixed point $\bar{x}$ is neither attracting nor repelling, and f is smooth with $|f'(\bar{x})| = 1$.

Problem 2. Give a graphical example where a fixed point $\bar{x}$ is neither attracting nor repelling, and f is not smooth at $\bar{x}$.

Problem 3. For the dynamical system represented in Figure 1, find the fixed points and say if they are attracting or repelling.

Remark. If at the fixed point $\bar{x}$ the derivative is 1 or -1, in order to know the stability property we have to investigate higher derivatives of f (if f is smooth at $\bar{x}$). In this case the fixed point $\bar{x}$ is said *non hyperbolic*.

2. Loss of stability: bifurcations

We have seen that the fixed points of the dynamical system (1) are the points x satisfying $f(x) = x$. Let us suppose that our system is given by a function f like that of Figure 1, and that such function belongs to a family f_a of functions depending continuously on a parameter a. Let $f = f_0$. Hence, for $a = 0$ there is only one attracting fixed point and two repelling fixed points at the extremes of the interval where f is defined. Let us suppose that all the functions of the family satisfy $f(0) = 0$, $f(1) = 1$ and $f'(1) = f'(1) > 1$.

By a continuous change of the function f_0 into f_a, the attracting fixed point $\bar{x}_a$ (satisfying $f_a(\bar{x}_a) = \bar{x}_a$ moves continuously. However, for some isolate value of the parameter a, something may happen which changes the dynamic.

2.1. **Saddle-node bifurcation.**

As shown in Figure 4, it may happen that the graph of f_a, for some isolated value a^* of a becomes tangent to the diagonal (the graph of the function $h(x) = x$). At the point of tangency, say x^*, the derivative of the function f_{a^*} is equal to 1, and therefore the equilibrium point x^* is non hyperbolic. For $a > a^*$ two new fixed points exist. Observe that *necessarily* one is stable and the other one is unstable.

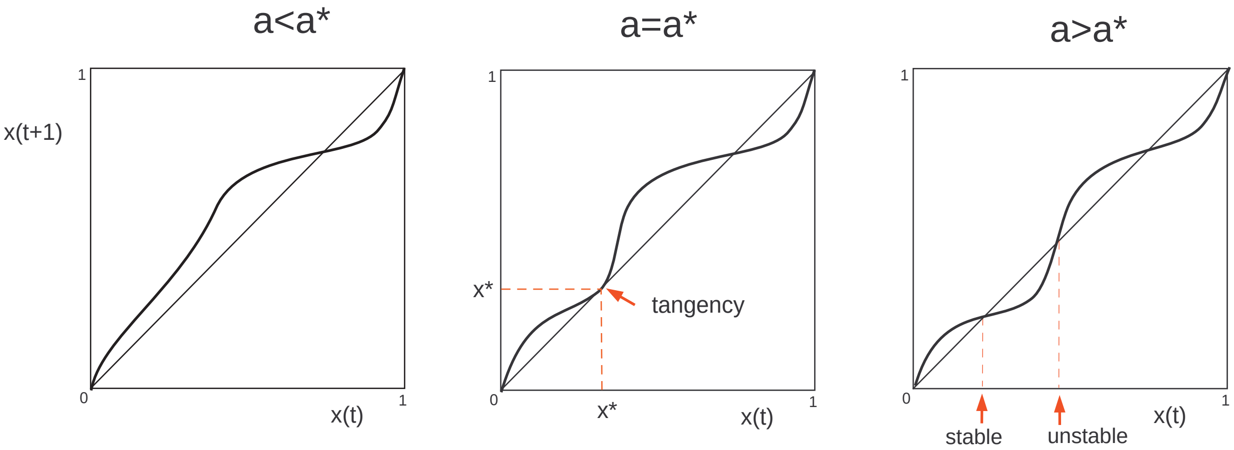

Figure 4. Saddle-node bifurcation in the family f_a.

The *bifurcation diagram* is the graph of a multivalued function, showing for every value of the parameter a in a neighbourhood of the bifurcation value a^* the fixed points of f_a in a neighbourhood of x^*. Stable fixed points are marked by a continuous line, unstable points by a dotted line.

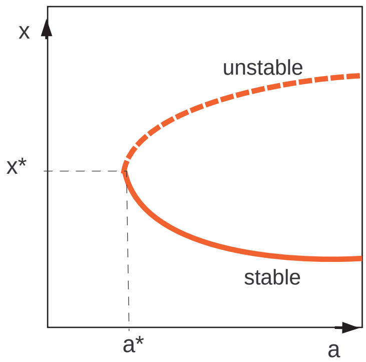

Figure 5. Bifurcation diagram of a saddle-node bifurcation.

2.2. **Pitch-fork bifurcation.** In this case the derivative of the fixed point $\bar{x}_a$ of f_a changes passing through the value 1 (or -1) (see Figure 6). At that point, say x^*, the graph of f_{a^*} is tangent to the diagonal, with an order-2 tangency. When a increases, the point of tangency disappears, the fixed point that was stable (derivative higher than zero and less than one) becomes unstable (derivative higher than one) and two other fixed points exist at right and at left of the unstable fixed point. These two points are stable.

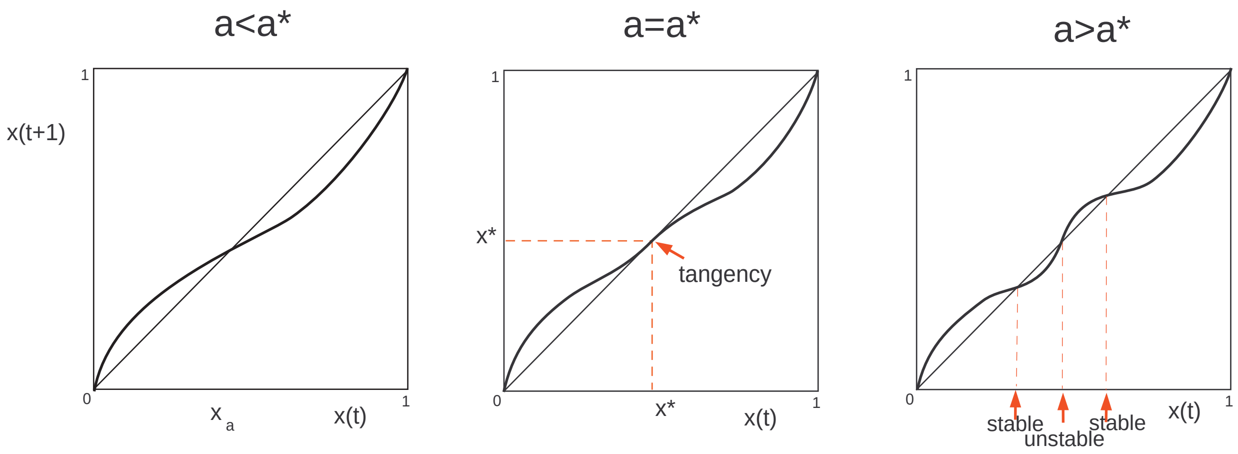

Figure 6. Pitch-fork bifurcation in the family f_a.

Figure 7 shows the bifurcation diagram. At x^* the stable fixed point becomes unstable and about it new stable fixed points appear.

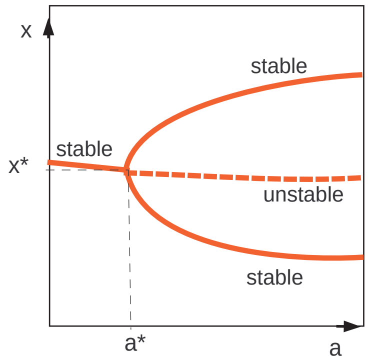

Figure 7. Bifurcation diagram of a pitch-fork bifurcation.

Exercise. Draw the graph of a function with an unstable fixed point that becomes stable by a pitch-fork bifurcation. Draw the corresponding bifurcation diagram.

3. Periodic points

In order to introduce another typical phenomenon of the discrete one-dimensional systems we study the dynamics determined by the family of smooth functions:

$$f_a = ax(1-x)$$

defined on the unit interval $I = [0, 1]$ for $a \in (0, 4]$.

Evidently, $x = 0$ is a fixed point, and since $f'(0) = a$, it is stable for all values of a less than 1.

For $a = 1$ the origin is therefore a non hyperbolic fixed point and for $a > 1$ it is unstable. We will denote by a_0 the value $a = 1$.

The equation $f_a(x) = x$ has as solution, besides $x = 0$, the point $\bar{x}_a = 1 - 1/a$, which is in the interval $[0, 1]$ for $a > 1$. The derivative at such point is $a(1 - 2\bar{x}_a) = 2 - a$, therefore $\bar{x}_a$ is stable for $1 < a < 3$.

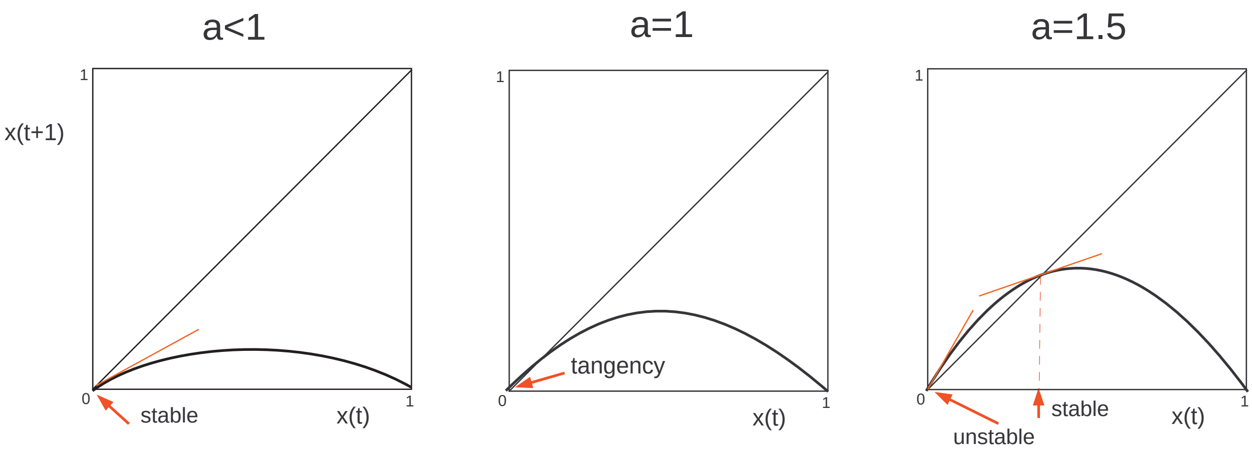

Figure 8. The logistic map for $a < 3$

The point $\bar{x}_a$ becomes unstable at $a = 3$. A trajectory starting near the equilibrium point $\bar{x}_a$ is like that in Figure 3, left, for $a < 3$ and like that in figure 3, right, for $a > 3$. We will denote the value $a = 3$ by a_1.

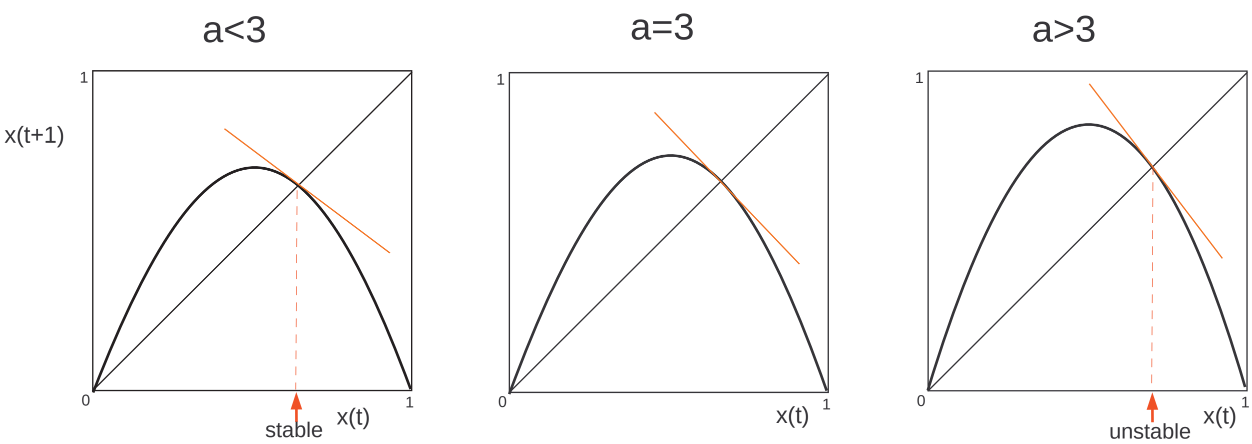

Figure 9. The logistic map about $a = 3$

But the question now is: where "the trajectory is going", i.e., does the succession $x(t)$, starting near $\bar{x}$, approach some set of points? In other words, does it exist an attracting set, which is not a fixed point?

The answer is yes. There is a value $a_2 > 3$ such that for $a_1 < a < a_2$, all trajectories starting at points different from 0 and non containing $\bar{x}_a$ are attracted towards a cycle of two points (see Figure 10).

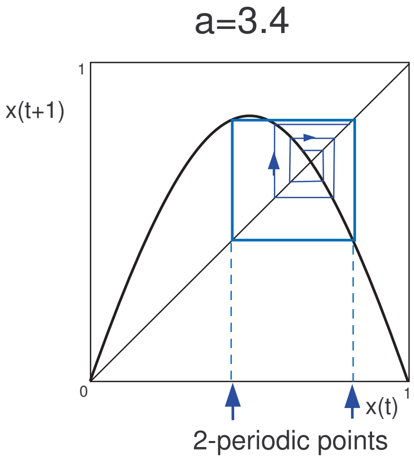

Figure 10. The attracting 2-cycle for $a = 3.4$

In fact, for every value of a between a_0 and a_1 there are two points x_1 and x_2 such that $f(x_1) = x_2$ and $f(x_2) = x_1$. The trajectory staring at x_1 or x_2 is therefore formed by $\{x_1, x_2, x_1, x_2, x_1, x_2, \dots\}$. Moreover, 'almost' all other trajectories tend to such a cycle. How to prove this?

We will consider, instead of the map f_a, the map $f_a^{(2)} := x \to f_a(f_a(x))$, the second iterate. It is evident that x_1 and x_2 satisfy

$$x_i = f_a^{(2)}(x_i) \quad i = 1, 2$$

i.e., they are fixed points for this map. We may apply Theorem 1 to valuate their stability: if they are attracting (repelling) for $f_a^{(2)}$, the cycle (x_1, x_2) will be attracting (repelling).

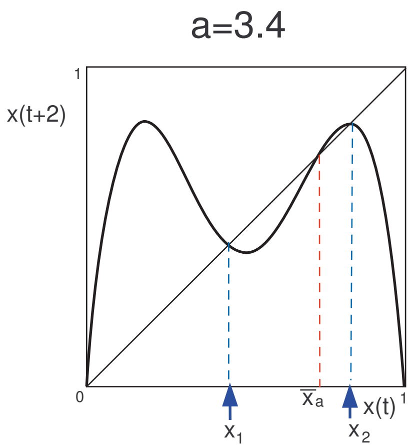

Figure 11. The stability of the attracting 2-cycle for $a = 3.4$

In Figure 11 we see that the absolute value of the slope of $f_a^{(2)}$ at x_1 and x_2 is less than 1.

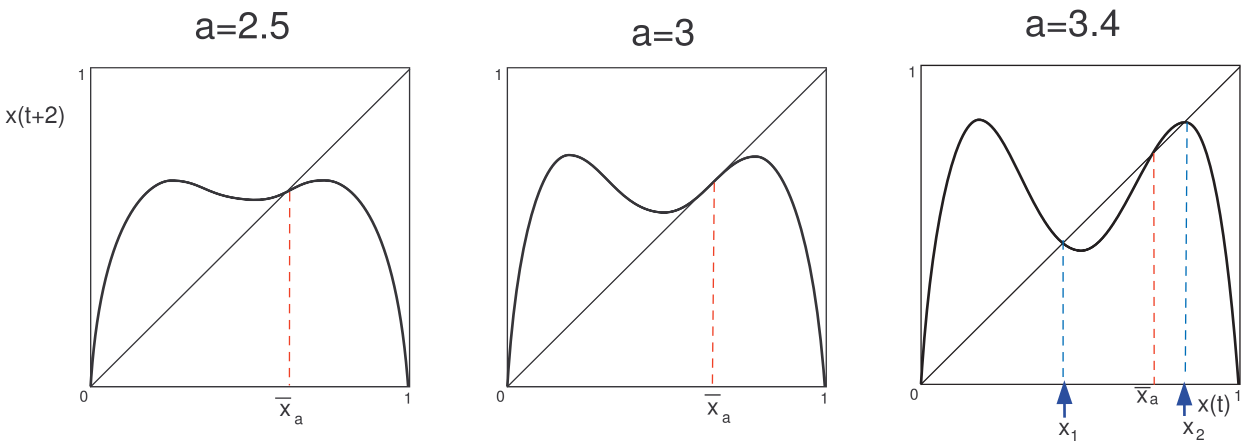

Figure 12. The pitchfork bifurcation of $f_a^{(2)}$ at $a = a_1$

Exercise. Prove that x_1, x_2 are the roots of the equation $a^2z^2 - a^2z - az + a + 1 = 0$. Find these roots.

4. Period doubling bifurcation

As a increases, the absolute value of the slope of $f_a^{(2)}$ at x_1 and x_2 increases (see Figure 12), till the value $a_2 = 1 + \sqrt{6} \approx 3.4495$ when it becomes equal to 1. For such a value of a the 2-cycle (x_1, x_2) loses stability. Observe that $f_a^{(2)}$, for $a > a_2$ has always four fixed points $(0, \bar{x}_a, x_1, x_2)$ but they are all unstable. Again, we ask: where the trajectories are going?

We observe that, locally, i.e. in a neighbourhood of x_1 or of x_2, the function $f_a^2(x)$ looks like the function $f_a(x)$ about $\bar{x}_a$ (its graph intersects the diagonal, the slope varying about -1). Therefore, if we now consider the iterate of $f_a^{(2)}(x)$, i.e. the fourth iterate $f_a^{(4)}(x) = f_a(f_a(f_a(f_a(x))))$, we expect a similar phenomenon near the points x_1 and x_2. I.e., for $a = a_2$ the function $f_a^{(4)}$ has contemporarily 2 pitchforks bifurcations in correspondence of the points x_1 and x_2, see Figure 13.

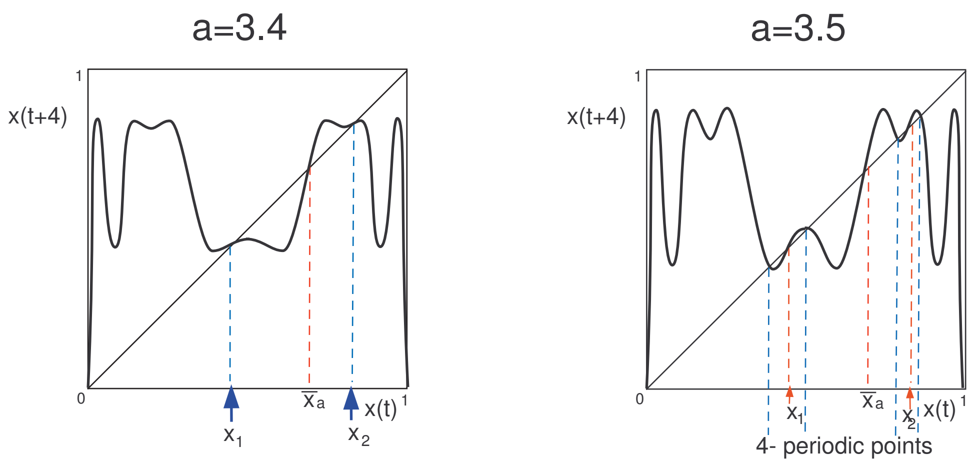

Figure 13. The loss of stability of x_1 and x_2 and birth of 4 stable 4-periodic points.

A cycle similar to that of Figure 10 continues to exist, but it is unstable and a double cycle of 4 points is the attracting set (see Figure 14).

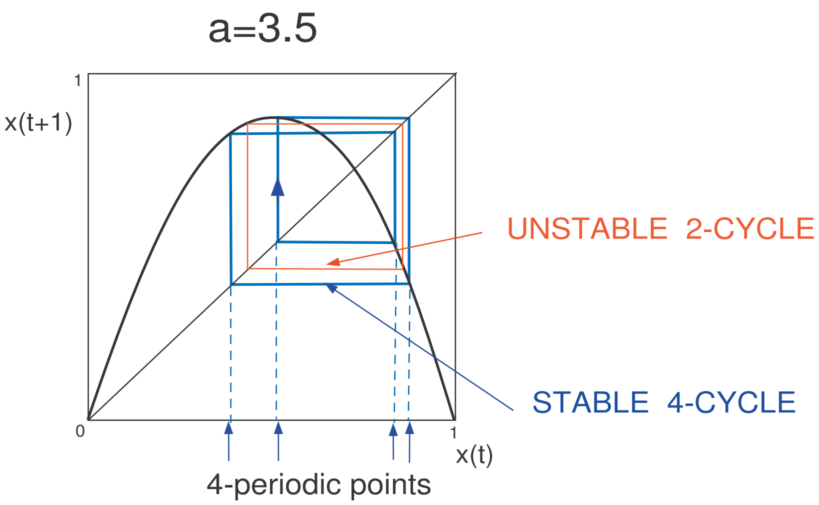

Figure 14. The stable 4-cycle and the unstable 2-cycle.

This phenomenon repeats when a increases: for a value $a_3 > a_2$ the 4-cycle loses stability: the map $f_a^{(4)}(f_a^{(4)}(x) = f^{(8)}(x)$ has 4 pitch-fork bifurcations and 8 new fixed points appear (i.e. 8-periodic points for f_a).

What we observe in the behaviour of the map f_a when a varies is not the pitch-fork bifurcation (which is visible in the 2^n-iterate of f_a), but a phenomenon which is called *period doubling bifurcation*, see figure 15.

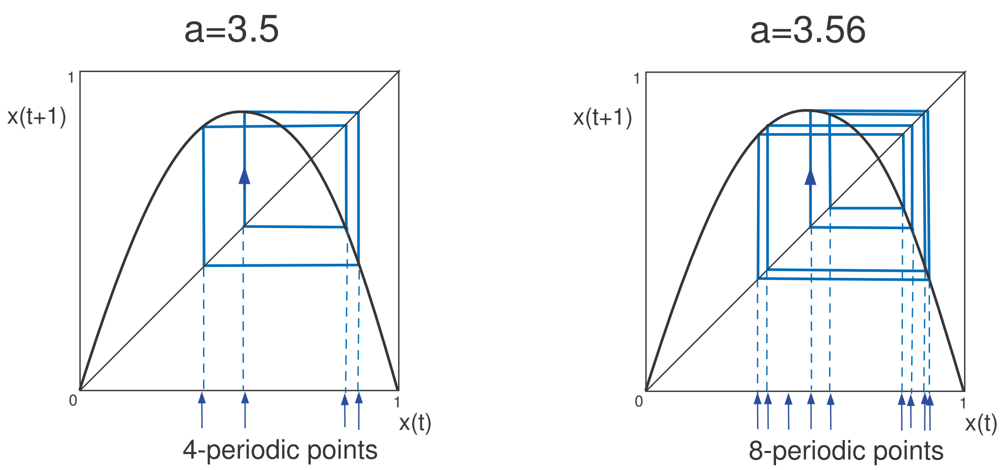

Figure 15. By a period doubling bifurcation a 4-cycle loses stability and appears a stable 8-cycle.

This phenomenon occurs for a succession of values a_i, (where the 2^{i-1}-cycle loses stability and the stable 2^i-cycle appears), which is converging to a value $a_\infty = 3.569946...$, and whose first values are

$$a_1 = 3, \quad a_2 \approx 3.49949, \quad a_3 \approx 3.54409, \quad a_4 \approx 3.5644, \quad a_5 \approx 3.5687.$$

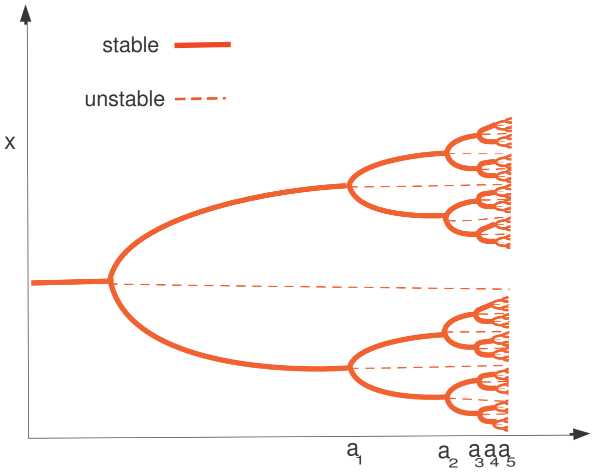

Figure 16. Scheme of the cascade of period doubling bifurcations.

5. Universality and Feigenbaum constants

This phenomenon of a succession of period adding bifurcations is not peculiar of the logistic map. Indeed, Feigenbaum proved in 1975 that every family $F_a = aF(x)$ of functions defined on the unit interval, such that F is at least 3 times differentiable and has a unique maximum in $[0, 1]$, exhibits the same behaviour. Such functions are said *unimodal*

Moreover, he found two 'universal constants', that are characteristics only of the cascade of doubling periods bifurcation, and not depend on the particular map we are using. These constants are denoted by δ and α:

$$\delta = \lim_{n\to\infty} \frac{a_n - a_{n-1}}{a_{n+1} - a_n} = 4.669201609102990671853203820...$$

The windows of the parameter values between successive bifurcation values decreases very rapidly.

The constant α is given by

$$\alpha = \lim_{n\to\infty} \frac{d_n}{d_{n+1}} = 2.502907875095892822283902873218...$$

where d_n is the distance between two branching points (coming from the preceding bifurcation) at the value $a = a_n$.

6. Chaos and other periods

At the value a_∞ the 'periodic cycle' is an infinite set of points which is called Feigenbaum attractor and has a fractal dimension equal to 0.538. This dimension is the same for unimodal maps.

For values of $a > a_\infty$ the map f_a has chaotic behaviour, but there are intervals where there are attracting stable cycles, as shown in this bifurcation diagram, where the stable attracting set is plotted versus a. The period 3 loses stability by a doubling period cascade, so that there are all $3 \cdot 2^n$ periodic points, characterised by the Feigenbaum constants.

Remark. The ratio between the diameters of successive circles on the real axis of the Mandelbrot set converges to the Feigenbaum constant δ.